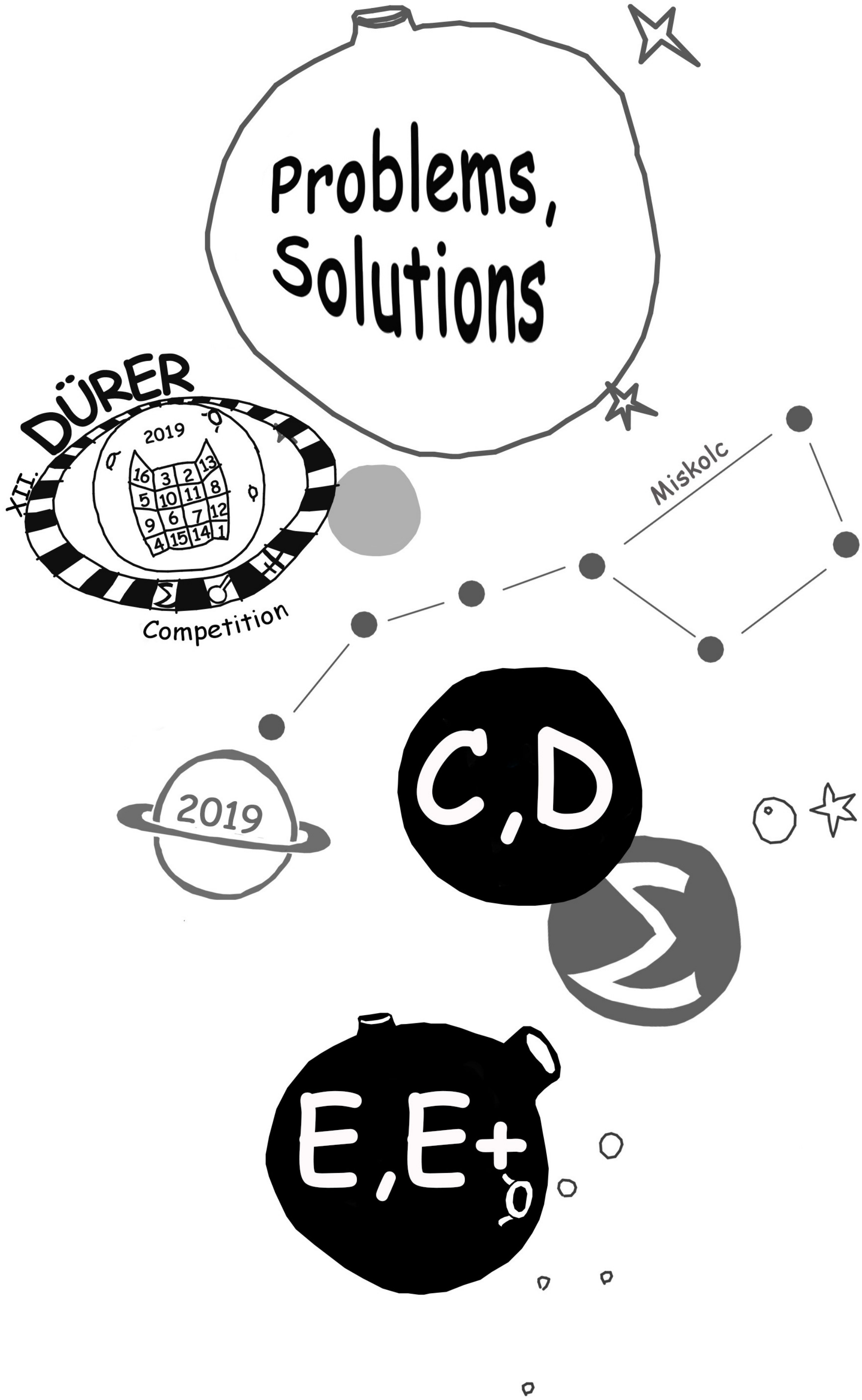
Problems,
Solutions
XII. DÜRER
2019
Competition
Miskolc
2019
C,D
E,E+

Contents

Introduction – About the Dürer Competition

Plenty of mathematics contests are traditionally held in Hungary. From primary schoolers to university students, everybody can find a contest that fits their age and qualifications. These are mostly individual contests where the participants sit down in a room for a few hours, working on the problems quietly. However at the Dürer Competition, there are teams of 3 taking part. For the duration of the contest each team works together to solve the problems, so the contestants can experience the benefits of cooperative thinking. Our experience shows that the majority of students are happier and more relaxed than during an individual contest.

Space was the theme of this year.

It is a very important goal for us to set interesting problems to show the beauty of mathematics and the joy of thinking to lots of students. We also wish to include as many original problems as possible. In each year, about 150 problems appear on the contest – of course not all can be original, but we invent most of the harder problems on our own.

At this point we definitely have to mention that the organising team traditionally consists of young people, mostly university students studying maths. This dates back to the early years of the competition, and ever since then, we can regularly welcome former competitors as new organisers. The success of the competition depends on this community, consisting of 30 to 70 people. Some of them have been organisers for 10 years already (and still take part enthusiastically, even alongside a full-time job) and some of them take important responsibilities as first-year undergraduates already.

This is the spirit in which we have been organising the contest for 12 years. The competition attracts more and more students and schools with each year. In the 2018–19 academic year, approximately 800 students competed in the high school maths categories.

Primary school students can take part in our competition in the following two categories: 5^{th} and 6^{th} grade students compete in *category A* while 7^{th} and 8^{th} grade students compete in *category B*. The contest is regional: it is organised in 6 cities in northeastern Hungary, but is open to anyone provided that they travel to one of the locations. (The problems of these two categories are not included in this booklet.)

Four categories are available for high school students:

> Category C is open to 9^{th} and 10^{th} graders who have never previously qualified for the final of any national math contest.
>
> Category D is open to 9^{th} to 12^{th} graders who are a bit more experienced, but do not come from a school that is outstanding in handling mathematical talents.
>
> Category E is open to 9^{th} to 12^{th} graders who already have good results from other contests, or come from a school outstanding in maths.
>
> Category E^{+} is designed for competitors who actively take part in olympiad training. In this category, most teams include some student who has taken part at an international

olympiad (IMO, MEMO, EGMO, RMM), or is about to qualify for one in the same academic year.

We also organise the contest in physics (*category F*) and chemistry (*categories K* and K^+), but these are also omitted from this booklet.

For high schoolers, the first round is a traditional olympiad-style contest, where detailed proofs have to be given. The teams have 3 hours to solve 5 problems. The contest can be sat in the whole country, at about 20 locations.

The final takes place in Miskolc every year. For high schoolers (categories C, D, E, E$^+$, F, K, K$^+$) we organise it on a weekend in early February from Thursday to Sunday. The first competition day is Friday, with the students working on five olympiad-style problems and a game. If a team thinks that they have found the winning strategy for the game, they can challenge us. If they can defeat us twice in a row, they get the maximal score for the problem. If they lose, they can still challenge us two more times for a partial score. On Saturday we hold a relay round consisting of 16 questions. The answer to each question is an integer between 0 and 9999. Initially each team gets the 1st question only. They have three attempts to submit an answer - if they get it right, they score a set number of points, and can proceed to the next question. Each wrong attempt to a question reduces the possible score by 1, and after 3 wrong attempts the team must move on to the next question without scoring. Rankings are based on a combined score from the two competition days.

At the weekend of the final, the students and teachers can participate in many educational and recreational activities, such as lectures, games and discussions about universities.

The competition is expanded year after year. In 2019–20 we plan to add an online round, so as to attract more entrants than ever before.

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1 Problems

1.1 First round

1.1.1 Category C

1. (Solution) My grandma believes she is getting younger since 5 years ago she was five times as old as I was back then and now she is only four times as old as I am.
a) How old is my grandma?
b) In how many years will she be three times as old as I will be then?

2. (Solution) The following 7 statements were made, concerning the same positive integer:
I. The number is less than 23. II. The number is less than 25. III. The number is less than 27. IV. The number is less than 29. V. The number is even. VI. The number is divisible by three. VII. The number is divisible by five.
We know that 4 of the statements are true and 3 are false. Find the largest possible value of the integer and justify why it cannot be larger.

3. (Solution) The best parts of grandma's 30 cm 30 cm square shaped pie are the edges. For this reason grandma's three grandchildren would like to split the pie between each other so that everyone gets the same amount (of the area) of the pie, but also of the edges. Can they cut the pie into three connected pieces like that?

4. (Solution) Albrecht constructed a strange machine. If one places a blue ball into the machine, it returns 5 red ones, whereas after putting in a red ball, it returns 5 blue ones.
a) Is it possible to get twice as many blue balls as red ones if we start with just one blue ball?
b) Is it possible to get twice as many blue balls as red ones if we start with two blue balls?
c) Is it possible to get the same number of red and blue balls if we start with just one blue ball?
d) Is it possible to get the same number of red and blue balls if we start with two blue balls?

5. (Solution) Three vampires keep biting each other for snack time. Whenever a vampire bites another, he collects his blood. If the targeted vampire has already got a mixture of blood from multiple vampires, then the vampire biting him "collects" all components of this mixture too. A vampire can never bite another one who has already collected the former's blood. (Bitings happen one at a time, never simultaneously.)
After they are finished with their snack, everyone gets as many cups of raspberry syrup as the number of fellow vampires whose blood they managed to collect. The goal of the three vampires is to get as much syrup altogether as they possibly can.
a) Give a sequence of bitings that guarantees the vampires as many cups of syrup as possible. *Even if you do not find the best solution, you can get partial credit.*
b) At most how many cups of syrup can they obtain? *Give a number as small as possible, so that they cannot get more cups than that number. Also justify why they cannot get more.*
c) What happens if there are six vampires? *Find an example when they get as many cups of syrup as possible. Similarly to the previous part, you should give a number as small as possible (with justification) so that they cannot get more than that.*

1.1.2 Category D

1. (Solution) The following 7 statements were made, concerning the same positive integer:
I. The number is less than 23. II. The number is less than 25. III. The number is less than 27. IV. The number is less than 29. V. The number is even. VI. The number is divisible by three. VII. The number is divisible by five.
We know that 4 of the statements are true and 3 are false. Find the largest possible value of the integer and justify why it cannot be larger.

2. (Solution) We would like to form four two-digit prime numbers using each of the digits $1, 2, 3, 4, 5, 6, 7$ and 9 exactly once.
a) Give an example of four such primes.
b) What are the possible sums of four such primes?

3. (Solution) Albrecht would like to choose some of the vertices of a regular 12-gon.
a) What is the maximal number of vertices he can choose such that no three of them form a right-angled triangle?
b) What is the maximal number of vertices he can choose such that no three of them form an obtuse triangle?

4. (Solution) Let ABC be an isosceles right-angled triangle, having the right angle at vertex C. Let us consider the line through C which is parallel to AB and let D be a point on this line such that $AB = BD$ and D is closer to B than to A. Find the angle CBD.

5. (Solution) Let a, b and c be natural numbers for which $a \mid b^2$, $b \mid c^2$ and $c \mid a^2$ hold.
a) With these conditions are there such a, b, c for which $abc \nmid (a + b + c)^6$?
b) Show that for any a, b, c satisfying the above conditions, $abc \mid (a + b + c)^7$ holds.
Note: $x \mid y$ means that x divides y, $x \nmid y$ means that x does not divide y.

1.1.3 Category E

1. (Solution) Non-negative integers a, b, c, d satisfy the equation $a + b + c + d = 100$ and there exists a non-negative integer n such that

$$a + n = b - n = c \cdot n = \frac{d}{n}.$$

Find all 5-tuples (a, b, c, d, n) satisfying all the conditions above.

2. (Solution) a) 11 kayakers row on the Danube from Szentendre to Kopaszi-gát. They do not necessarily start at the same time, but we know that they all take the same route and that each kayaker rows with a constant speed. Whenever a kayaker passes another one, they do a high five. After they all arrive, everybody claims to have done precisely 10 high fives in total. Show that it is possible for the kayakers to have rowed in such a way that this is true.
b) At a different occasion 13 kayakers rowed in the same manner; now after arrival everybody claims to have done precisely 6 high fives. Prove that at least one kayaker has miscounted.

3. (Solution) a) We are playing the following game on this table:
In each move we select a row or a column of the table, reduce two neighbouring numbers in that row or column by 1 and increase the third one by 1. After some of these moves can we get to a table with all the same entries?

1	2	3
4	5	6
7	8	9

b) This time we have the choice to arrange the integers from 1 to 9 in the 3×3 table. Still using the same moves now our aim is to create a table with all the same entries, maximising the value of the entries. What is the highest possible number we can achieve?

4. (Solution) Albrecht writes numbers on the points of the first quadrant with integer coordinates in the following way: If at least one of the coordinates of a point is 0, he writes 0; in all other cases the number written on point $(a; b)$ is one greater than the average of the numbers written on points $(a + 1; b - 1)$ and $(a - 1; b + 1)$. Which numbers could he write on point $(121; 212)$?
Note: The elements of the first quadrant are points where both of the coordinates are non-negative.

5. (Solution) Let ABC be a non-right-angled triangle, with $AC \neq BC$. Let F be the midpoint of side BC. Let D be a point on line AB satisfying $CA = CD$, and let E be a point on line BC satisfying $EB = ED$. The line passing through A and parallel to ED meets line FD at point I. Line AF meets line ED at point J. Prove that points C, I and J are collinear.

1.1.4 Category E$^+$

1. (Solution) a) We are playing the following game on this table:
In each move we select a row or a column of the table, reduce two neighbouring numbers in that row or column by 1 and increase the third one by 1. After some of these moves can we get to a table with all the same entries?

1	2	3
4	5	6
7	8	9

b) This time we have the choice to arrange the integers from 1 to 9 in the 3×3 table. Still using the same moves now our aim is to create a table with all the same entries, maximising the value of the entries. What is the highest possible number we can achieve?

2. (Solution) For a positive integer n let $P(n)$ denote the set of primes p for which there exist positive integers a, b such that $n = a^p + b^p$. Is it true that for any finite set H consisting of primes, there is an n such that $P(n) = H$?

3. (Solution) Anne has thought of a finite set $A \subset \mathbb{R}^2$. Bob does not know how many elements A has, but his goal is to completely determine A. To achieve this, Bob can choose any point $b \in \mathbb{R}^2$ and ask Anne how far it is from A. Anne replies with the distance, defined as $\min\{d(a; b) \mid a \in A\}$. (Here $d(a; b)$ denotes the distance between points $a; b \in \mathbb{R}^2$.) Bob can ask as many questions of this type as he wants, until he can determine A with certainty.
a) Can Bob achieve his goal with finitely many questions?
b) What if Anne tells Bob in advance that all points of A have both coordinates in the interval $[0; 1]$?
Note: $\mathbb{R}^2$ *is the set of points in the plane.*

4. (Solution) An n-tuple $(x_1, x_2, \dots, x_n)$ is called *unearthly* if $q_1x_1 + q_2x_2 + \dots + q_nx_n$ is irrational for any non-negative rational coefficients $q_1, q_2, \dots, q_n$ where q_i's are not all zero. Prove that it is possible to select an unearthly n-tuple from any $2n - 1$ distinct irrational numbers.

5. (Solution) Let ABC and $A'B'C'$ be similar triangles with different orientation such that their orthocentres coincide. Show that lines AA', BB', CC' are concurrent or parallel.

1.2 Final round – day 1

1.2.1 Category C

1. (Solution) At the end of the Dürer Competition, Hilda wanted to buy an asteroid, a shuttle and a space dog on her Dürer Dollars. Unfortunately neither of these items is sold in the Dürer Shop, so she decided to visit the Space Market instead. She heard that Sophie bought one asteroid, three shuttles and five space dogs and paid 200 Dürer Dollars in total. Lucy told her that she could buy one asteroid, four shuttles and seven space dogs for 225 Dürer Dollars. How much money should Hilda take with her if she wants to pay the exact amount for her purchase?

2. (Solution) Mr. Wilson, 12 actors and a reporter were invited to a party. The reporter knows that Mr. Wilson knows all the actors, but none of the actors know Mr. Wilson. She knows nothing about the relationships between the actors and she does not know who Mr. Wilson is either. This is what she wants to find out. To accomplish this, she can go up to an arbitrary guest and ask if he knows another arbitrary guest. Show that the reporter can have a strategy that after 12 such questions she can be sure about who Mr. Wilson is.

3. (Solution) a) Can we fill a 4×4 table with digits in a way that reading from left to right every four-digit number we get is even, and reading from top to bottom every four-digit number is odd?
b) Can we fill a 4×4 table with digits in a way that reading from left to right every four-digit number we get is divisible by 3, and reading from top to bottom every four-digit number divided by 3 has remainder 1?
c) Can we fill a 4×4 table with only the digits 1 and 2 in a way that reading from left to right and reading from top to bottom the eight four-digit numbers we get all have different remainders divided by 8?

4. (Solution) A card game consists of cards featuring various symbols. Each card contains the same number of symbols, no card shows the same symbol twice, and any two cards have exactly one symbol in common.
a) What is the least possible number of different symbols in the game if there are seven cards in total and each of them shows six symbols?
b) What is the least possible number of different symbols in the game if there are 100 cards in total and each of them shows 99 symbols?

5. (Solution) A, B, C, D are four distinct points such that triangles ABC and CBD are both equilateral. Find as many circles as you can, which are equidistant from the four points. How can these circles be constructed?
Remark: The distance between a point P and a circle c is measured as follows: we join P and the centre of the circle with a straight line, and measure how much we need to travel along this line (starting from P) to hit the perimeter of the circle. If P is an internal point of the circle, the distance is the length of the shorter such segment. The distance between a circle and its centre is the radius of the circle.

6. (Solution) Game: Vincent and Henry take turns to place 1×2 dominos on a 4×4 board. Vincent always places a vertical domino whereas Henry always places a horizontal one. Vincent goes first, and a player loses if he cannot make a legal move on his turn.
Defeat the organisers in this game twice in a row! You can decide whether you want to go first or second.

1.2.2 Category D

1. (Solution) Dim-witted Douglas was taking a walk when he reached a fork in the road. Although he knew he would get to the same place whichever way he chose, he still had to decide which way to go. The road on the left – although it was straight – seemed very long, so he chose the road on the right without hesitation. This met the road on the left perpendicularly and consisted of two straight segments, both of length of a whole number of miles. Having chosen the road on the right, he walked 99 miles in total. We know that the length of the left road is also a whole number of miles. How many miles would have Douglas walked if he had chosen the road on the left?

2. (Solution) A card game consists of cards featuring various symbols. Each card contains the same number of symbols, no card shows the same symbol twice, and any two cards have exactly one symbol in common.
a) What is the least possible number of different symbols in the game if there are seven cards in total and each of them shows six symbols?
b) What is the least possible number of different symbols in the game if there are 100 cards in total and each of them shows 99 symbols?

3. (Solution) a) Does there exist a quadrilateral with (both of) the following properties: three of its edges are of the same length, but the fourth one is different, and three of its angles are equal, but the fourth one is different?
b) Does there exist a pentagon with (both of) the following properties: four of its edges are of the same length, but the fifth one is different, and four of its angles are equal, but the fifth one is different?

4. (Solution) a) Is it possible to colour the positive rational numbers with blue and red in a way that there are numbers of both colours and the sum of any two numbers of the same colour is the same colour as them?
b) Is it possible to colour the positive rational numbers with blue and red in a way that there are numbers of both colours and the product of any two numbers of the same colour is the same colour as them?
Note: When forming a sum or a product, it is allowed to pick the same number twice.

5. (Solution) a) Show that it is possible to arrange the integers between 1 and 16 into 8 fractions such that every number is used precisely once, and the sum of the 8 fractions is an integer.
b) Show that it is possible to arrange the integers between 1 and 64 into 32 fractions such that every number is used precisely once, and the sum of the 32 fractions is an integer.
Note: for example the integers between 1 and 6 can be arranged in the desired manner as follows: $5/1 + 3/2 + 6/4 = 8$.

6. (Solution) Game: Vincent and Henry take turns to place 1×2 dominos on a 4×4 board. Vincent always places a vertical domino whereas Henry always places a horizontal one. Vincent goes first, and a player loses if he cannot make a legal move on his turn.
Defeat the organisers in this game twice in a row! You can decide whether you want to go first or second.

1.2.3 Category E

1. (Solution) a) Is it possible to colour the positive rational numbers with blue and red in a way that there are numbers of both colours and the sum of any two numbers of the same colour is the same colour as them?
b) Is it possible to colour the positive rational numbers with blue and red in a way that there are numbers of both colours and the product of any two numbers of the same colour is the same colour as them?
Note: When forming a sum or a product, it is allowed to pick the same number twice.

2. (Solution) Albrecht fills in each cell of an 8×8 table with a 0 or a 1. Then at the end of each row and column he writes down the sum of the 8 digits in that row or column, and then he erases the original digits in the table. Afterwards, he claims to Berthold that given only the sums, it is possible to restore the 64 digits in the table uniquely. Show that the 8×8 table contained either a row full of 0's or a column full of 1's.

3. (Solution) Determine all triples (p, q, r) of prime numbers for which $p^q + p^r$ is a perfect square.

4. (Solution) Let ABC be an acute-angled triangle having angles α, β, γ at vertices A, B, C respectively. Let isosceles triangles BCA_1, CAB_1, ABC_1 be erected outwards on its sides, with apex angles 2α, 2β, 2γ respectively. Let A_2 be the intersection point of lines AA_1 and B_1C_1 and let us define points B_2 and C_2 analogously. Find the exact value of the expression

$$\frac{AA_1}{A_2A_1} + \frac{BB_1}{B_2B_1} + \frac{CC_1}{C_2C_1}.$$

5. (Solution) In one of the hotels of the wellness planet Oxys, there are 2019 saunas. The managers have decided to accommodate k couples for the upcoming long weekend. We know the following about the guests: if two women know each other then their husbands also know each other, and vice versa. There are several restrictions on the usage of saunas. Each sauna can be used by either men only, or women only (but there is no limit on the number of people using a sauna at once, as long as they are of a single gender). Each woman is only willing to share a sauna with women whom she knows, and each man is only willing to share a sauna with men whom he does not know. What is the greatest possible k for which we can guarantee, without knowing the exact relationships between the couples, that all the guests can use the saunas simultaneously while respecting the restrictions above?

6. (Solution) Game: At the beginning of the game, the organisers place paper disks on the table, grouped into piles which may contain various numbers of disks. The two players take turns. On a player's turn, their opponent selects two piles (one if there is only one pile left), and the player must remove some number of disks from one of the piles selected. This means that at least one disk has to be removed, and removing all disks in the pile is also permitted. The player removing the last disk from the table wins.
Defeat the organisers in this game twice in a row! A starting position will be given and then you can decide whether you want to go first or second.

1.2.4 Category E$^+$

1. (Solution) Let $a_0; a_1; \dots; a_n$ be a non-decreasing sequence of $n+1$ real numbers where $a_0 = 0$ and for every $j > i$ we have $a_j - a_i \le j - i$. Show that

$$\left(\sum_{i=0}^{n} a_i\right)^2 \ge \sum_{i=0}^{n} a_i^3 .$$

2. (Solution) Prove that if a triangle has integral side lengths and its circumradius is a prime number then the triangle is right-angled.

3. (Solution) For every positive integer $n \ge 2$, let $f(n)$ be the sum of all positive integers between 1 and n inclusive which are not coprime to n. Let $p > 2$ be a prime and $n \ge 2$ such that $p \nmid n$.
Show that $f(n+p) \neq f(n)$.

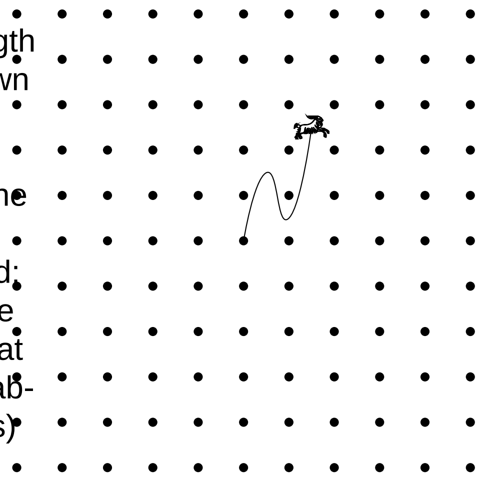
4. (Solution) In the Intergalactic Lottery, 7 numbers are drawn out of 55. R2-D2 and C-3PO decide that they want to win this lottery, so they fill out lottery tickets separately such that for each possible draw one of them does have a winning ticket for that draw.
Prove that one of them has 7 tickets with all different numbers.

5. (Solution) Let ABC be an acute triangle and let X, Y, Z denote the midpoints of the shorter arcs BC, CA, AB of its circumcircle, respectively. Let M be an arbitrary point on side BC.
The line through M, parallel to the inner angular bisector of $\angle CBA$ meets the outer angular bisector of $\angle BCA$ at point N. The line through M, parallel to the inner angular bisector of $\angle BCA$ meets the outer angular bisector of $\angle CBA$ at point P.
Prove that lines XM, YN, ZP pass through a single point.

6. (Solution) Game: At the beginning of the game, the organisers place paper disks on the table, grouped into piles which may contain various numbers of disks. The two players take turns. On a player's turn, their opponent selects two piles (one if there is only one pile left), and the player must remove some number of disks from one of the piles selected. This means that at least one disk has to be removed, and removing all disks in the pile is also permitted. The player removing the last disk from the table wins.
Defeat the organisers in this game twice in a row! A starting position will be given and then you can decide whether you want to go first or second.

1.3 Final round day 2

1.3.1 Category C



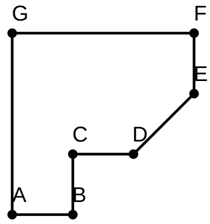
C-1. Simple John has a square cabbage field of side length 10 m. The cabbages in the field form a square grid as shown in the figure. In each row and column there are 11 cabbages, with each neighbouring pair being 1 metre apart, except that there is no cabbage in the exact centre of the field.
One day John decided to let his goat take care of the field; with a 5-metre long rope he tethered it to a stake at the centre of the square. By the time John woke up, the goat had eaten all the cabbages it could reach. How many cabbages did John still have remaining? (3 points)

C-2. What is the total number of minutes in a day when at least one digit 2 appears on the display of a 24-hour digital clock? (A digital clock displays the time in the 24-hour format, ranging from 00:00 to 23:59.) (3 points)

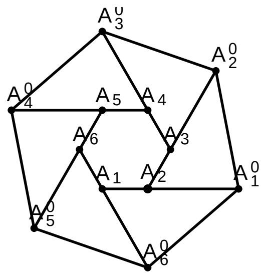
C-3. What is the smallest positive integer in which the product of all digits is 200? (3 points)

C-4. Bob wrote down all positive integers from 1 to 2019 in ascending order without thousands separators or spaces. What is the 2019^{th} digit in the resulting string? (3 points)

C-5. What is the maximum possible number of 90-degree angles that a polygon with 12 vertices can have?
Note: The polygon cannot intersect itself. We only count interior right angles, so a 270-degree angle does not count. For example, the heptagon in the figure has four 90-degree angles, namely at the vertices A, B, F and G. (4 points)



C-6. Let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon. Extend side A_iA_{i+1} beyond vertex A_{i+1} until a new endpoint A_i' such that A_{i+1} is the trisecting point of segment A_iA_i' closer to A_i (where $1 \le i \le 6$, $A_7 = A_1$). If the original hexagon has area 1, what is the area of the hexagon $A_1'A_2'A_3'A_4'A_5'A_6'$? (4 points)



C-7. A bicycle race consists of 15 days. A cyclist's final result is determined by adding together the times achieved on each race day, and overall rankings are determined by comparing these total times. This year there were 500 cyclists competing, and everybody managed to complete every race day. Albrecht crossed the finish line in 7^{th} position on each day. What is the worst possible overall ranking he may have achieved? (4 points)

C-8. The following 100 statements are written on a piece of paper:

1. At least 1 of the statements is false.
2. Exactly 2 of the statements are true.
3. At least 3 of the statements are false.
4. Exactly 4 of the statements are true.

$\vdots$

99. At least 99 of the statements are false.
100. Exactly 100 of the statements are true.

How many of the 100 statements are true? (4 points)

C-9. What remainder do we get if we divide the value of the following expression by 2019?

$$0 - 1 - 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 2019$$

The terms with negative sign are precisely the powers of 2. (5 points)

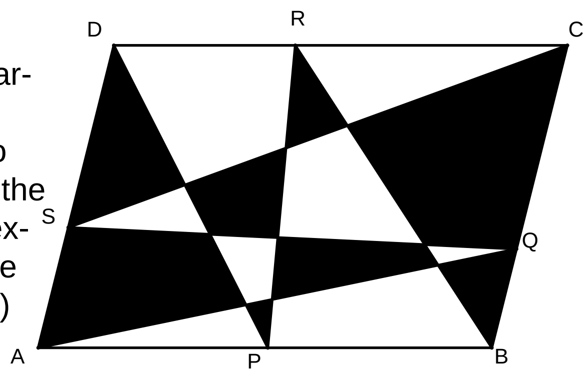
C-10. In Miskolc there are two tram lines: line 1 runs between Tiszai railway station and Upper-Majláth, while line 2 runs between Tiszai railway station and the Ironworks. The timetable for trams leaving Tiszai railway station is as follows: tram 1 leaves at every minute ending in a 0 or 6, and tram 2 leaves at every minute ending in a 3. There are three types of passengers waiting for the trams: those who will take tram 1 only, those who will take tram 2 only and those who will take any tram. Every minute there is a constant number of passengers of each type arriving at the station. (This number is not necessarily the same for the different types.) Also, every tram departs with an equal number of passengers from Tiszai railway station. How many passengers are there on a departing tram, if we know that every minute there are 3 passengers arriving at the station who will take tram 2 only? (5 points)

C-11. We call a non-negative integer *nice* if it can be obtained by taking some number n and subtracting the sum of the digits of n from n. (So for example 27 is nice, as it can be written e.g. as $34 - 3 - 4$.) How many nice numbers are there between 0 and 10000? (5 points)

C-12. Let O be the incenter of triangle ABC. Given that the angle BAC is one-third of the angle BOC, find the angle BAC. (5 points)

C-13. We write down all natural numbers from 1 to 100000 in ascending order, without spaces or thousands separators. In the so-obtained string, how long is the longest sequence of digits that appears more than once? (6 points)

C-14. We choose a point on each side of a parallelogram $ABCD$; let these four points be P, Q, R and S. Then we divide the parallelogram into several regions using line segments as shown in the figure. The areas of the grey regions are given, except for one (see the figure). Find the area of the region marked with a question mark. (6 points)

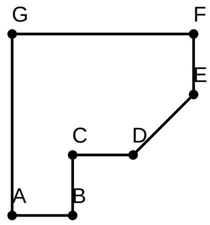


C-15. How many ways are there to arrange the numbers $1, 2, 3, 4, 5, 6, 7$ in some order such that for any two numbers which are 2 or 3 positions apart, the one on the left is greater? (6 points)

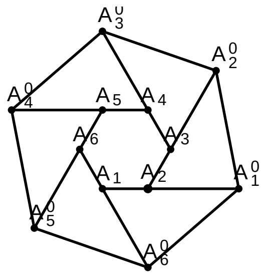
C-16. How many ways are there to go from point $(-4; 0)$ to point $(4; 0)$ such that our route fully stays on the integer grid-lines, has length 16, and does not touch any point $(x; y)$ for which $|x| + |y| \quad 3$? (6 points)

1.3.2 Category D

D-1. What is the maximum possible number of 90-degree angles that a polygon with 12 vertices can have?
Note: The polygon cannot intersect itself. We only count interior right angles, so a 270-degree angle does not count. For example, the heptagon in the figure has four 90-degree angles, namely at the vertices A, B, F and G. (3 points)



D-2. Let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon. Extend side A_iA_{i+1} beyond vertex A_{i+1} until a new endpoint A_i' such that A_{i+1} is the trisecting point of segment A_iA_i' closer to A_i (where $1 \le i \le 6$, $A_7 = A_1$). If the original hexagon has area 1, what is the area of the hexagon $A_1'A_2'A_3'A_4'A_5'A_6'$? (3 points)



D-3. A bicycle race consists of 15 days. A cyclist's final result is determined by adding together the times achieved on each race day, and overall rankings are determined by comparing these total times. This year there were 500 cyclists competing, and everybody managed to complete every race day. Albrecht crossed the finish line in 7^{th} position on each day. What is the worst possible overall ranking he may have achieved? (3 points)

D-4. The following 100 statements are written on a piece of paper:

1. At least 1 of the statements is false.
2. Exactly 2 of the statements are true.
3. At least 3 of the statements are false.
4. Exactly 4 of the statements are true.

$\vdots$

99. At least 99 of the statements are false.
100. Exactly 100 of the statements are true.

How many of the 100 statements are true? (3 points)

D-5. What remainder do we get if we divide the value of the following expression by 2019?

$$0 - 1 - 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 2019$$

The terms with negative sign are precisely the powers of 2. (4 points)

D-6. Let O be the incenter of triangle ABC. Given that the angle BAC is one-third of the angle BOC, find the angle BAC. (4 points)

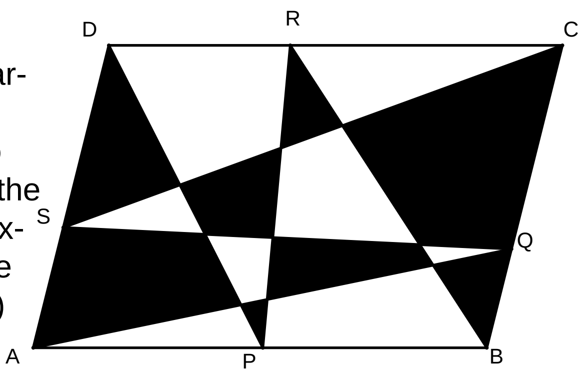
D-7. We call a non-negative integer n nice if it can be obtained by taking some number n and subtracting the sum of the digits of n from n. (So for example 27 is nice, as it can be written e.g. as 34 3 4.) How many nice numbers are there between 0 and 10000? (4 points)

D-8. In Miskolc there are two tram lines: line 1 runs between Tiszai railway station and Upper-Majláth, while line 2 runs between Tiszai railway station and the Ironworks. The timetable for trams leaving Tiszai railway station is as follows: tram 1 leaves at every minute ending in a 0 or 6, and tram 2 leaves at every minute ending in a 3. There are three types of passengers waiting for the trams: those who will take tram 1 only, those who will take tram 2 only and those who will take any tram. Every minute there is a constant number of passengers of each type arriving at the station. (This number is not necessarily the same for the di erent types.) Also, every tram departs with an equal number of passengers from Tiszai railway station. How many passengers are there on a departing tram, if we know that every minute there are 3 passengers arriving at the station who will take tram 2 only? (4 points)

D-9. We write down all natural numbers from 1 to 100000 in ascending order, without spaces or thousands separators. In the so-obtained string, how long is the longest sequence of digits that appears more than once? (5 points)

D-10. Find the smallest multiple of 81 that only contains the digit 1. How many 1's does it contain? (5 points)

D-11. We choose a point on each side of a parallelogram $ABCD$; let these four points be P, Q, R and S. Then we divide the parallelogram into several regions using line segments as shown in the gure. The areas of the grey regions are given, except for one (see the gure). Find the area of the region marked with a question mark. (5 points)



D-12. A chess piece is placed on one of the squares of an 8 8 chessboard where it begins a tour of the board: it moves from square to square, only moving horizontally or vertically. It visits every square precisely once, and ends up exactly where it started. What is the maximum number of times the piece can change direction along its tour? (5 points)

D-13. We call a positive integer n a yo-yo number if any two adjacent digits in it di er by at least 7. How many six-digit yo-yo numbers are there? (6 points)

D-14. The sum of all positive integers that have exactly two proper divisors, with both of them being less than 30, is 2397. What is the sum of all positive integers that have exactly three proper divisors, with all three of them being less than 30?
A proper divisor of a number n is a positive integer that divides n and is not equal to n.
(6 points)

D-15. In an isosceles, obtuse-angled triangle, the lengths of two internal angle bisectors are in a 2:1 ratio. Find the obtuse angle of the triangle. (6 points)

D-16. How many ways are there to arrange the numbers $1, 2, 3, \ldots, 15$ in some order such that for any two numbers which are 2 or 3 positions apart, the one on the left is greater?
(6 points)

1.3.3 Category E

E-1. Find the number of non-isosceles triangles (up to congruence) with integral side lengths, in which the sum of the two shorter sides is 19. (3 points)

E-2. Anne multiplies each two-digit number by 588 in turn, and writes down the so-obtained products. How many perfect squares does she write down? (3 points)

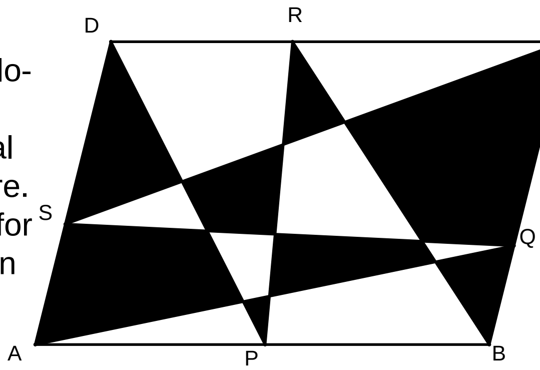
E-3. On a piece of paper we have 2019 statements numbered from 1 to 2019. The n^{th} statement is the following: "On this piece of paper there are at most n true statements". How many of the statements are true? (3 points)

E-4. In Miskolc there are two tram lines: line 1 runs between Tiszai railway station and Upper-Majláth, while line 2 runs between Tiszai railway station and the Ironworks. The timetable for trams leaving Tiszai railway station is as follows: tram 1 leaves at every minute ending in a 0 or 6, and tram 2 leaves at every minute ending in a 3. There are three types of passengers waiting for the trams: those who will take tram 1 only, those who will take tram 2 only and those who will take any tram. Every minute there is a constant number of passengers of each type arriving at the station. (This number is not necessarily the same for the different types.) Also, every tram departs with an equal number of passengers from Tiszai railway station. How many passengers are there on a departing tram, if we know that every minute there are 3 passengers arriving at the station who will take tram 2 only? (3 points)

E-5. We want to write down as many distinct positive integers as possible, so that no two numbers on our list have a sum or a difference divisible by 2019. At most how many integers can appear on such a list? (4 points)

E-6. Find the smallest multiple of 81 that only contains the digit 1. How many 1's does it contain? (4 points)

E-7. We choose a point on each side of a parallelogram $ABCD$; let these four points be P, Q, R and S. Then we divide the parallelogram into several regions using line segments as shown in the figure. The areas of the grey regions are given, except for one (see the figure). Find the area of the region marked with a question mark. (4 points)



E-8. A chess piece is placed on one of the squares of an 8×8 chessboard where it begins a tour of the board: it moves from square to square, only moving horizontally or vertically. It visits every square precisely once, and ends up exactly where it started. What is the maximum number of times the piece can change direction along its tour? (4 points)

E-9. A cube has been divided into 27 equal-sized sub-cubes. We take a line that passes through the interiors of as many sub-cubes as possible. How many does it pass through? (5 points)

E-10. In an isosceles, obtuse-angled triangle, the lengths of two internal angle bisectors are in a 2:1 ratio. Find the obtuse angle of the triangle. (5 points)

E-11. What is the smallest possible value of the least common multiple of a, b, c, d if we know that these four numbers are distinct and $a + b + c + d = 1000$? (5 points)

E-12. How many ways are there to arrange the numbers $1, 2, 3, \ldots, 15$ in some order such that for any two numbers which are 2 or 3 positions apart, the one on the left is greater? (5 points)

E-13. Let $k > 1$ be a positive integer and $n \geq 2019$ be an odd positive integer. The non-zero rational numbers $x_1; x_2; \ldots; x_n$ are not all equal, and satisfy the following chain of equalities:

$$x_1 + \frac{k}{x_2} = x_2 + \frac{k}{x_3} = x_3 + \frac{k}{x_4} = \ldots = x_{n-1} + \frac{k}{x_n} = x_n + \frac{k}{x_1}:$$

What is the smallest possible value of k? (6 points)

E-14. Seven classmates are comparing their end-of-year grades in 12 subjects. They observe that for any two of them, there is some subject out of the 12 where the two students got different grades. It is possible to choose n subjects out of the 12 such that if the seven students only compare their grades in these n subjects, it will still be true that for any two, there is some subject out of the n where they got different grades. What is the smallest value of n for which such a selection is surely possible?
Note: In Hungarian high schools, students receive an integer grade from 1 to 5 in each subject at the end of the year. (6 points)

E-15. ABC is an isosceles triangle such that $AB = AC$ and $\angle BAC = 96^\circ$. D is the point for which $\angle ACD = 48^\circ$, $AD = BC$ and triangle DAC is obtuse-angled. Find $\angle DAC$. (6 points)

E-16. How many ways are there to paint the squares of a 6×6 board black or white such that within each 2×2 square on the board, the number of black squares is odd? (6 points)

1.3.4 Category E $^+$

E$^+$-1. We want to write down as many distinct positive integers as possible, so that no two numbers on our list have a sum or a difference divisible by 2019. At most how many integers can appear on such a list? (3 points)

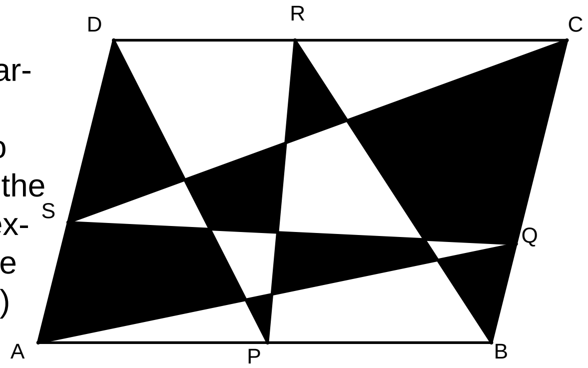
E$^+$-2. Find the smallest multiple of 81 that only contains the digit 1. How many 1's does it contain? (3 points)

E$^+$-3. Let P be an interior point of triangle ABC. The lines AP, BP and CP divide each of the three sides into two segments. If the so-obtained six segments all have distinct integer lengths, what is the minimum possible perimeter of ABC? (3 points)

E$^+$-4. A cube has been divided into 27 equal-sized sub-cubes. We take a line that passes through the interiors of as many sub-cubes as possible. How many does it pass through? (3 points)

E$^+$-5. How many permutations s does the set $\{1, 2, \ldots, 15\}$ have with the following properties: for every $1 \le k \le 13$ we have $s(k) < s(k+2)$ and for every $1 \le k \le 12$ we have $s(k) < s(k+3)$? (4 points)

E$^+$-6. We choose a point on each side of a parallelogram $ABCD$; let these four points be P, Q, R and S. Then we divide the parallelogram into several regions using line segments as shown in the figure. The areas of the grey regions are given, except for one (see the figure). Find the area of the region marked with a question mark. (4 points)



E$^+$-7. Find the smallest positive integer n with the following property: if we write down all positive integers from 1 to 10^n and add together the reciprocals of every non-zero digit written down, we obtain an integer. (4 points)

E$^+$-8. Let N be a positive integer such that N and N^2 both end in the same four digits $\overline{abcd}$, where $a \neq 0$. What is the four-digit number $\overline{abcd}$? (4 points)

E$^+$-9. How many ways are there to paint the squares of a 6×6 board black or white such that within each 2×2 square on the board, the number of black squares is odd? (5 points)

E$^+$-10. What is the smallest possible value of the least common multiple of a, b, c, d if we know that these four numbers are distinct and $a + b + c + d = 1000$? (5 points)

E$^+$-11. What is the smallest N for which

$$\sum_{k=1}^{N} k^{2018}$$

is divisible by 2018? (5 points)

E$^+$-12. P and Q are two different non-constant polynomials such that $P(Q(x)) = P(x)Q(x)$ and $P(1) = P(-1) = 2019$. What are the last four digits of $Q(P(-1))$? (5 points)

E^+-13. There are 12 chairs arranged in a circle, numbered from 1 to 12. How many ways are there to select some of the chairs in such a way that our selection includes 3 consecutive chairs somewhere? (6 points)

E^+-14. Let S be the set of all positive integers less than 10,000 whose last four digits in base 2 are the same as its last four digits in base 5. What remainder do we get if we divide the sum of all elements of S by 10000? (6 points)

E^+-15. The positive integer m and non-negative integers $x_0, x_1, \dots, x_{1001}$ satisfy the following equation:

$$m^{x_0} = \sum_{i=1}^{1001} m^{x_i}.$$

How many possibilities are there for the value of m? (6 points)

E^+-16. Triangle ABC has side lengths 13, 14 and 15. Let k, k_A, k_B, k_C be four circles of radius r inside the triangle such that k_A is tangent to sides AB and AC, k_B is tangent to sides BA and BC, k_C is tangent to sides CA and CB, and k is externally tangent to circles k_A, k_B and k_C. Let $r = \frac{m}{n}$ where m and n are coprime. Find $m + n$. (6 points)

2 Solutions

2.1 First round

2.1.1 Category C

1. (Back to problem) a) Let us consider the age difference between me and grandma. 5 years ago my age was one fourth of the difference, now one third. This means that one third of the difference is 5 more than the one fourth of it. Since $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, one twelfth of the age difference is 5 years. So the age difference is 60 years. The third of it is 20 years, this means that I am now 20 years old and my grandma is 80 years old.

b) We already know that the age difference is 60 years. When she is going to be exactly three times as old as I will be, the age difference will be twice my age. This will happen when I am 30 years old, which is in 10 years, and indeed grandma will be 90 years old.

2. (Back to problem) Note that if one of the first 4 statements is false, then all previous statements are also false. Therefore statement IV. must be true (if it was false, then we would have at least 4 false statements). So the number is "< 29", i.e. at most 28.

Let us check the integers from 28 downwards and find the first one for which exactly 4 of the statements are true. (See the table below.)

	< 23	< 25	< 27	< 29	even	$3 \mid$	$5 \mid$	# true statements
28	False	False	False	True	True	False	False	2
27	False	False	False	True	False	True	False	2
26	False	False	True	True	True	False	False	3
25	False	False	True	True	False	False	True	3
24	False	True	True	True	True	True	False	5
23	False	True	True	True	False	False	False	3
22	True	True	True	True	True	False	False	5
21	True	True	True	True	False	True	False	5
20	True	True	True	True	True	False	True	6
19	True	True	True	True	False	False	False	4

We can see that 19 is the first number for which exactly 4 of the statements hold, so the answer is 19.

3. (Back to problem) It is possible to split the pie as required. See an example below.

Firstly we will divide the circumference of the pie into 3 connected pieces of equal length and then we will try to split the pie into three pieces of equal area so that they cut the above pieces out of the circumference.

The perimeter of the pie is 120 cm, therefore each circumference-part should be 40 cm long. Let the vertices of the pie be denoted by A, B, C and D (in this order). Let E be the point on segment BC which is $\frac{1}{3}$ way from B (so the distance between E and B is 10 cm). Let F be the point on segment CD which is $\frac{1}{3}$ way from D. Then broken lines ABE, ADE and FCE all have length 40 cm.

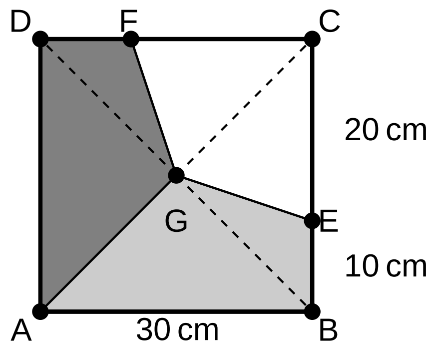
Now let us cut the interior of the pie as required. Let G be the midpoint of the square (i.e. the point which is of distance 15 cm from each of the sides). We claim that quadrilaterals $ABEG$, $ECFG$ and $FDAG$ have the same area.

We can prove it as follows: Let us consider segments BG, CG and DG. The area of quadrilateral $ABEG$ is the sum of the areas of triangles ABG and BEG. We know that AB has length 30 cm and G is of distance 15 cm from side AB, therefore the area of triangle ABC is $\frac{30\ 15}{2} = 225$ cm^2.

Similarly the length of BE is 10 cm and G is of distance 15 cm from side BE, so the area of triangle BEG is $\frac{10\ 15}{2} = 75$ cm^2. Therefore the total area of quadrilateral $ABEG$ is 300 cm^2.

We can nd the area of the other two pieces similarly. (We could also say that the area of $ADFG$ is the same as the area of $ABEG$ by symmetry, and then the area of the remaining part is also $30\ 30\quad 300\quad 300 = 300$ cm^2.)

So the three pieces are of equal area and they contain equal lengths of the boundary of the pie.



4. (Back to problem) a) Answer: It is possible.

One possible way is shown in the table (the rst two columns show the number of blue and red balls in each step and the last column shows the color of ball to be put into the machine).

number of blue balls	number of red balls	color to be put in the machine
1	0	blue
0	5	red
5	4	red
10	3	blue
9	8	red
14	7	

b) Answer: It is possible.

See an example in the table below.

number of blue balls	number of red balls	color to be put in the machine
2	0	blue
1	5	blue
0	10	red
5	9	red
10	8	red
15	7	red
20	6	red
25	5	red
30	4	blue
29	9	blue
28	14	

c) Answer: It is not possible.

Initially we have 1 ball. At each step the number of balls is first decreased by one, and then increased by 5, so eventually it is increased by 4, which is an even number. We start with an odd number of balls and at each step we increase it with an even number, so after each step we will have an odd number of balls.

To have the same number of red and blue balls we would need an even number of balls in total, which can not be attained.

d) Answer: It is not possible.

Let us consider the difference of the number of blue and red balls, and see what remainder it gives when divided by 6. We will show that this remainder is unchanged.

After a few steps let us have r red and b blue balls. In the next step we can place a red or a blue ball into the machine.

If we put in a red one: we will get $b' = b + 5$ blue and $r' = r - 1$ red balls. Then

$$b' - r' = (b + 5) - (r - 1) = (b - r) + 6:$$

If we put in a blue one: we will get $b' = b - 1$ blue and $r' = r + 5$ red balls. Then

$$b' - r' = (b - 1) - (r + 5) = (b - r) - 6:$$

We can see that in both cases $b' - r'$ has the same remainder mod 6 as $b - r$.

Initially we have 2 blue balls and no red ball, so the difference is 2. Therefore after each step the difference will give remainder 2 mod 6. To get the same number of blue and red balls, we would need the difference to be 0, which does not give remainder 2, so it cannot be attained.

5. (Back to problem) a) Let the three vampires be A, B and C. In the following table the first row describes the initial state and what happens after that. (Similarly for the other rows.)

Blood that A has	Blood that B has	Blood that C has	Who bites whom
A	B	C	A bites B
A and B	B	C	B bites C
A and B	B and C	C	C bites A
A and B	B and C	C and A and B	A bites B
A and B and C	B and C	C and A and B	-

One can see that this way they obtain 5 cups of syrup. Next we will show that this is the maximum.

b) Every vampire can have the blood of at most 2 other vampires in the end, so they can obtain no more than $3 \cdot 2 = 6$ cups of syrup. We will show that this is not possible. For now let us assume that they can collect 6 cups of syrup.

Let us consider the last bite. We can assume that A has bitten B or C. However B and C cannot collect more blood after this, so they must have the blood of A as well. So A could not bite them as the last bite, thus we have arrived at a contradiction, they cannot collect 6 cups of syrup.

c) We can give a similar construction in the case of 6 vampires as well. Let the vampires be $1; 2; \dots; 6$. Let the first bite be that 1 bites 2, then 2 bites 3, and so on, finally 6 bites 1. By repeating these 6 bites until it is possible, everyone will have everyone else's blood, except for 2, will not have the blood of 1. So with this method they can collect $5 \cdot 5 + 4 = 29$ cups of syrup.

We will now prove they cannot obtain more blood. It is clear that they cannot collect more than 30 since one vampire can collect at most 5 other types of blood and there are 6 vampires. So now let us assume that they can obtain 30 cups of syrup. Let X be the vampire performing the last bite. Since the one X bites cannot have the blood of X and there are no more bites after the last one, they cannot collect 30 cups of syrup.

2.1.2 Category D

1. (Back to problem) For the solution, see Category C Problem 2.

2. (Back to problem) a) One can notice that the digits $2; 4; 5; 6$ cannot appear in the ones place since then at least one of the two-digit numbers would be divisble by 2 or 5, hence it would not be a prime. After this by trial and error one can find for example $23; 41; 59; 67$, which is a solution.

b) Since in the case of any such four primes, digits $1; 3; 7; 9$ will appear in the ones place and $2; 4; 5; 6$ in the tens place, therefore the sum of the four numbers can only be $1 + 3 + 7 + 9 + 10 \cdot (2 + 4 + 5 + 6) = 190$.

3. (Back to problem) a) Let us denote the vertices by $A_1; A_2; \dots; A_{12}$ respectively. Note that they all lie on one circle, the circumcircle of the 12-gon.

From Thales's theorem (and its converse) three vertices form a right-angled triangle if and only if two of them are opposite points of the circumcircle, i.e. they are of form A_i, A_{i+6}.

Therefore if Albrecht wants to choose at least 3 vertices, he can choose at most one vertex from each of the pairs $(A_1; A_7)$, $(A_2; A_8)$, $(A_3; A_9)$, $(A_4; A_{10})$, $(A_5; A_{11})$, $(A_6; A_{12})$. So he can choose at most 6 vertices.

It is possible to choose 6 vertices, e.g. $A_1; A_2; \dots; A_6$.

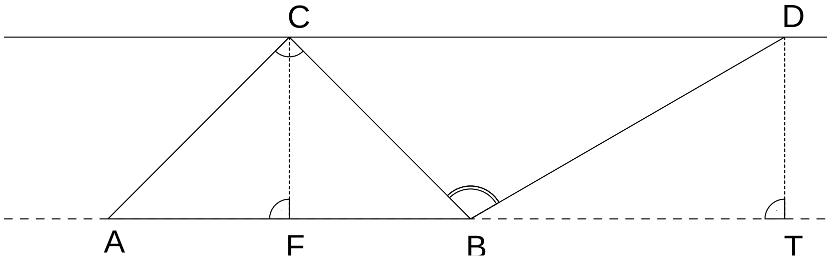
b) Let us use the above notations. Albrecht can choose 4 vertices, e.g. $A_1; A_4; A_7; A_{10}$, which form a square. (Also: any four-tuple forming a rectangle works.)

Assume that Albrecht chooses at least 5 vertices. If he has chosen more than 5, consider only five of them. Consider the pentagon formed by these five vertices. The sum of the internal angles of a pentagon is 540°, so there is a vertex with internal angle at least $540^\circ/5 = 108^\circ > 90^\circ$.

The pentagon is convex, so this internal angle is also less than 180°. Then this vertex and its two neighbouring vertices (in the pentagon) form an obtuse triangle, which is a contradiction.

So Albrecht cannot choose 5 or more vertices.

4. (Back to problem) Consider the figure below. Let F and T be the feet of perpendiculars onto AB from C and D respectively. By definition CF and DT are perpendicular to AB.

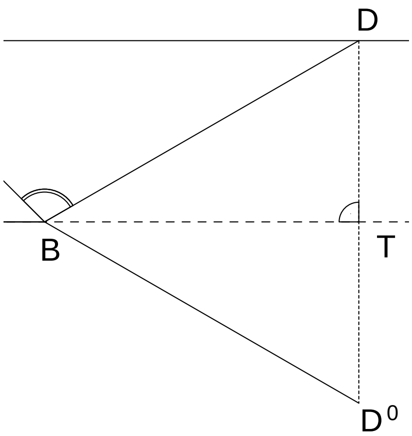


Since ABC is an isosceles right triangle, $AC = BC$ and $\angle CAB = \angle ABC = 45^\circ$ (because the right angle is at vertex C, and the sum of the angles in a triangle is 180°). Similarly we get that in triangles AFC and CFB $\angle FCA = \angle FCB = 45^\circ$. This means that these triangles are also isosceles and right (with the right angle at F in both triangles), so $AF = FC = FB$, and F is the midpoint of segment AB.

We know that $CD \parallel AB$, this means that $CF \perp CD$ and $DT \perp CD$. But then $CF \parallel DT$ (both are perpendicular to AB), and by the previous observations and the conditions in the problem it follows that

$$DT = CF = \frac{1}{2} AB = \frac{1}{2} BD \tag{1}$$

Now consider triangle BTD. Let D' be the reflection D over line AB. Since T is the foot of the perpendicular from D onto AB, due to the fact that reflection conserves distance, $DT = TD'$. Similarly we get that $DB = BD'$. Now from equation (1) it follows that $DB = BD' = DD'$. This means that BDD' is an equilateral triangle, so all of its angles are 60°. Since reflection conserves the angles as well, $\angle DBT = \angle TBD' = \frac{60^\circ}{2} = 30^\circ$.



Finally the angle we are looking for: $\angle CBD = 180^\circ - \angle ABC - \angle DBT = 180^\circ - 45^\circ - 30^\circ = 105^\circ$.

5. (Back to problem) a) Triple $a = 4$, $b = 2$, $c = 16$ is a counterexample. We can see that $4 \mid 2^2$, $2 \mid 16^2$ and $16 \mid 4^2$, but $abc = 2^7$ does not divide $(a + b + c)^6 = 22^6$.

b) Expanding $(a + b + c)^7$, we get the following sum:

$$\sum_{i+j+k=7} c_{i,j,k}\, a^i b^j c^k,$$

where $0 \le i, j, k \le 7$ and each $c_{i,j,k}$ is some positive integer. We will show that each term in the above sum is divisible by abc.

Case 1: If $i, j, k \ge 1$, then $abc \mid a^i b^j c^k$ obviously holds.

Case 2: If two of i, j, k are 0, then by symmetry we can assume $i = j = 0$ and so $k = 7$. $b \mid c^2$, thus $b^2 \mid c^4$ and so $a \mid b^2 \mid c^4$. Therefore $abc \mid c^4 \cdot c^2 \cdot c = c^7 = a^i b^j c^k$.

Case 3: If exactly one of i, j, k is 0, then we can assume it is i. In this case $j + k = 7$. If $j \ge 3$, then $ab \mid b^j$ and $c \mid c^k$, therefore $abc \mid b^j c^k = a^i b^j c^k$. If $j \le 2$, then $k \ge 5$. In this case $ac \mid c^4 \cdot c \mid c^k$, and $b \mid b^j$, so again $abc \mid b^j c^k = a^i b^j c^k$.

So each term in $\sum_{i+j+k=7} c_{i,j,k}\, a^i b^j c^k$ is divisible by abc, so $(a + b + c)^7$ is also divisible by abc.

2.1.3 Category E

1. (Back to problem) Let $a + n = b - n = c \cdot n = \frac{d}{n} = x$, where x is a positive integer. Then $a = x - n$, $b = x + n$, $c = \frac{x}{n}$ and $d = n \cdot x$.

By the condition $a + b + c + d = 100$, we get $(x - n) + (x + n) + \frac{x}{n} + nx = 100$, giving $x(n^2 + 2n + 1) = x \cdot (n + 1)^2 = 100n$.

By the fact that $x = c \cdot n$, we get that $c \cdot (n + 1)^2 = 100$, so both factors have to be square numbers. As $c \mid 100$, there are four cases for the value of c:

Case 1: If $c = 1$ then $n = 9$ and $x = 9$. So we get $a = 0$, $b = 18$, $c = 1$ and $d = 81$. This is indeed a solution.

Case 2: If $c = 4$ then $n = 4$ and $x = 16$. So we get $a = 12$, $b = 20$, $c = 4$ and $d = 64$. This is also a good solution.

Case 3: If $c = 25$ then $n = 1$ and $x = 25$. So we get $a = 24$, $b = 26$, $c = 25$ and $d = 25$. This is also good.

Case 4: If $c = 100$ then $n = 0$. This is impossible as $\frac{d}{n}$ has to be well-defined.

So the only three solutions $(a; b; c; d; n)$ satisfying the conditions of the problem are $(0; 18; 1; 81; 9)$, $(12; 20; 4; 64; 4)$ and $(24; 26; 25; 25; 1)$.

We can easily check that these are valid solutions indeed.

2. (Back to problem) a) One possible solution is the following: the kayakers started in reverse order of their speeds, so the slowest kayaker was the first one to start and the fastest kayaker was the last one. Note that for any two competitors, if the course is long enough then the faster competitor who starts later, will at some point overtake the slower competitor who starts earlier. So if the course is long enough, the order of the kayakers at the finish will be the same as the order at the start but reversed, so all possible overtakings will happen, so each kayaker will do a high five with all other kayakers. So if they start in this order immediately after one another, then any two kayakers will do a high five.

b) Suppose that nobody has miscounted and everybody did high fives with precisely 6 other people. The kayaker who started the earliest made a high five with another kayaker if and only if the latter overtook him in the order of finishing. As the earliest kayaker made 6 high fives, this means that precisely 6 people overtook him so he finished in 7^{th} place. Similarly, the kayaker who started the latest could only do a high five with someone he overtook. So he overtook precisely 6 people, and so finished in 7^{th} place. But then we have two people who both finished seventh, which is a contradiction. So somebody must have made miscounted, not having made 6 high fives exactly.

3. (Back to problem) a) First we can notice that the sum of the 9 numbers decreases by 1 with every move, since we reduce two numbers by 1 and increase one by 1. We can also notice that the sum of the 4 numbers in the corners is constant, since if we change any of them, another one of them also has to change and since they are not neighbouring, they must change in opposite directions.

Now assume that we can achieve a table with all the same entries. Then, if all the entries are n, the sum of the 4 numbers in the corners is $4n = 20 = 1 + 3 + 7 + 9$, since it does not change. This means that $n = 5$, the sum all the numbers in table has to be 45. We can also notice that the sum of all numbers is 45 in the starting position. Since this sum decreases with every move, and in the starting position not all entries are the same, the desired table is not achievable.

b) From the sction a) we got that if all entries are n, then $n < 5$. Now we give an example of achieving $n = 4$, so this is the highest possible value.

Starting position:	1	8	5
	6	9	4
	7	2	3
Move in the third row, increasing the rightmost number:	1	8	5
	6	9	4
	6	1	4
Three moves in the second column, increasing the bottom number:	1	5	5
	6	6	4
	6	4	4
Move in the first row, increasing the leftmost number:	2	4	4
	6	6	4
	6	4	4
Two moves in the first column, increasing the top number:	4	4	4
	4	6	4
	4	4	4
Move in the second row, increasing the leftmost number:	4	4	4
	5	5	3
	4	4	4
Finally another move in the second row, increasing the rightmost number:	4	4	4
	4	4	4
	4	4	4

4. (Back to problem) Our goal is to show that the number written on point $(x; y)$ is the product of x and y. For some fixed x and y: let $x + y = c$ and let the number on point $(c - n; n)$ be denoted by a_n. Then we have numbers $a_0; a_2 \ldots a_c$, which satisfy the followings:

$$a_0 = a_c = 0$$

$$a_n = \frac{a_{n+1} + a_{n-1}}{2} + 1; \quad n = 1; 2; \ldots; c - 1:$$

Let $a_1 = c - 1 + d$. (Remember that a_1 is the number at point $(c - 1; 1)$, which we expect to be $c - 1$. So d is the 'error'.) We will prove by induction on n that

$$a_n = (c - n)n + nd; \quad n = 0; 1; \ldots; c:$$

Then considering $n = c$, we get that $cd = a_c = 0$, i.e. $d = 0$. Therefore $a_n = (c - n)n + nd = (c - n)n$, i.e. $a_y = (c - y)y = xy$, as we wanted to prove.

Induction: The statement is true for $n = 0$ and $n = 1$.

Let us rearrange $a_n = \frac{a_{n+1} + a_{n-1}}{2} + 1$ and use the statement for n and $n - 1$ to get that

$$a_{n+1} = 2a_n - a_{n-1} - 2 = 2(c - n)n + 2dn - (c - n + 1)(n - 1) - d(n - 1) - 2 =$$

$$(c - n - 1)(n + 1) + d(n + 1)$$

i.e. the statement for $n + 1$.

Apply the formula to get that the number written on point (121,212) is $121 \cdot 212 = 25652$

Second solution: Note that writing $x \cdot y$ on point $(x; y)$ works. We will show that this is the only possibility.

Assume we write numbers on the points two different ways. Again consider the numbers on points $(c - n; n)$, where c is fixed and $n = 0; \ldots; c$. Let these numbers be $a_0; a_1; \ldots a_c$ according to the first scheme and $b_0; b_1; \ldots b_c$ according to the second one. Let us consider the differences $d_n = a_n - b_n$.

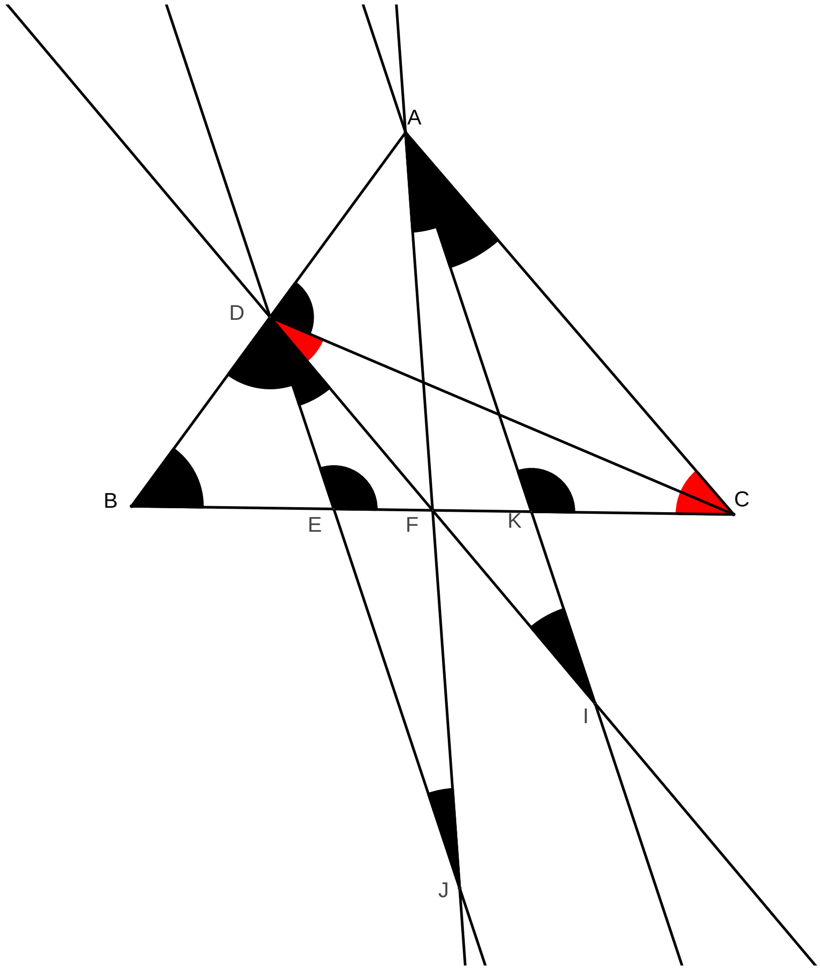
For $n = 1; 2; \ldots; c - 1$:

$$d_n = a_n - b_n = \frac{a_{n+1} + a_{n-1}}{2} + 1 - \frac{b_{n+1} + b_{n-1}}{2} - 1 = \frac{(a_{n+1} - b_{n+1}) + (a_{n-1} - b_{n-1})}{2} = \frac{d_{n-1} + d_{n+1}}{2};$$

therefore d_n form an arithmetic progression. But $d_0 = d_c = 0 - 0 = 0$, so two elements in the arithmetic progression are the same, so the whole d_n will be constant. This means that for each n we have $d_n = 0$, so $a_n = b_n$, i.e. the two schemes agree everywhere. So the numbers written on the points are unique.

Therefore the number written on point $(121; 212)$ must be $121 \cdot 212 = 25652$

5. (Back to problem)



Denote the angles of the triangle at vertices A, B, C with , , respectively. Since $ADC\triangle$ and $DBE\triangle$ are isosceles, therefore $\angle EDC =$ and $\angle CED = 2$. Let $AI \cap BC = K$. Since $AK \parallel DE$, therefore $\angle KAB = \angle EDB = \angle BDE =$. So $\angle KAC =$, this means that $\angle CKA = 180$ $+$ $= 2$. Then the angles of $DEC\triangle$ and $CKA\triangle$ are pairwise the same, moreover $DC = AC$, so these triangles are congruent. This means that $DE = KC$ and $EC = AK$. But since $DE = BE = KC$, so (F is the midpoint of side BC) $EF = FK$.

Since $DE \parallel AK$, therefore $\angle KAF = \angle FJE$ and $\angle AIF = \angle DEF$. This means that since their angles and sides are the same $AFK\triangle = EFJ\triangle$ and $DEF\triangle = FIK\triangle$, from which follows that

$$EJ = AK = EC$$

and

$$KI = ED = KC$$

From here we get that $\angle ECI = \angle ECJ$, so points C, I and J are indeed collinear.

2.1.4 Category E$^+$

1. (Back to problem) For the solution, see Category E Problem 3.

2. (Back to problem) We will show that for any finite set H of primes, there exist an n such that $P(n) = H$.

If H is empty, then $n = 3$ is sufficient. (Assume that $3 = a^p + b^p$. Then $a, b < 3$ and they are not both 1, so a and b are 1 and 2 in some order. But $2^p > 3$, which is a contradiction.)

If $H = \{p_1, \ldots, p_k\}$ (non-empty), then $n = 2^{1+p_1 \cdots p_k}$ is sufficient.

For any $p \in H$: $n = a^p + b^p$, where $a = b = 2^{p_1 \cdots p_k / p}$ are positive integers. So $p \in P(n)$.

Now assume that $n = a^q + b^q$ where $q \notin H$ is a prime.

Case 1: $q = 2$: All p_i are odd ($2 \notin H$), therefore $n = 4^k$ for some positive integer k, and $4^k = a^2 + b^2$. Square numbers give remainder 0 or 1 when modulo 4 and the left-hand side is divisible by 4, therefore a^2 and b^2 must both be divisible by 4. Dividing both sides by 4 we get $4^{k-1} = a_1^2 + b_1^2$ where $a_1 = \frac{a}{2}$, $b_1 = \frac{b}{2}$ are both positive integers. Proceeding similarly we can get a sequence of equations of form $4^{k-i} = a_i^2 + b_i^2$ with a_i, b_i positive integers, $i = 1, 2, \ldots, k$. But then $1 = a_k^2 + b_k^2$ where a_k, b_k are both positive integers, which is impossible. So we reached a contradiction.

Case 2: q is odd: Assume that $n = 2^k = a^q + b^q = (a + b)(a^{q-1} - a^{q-2}b \ldots b^{q-1})$. We know that a and b have the same parity. If they were both odd, then the term in the second bracket would be odd and a divisor of 2^k, so it would be 1. In this case we would have $a + b = a^q + b^q = n > 2$ with $a, b > 0$ and q a prime, which is impossible. So a and b must be both even. We can repeat the aboves for the equation $2^{k-2} = a_1^q + b_1^q$, where $a_1 = \frac{a}{2}$, $b_1 = \frac{b}{2}$ are positive integers. Proceeding similarly we will get that $a_i = \frac{a}{2^i}$, $b_i = \frac{b}{2^i}$ are integers for $i = 1, 2, \ldots$, which is impossible. So again we arrived at a contradiction.

This means that for all primes $q \notin H$, we have $q \notin P(n)$.

3. (Back to problem) a) Bob cannot achieve his goal.

Assume that after some finite number of questions Bob stops and determines A.

Let a be the largest of Anne's replies. Let P be a point of the plane which has distance greater than a from each of the points Bob asked about. (It exists, as finitely many discs of radius a can cover only a finite area of the plane.) Bob would get the same replies whether or not P is an element of A, so he cannot decide whether P is in A. This is a contradiction.

b) Bob can achieve his goal. In the following we will present an algorithm to find A.

Initially let $k = 1$. (We will run the following steps with the given value of k and if they are inconclusive, we will increase the value of k and run the steps again with this new value. We will keep doing this until we find A. So k is like a loop variable.)

Step 1: Divide the square $[0, 1]^2$ into $k \times k$ little squares of side length $\frac{1}{k}$.

Step 2: Ask about each of the arising $(k+1)^2$ grid points. If a grid point has distance less than $r_k = \frac{\sqrt{2}}{2} \cdot \frac{1}{k}$ from A, mark it with green.

Important side note: Let us consider how the green points will be arranged for sufficiently large values of k (so for a sufficiently fine grid).

Let A be a finite subset of $\mathbb{R}^2$. It is possible to draw congruent circles around the points of A, such that for each point on or within a circle, the closest point of A is the centre of the circle. (E.g. let the shortest distance between two distinct points of A be d, and let the circles have radius $\frac{d}{3}$.) It is also possible to draw congruent red squares around the points of A, such that for each point of a red square, the closest point of A is the centre of the square. (E.g. write a square into each of the above circles, and colour it with red. :)) Let t be the side length of these squares.

If the distance between two grid lines is less than $\frac{f}{100}$ (i.e. $k > \frac{100}{f}$), then for each $P \in A$, one to four of the vertices of the grid square containing P will be green, and no other grid point within the red square will be green.

It means that for sufficiently fine grids, the green points will be positioned in groups of one to four (which are the vertices of a little grid square), and there will be no other green point which is at most 48 rows and at most 48 columns away from a member of such a group. So between points of two different such groups, there will be at least 96 (horizontal or vertical) grid lines.

Step 3: Let us examine whether the structure of the green points is as above. (I.e. they appear in groups of one to four, which are vertices of a little grid square, and between any two such groups there are at least 96 grid lines.) If not, let us increase k and repeat all the aboves. If yes, proceed to Step 4.

Step 4: If the structure of the green points is as above, we have some hope to find A.

For each group of green points, let us determine whether it corresponds to one or more points of A. For each group let us consider a square $QRST$ of grid points around it, such that each green point is of distance at least $\frac{3}{k}$ and at most $\frac{10}{k}$ from each of Q, R, S, T. (Then for each of Q, R, S, T the closest point of A will be close to the points of the green group, so it will be within the square $QRST$.)

If there is no point P of the plane with $d(P,Q) = d(A,Q)$, $d(P,R) = d(A,R)$, $d(P,S) = d(A,S)$, $d(P,T) = d(A,T)$, then there are multiple points of A within square $QRST$.
If there exists a point P with $d(P,Q) = d(A,Q)$, etc, then there is no point of A in the interior of any of the four discs $D(Q, d(Q,P))$, $D(R, d(R,P))$, $D(S, d(S,P))$, $D(T, d(T,P))$. The interiors of these discs cover the square $QRST$, except for point P, thus the only point of A within square $QRST$ can be P. (Note: $D(X,r)$ denotes the disc of centre X, radius r.)

So we can determine for each green group whether it corresponds to one or more points of A. If each green group corresponds to one point, we can find the points of A as above and we are done. If not, let us increase k and repeat all of the aboves.

The process will terminate, as for sufficiently large values of k (say $k > \frac{100}{f}$, where f is as in the side note) the green points will be positioned as required and each green group will correspond to precisely one point of A.

4. (Back to problem) Let $\sim$ be a relation on the set of irrationals, such that $a \sim b$ if and only if $p = qa - rb$ for some rationals p and $0 < q, r$. We can see that $\sim$ is symmetric, reflexive and transitive, i.e. it is an equivalence relation.

Let us arrange the equivalence classes into pairs, such that if x is an element of one class, then $-x$ is in the pair of that class.

Let us consider $2n-1$ distinct irrational numbers. Let $a_1, \dots, a_k$ be irrational numbers such that $a_1, \dots, a_k, -a_1, \dots, -a_k$ are from different equivalence classes and each of our numbers fall in the equivalence class of an a_i or $-a_i$.

Lemma. It is possible to choose k of the numbers $a_1, \dots, a_k, -a_1, \dots, -a_k$ which form an unearthly k-tuple, such that for each i exactly one of a_i and $-a_i$ is chosen (call it b_i).

Proof. Let us use induction on k. For $k = 1$ we can choose a_1, so the statement is true.

Assume that $b_1, \dots, b_{k-1}$ form an unearthly $(k-1)$-tuple where each b_i is a_i or $-a_i$. Assume that neither a_k, nor $-a_k$ makes it into an unearthly k-tuple. It means that there exist non-negative rational numbers $q_1, \dots, q_k$ and $r_1, \dots, r_k$ such that not all q_i are 0, not all r_i are 0, and

$\sum_{i=1}^{k} q_i b_i + q_k a_k$, $\sum_{i=1}^{k} r_i b_i - r_k a_k$ are both rational. Then

$$r_k\left(\sum_{i=1}^{k} q_i b_i + q_k a_k\right) + q_k\left(\sum_{i=1}^{k} r_i b_i - r_k a_k\right) = \sum_{i=1}^{k} (r_k q_i + q_k r_i) b_i$$

is also rational, which contradicts the assumption that the b_i form an unearthly $(k-1)$-tuple. (All the coefficients $r_k q_i + q_k r_i$ are non-negative rationals and they are not all zero, as q_k is non-zero and r_i are not all zero.)

So a_k or $-a_k$ extends $b_1, \dots, b_{k-1}$ to an unearthly k-tuple. □

Note that $-b_1, \dots, -b_k$ also from an unearthly k-tuple.

Lemma. If $b_1 \dots b_k$ form an unearthly k-tuple and $c_1, \dots, c_m$ are irrational numbers from the equivalence class of b_k, then $b_1, \dots, b_{k-1}, c_1, \dots, c_m$ form an unearthly $(k+m-1)$-tuple.

Proof. Assume that $\sum_{i=1}^{k-1} q_i b_i + \sum_{i=1}^{m} r_i c_i$ is rational for some non-negative rationals $q_1, \dots, q_{k-1}, r_1, \dots, r_m$ which are not all zero. Then $q_1, \dots, q_{k-1}, \sum_{i=1}^{m} r_i c_i / b_k$ are non-negative rationals and they are not all zero. Also:

$$\sum_{i=1}^{k} q_i b_i + \left(\sum_{i=1}^{m} r_i c_i / b_k\right) b_k = \sum_{i=1}^{k} q_i b_i + \sum_{i=1}^{m} r_i c_i$$

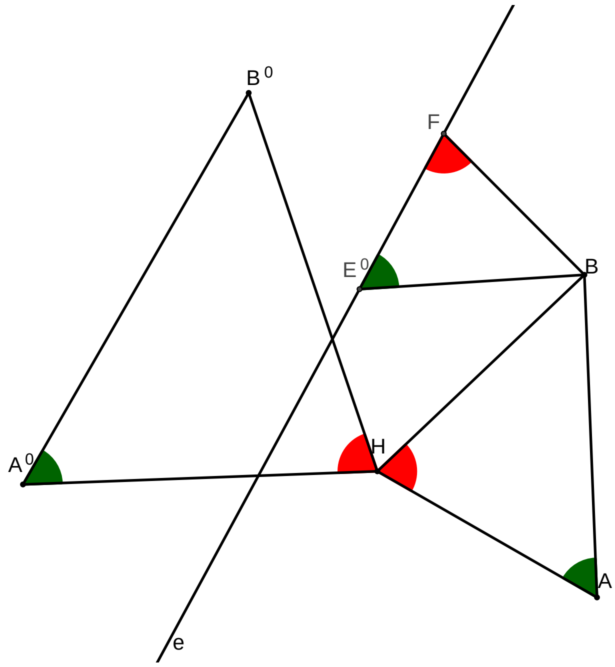
is rational, which contradicts the assumption that $b_1 \dots b_k$ is unearthly. □

If at least n of our $2n-1$ numbers are from the equivalence classes of $b_1, \dots, b_k$, then we can choose n of them and they will form an unearthly n-tuple.

If less than n of them are from the equivalence classes of $b_1, \dots, b_k$, then at least n of them are from the equivalence classes of $-b_1, \dots, -b_k$. We can choose n of them to get an unearthly n-tuple.

5. (Back to problem) Let us use directed angles in the solution. Let $\measuredangle(e, f)$ denote the directed angle defined by lines e and f. First let us prove a lemma:

Lemma. Let HAB and $HA'B'$ be similar triangles with different orientation, let the intersection point of the perpendicular bisectors of segments AA' and BB' meet at E, let e denote the perpendicular bisector of BB'. Then $\measuredangle(BE, e) = \measuredangle(AB, AH)$.

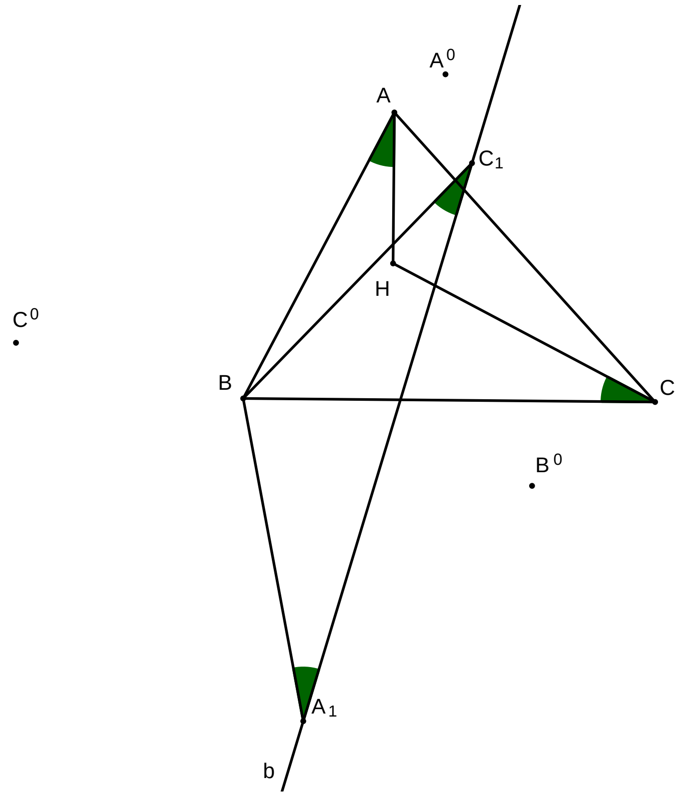


Proof. Let E' be the point on line e for which $\angle(BE', e) = \angle(AB, AH)$. Let F be the point on line e for which $\angle(e, BF) = \angle(HA, HB)$. Then triangles BFE' and BHA are similar. Since E' and F lie on line e, this means that triangles BFE' and $B'FE'$ congruent since the reflection onto e maps the triangles into each other. By the conditions of the lemma HAB and $HA'B'$ are similar, this means that $HA'B'$ and $B'E'F$ are also similar. The spiral symmetry with center B that maps triangle BFE' to triangle BHA, takes F to H and the image of E' is A, so the triangle BFH is similar to triangle $BE'A$. Similarly we get that triangles $B'FH$ and $B'E'A'$ are similar. From these similarities we obtain that

$$\frac{AE'}{HF} = \frac{BE'}{BF} = \frac{B'E'}{B'F} = \frac{A'E'}{HF}$$

this means that $AE' = A'E'$, so E' lies on the perpendicular bisector of AA', this means that $E = E'$, so indeed $\angle(BE, e) = \angle(AB, AH)$ holds, thus we have proven the lemma. □

Now considering the problem:



Let the perpendicular bisector of AA' be line a, the perpendicular bisector of BB' be line b, the perpendicular bisector of CC' be line c. Let a and b intersect at C_1-ben, b and c at A_1, c and a at B_1. Since HAB and $HA'B'$ are similar triangles with different orientation and the intersection of the perpendicular bisectors of AA' and BB' is C_1, by the lemma we obtain that $\angle(BC_1; b) = \angle(AB; AH)$. Similarly $\angle(BA_1; b) = \angle(CB; CH)$. We know that $\angle(AB; AH) = 90^\circ - \beta = \angle(CH; CB)$ (where β is the angle at B in triangle ABC), so $\angle(AB; AH) = -\angle(CB; CH)$ this means that $\angle(BC_1; b) = -\angle(BA_1; b)$, so triangle BA_1C_1 is isosceles, this means that $BA_1B'C_1$ rhombus, so line BB' is the perpendicular bisector of segment A_1C_1. Similarly we obtain that line AA' is the perpendicular bisector of segment B_1C_1 and line CC' is the perpendicular bisector of segment A_1B_1. This means that lines AA', BB', CC' have a common point, which is the orthocenter of triangle $A_1B_1C_1$. With this we are done with the proof.

2.2 Final round – day 1

2.2.1 Category C

1. (Back to problem) Let the price of an asteroid be a, the price of a shuttle be b, and the price of a space dog be c Dürer Dollars. From Sophie we have the information that $a + 3b + 5c = 200$ and from Lucy we know that $a + 4b + 7c = 225$. Hilda would like to find the value of $a + b + c$.

Let us take the difference of the two equations above, to get $b + 2c = 25$. Notice that Lucy has payed $a + b + c + 2b + 4c$ Dürer Dollars. From the above $2b + 4c = 50$, so Hilda should take $200 - 50 = 150$ Dürer Dollars with her.

2. (Back to problem) Let us consider what information the reporter gets from an answer. Assume she goes to a person X and asks whether he knows person Y. If the answer is yes, she will certainly know that Y is not Mr. Wilson (as none of the actors know Mr. Wilson), whereas if the answer is no, she will certainly know that X is not Mr. Wilson (as Mr. Wilson knows everyone else).

Initially there are 13 candidates for being Mr. Wilson. The reporter can go by the following strategy: She walks to one of the 13 people in the room, points at an arbitrary other person and asks whether he knows him. From the answer, she will know about one of the two people that he cannot be Mr. Wilson. This way only 12 candidates remain for being Mr. Wilson. Now she can go to any of these 12 people and ask about any of the other 11. From the answer, she will exclude one of the two people, so only 11 candidates remain. She can proceed like this: If there are $k > 1$ candidates left, she can ask any of them about any of the other $k-1$, and then only $k-1$ candidates remain. With each question she can reduce the number of candidates by one. After 12 questions she will be left with a single candidate, who must be Mr. Wilson himself.

3. (Back to problem) a) The table cannot be filled as required: To get an even number in the last row we would need an even digit in the bottom-right cell, but to get an odd number in the last column we would need an odd digit in the same cell.

b) The table cannot be filled as required: We know that the remainder of a number divided by three is the remainder of the sum of its digits when divided by three. To get numbers divisible by three in each row, we need the sum of the digits in each row to be divisible by three. Then the sum of all digit in the table would also be divisible by three. To get numbers giving remainder 1 in each column, we need the sum of digits in each column to give remainder 1. Then the sum of all digits in the table would give remainder 1. We cannot satisfy the two conditions simultaneously.

c) The table can be filled as required, see an example below.

1	2	1	1
2	1	1	1
2	2	1	2
2	2	2	1

4. (Back to problem) We shall solve the problem generally for $n+1$ with n pictures on each card. This will yield the solution for both parts.

We will use mathematical induction to prove the following statement: For $n+1$ cards we need at least $\frac{n(n+1)}{2}$ symbols.

For $n = 1$ the statement is trivial, we have 2 cards, and 1 common symbol.

We now assume that n the statement is true.

Let us now take $n+1$ cards and let them be numbered from 1 to $n+1$. Set aside the card $n+1$ and observe the other n cards. Any of these cards have at most $n-1$ symbols on them which are on the other $n-1$ cards, so we may cover one symbol on each card from the n obtaining the condition of the problem (each card has exactly one common symbol with any other card, and there are no two same symbols on a card). We may now use the inductive hypothesis for n cards. So among the not covered symbols there are at least $\frac{n(n-1)}{2}$ distinct symbols. Now let us put back the last card and uncover the covered symbols. If a symbol was covered until now, it must be different from all the not covered pictures on the first n cards, so otherwise the condition of the problem would not be true. So there are at least

$$\frac{n(n-1)}{2} + \frac{2n}{2} = \frac{n(n+1)}{2}$$

distinct symbols on the cards. For this number a trivial construction yields, let each of the pairs of cards have a distinct symbol given to them. Each card has those symbols on it, which were given to a pair, in which that card is present. Each pair of cards have a common symbol, and each card have n distinct symbols on them. So the solutions to our parts are a) 21 and b) 4950.

5. (Back to problem) Let $AB = AC = BC = CD = BD = 1$, then $AD = \sqrt{3}$.

Let the centre of the equilateral triangle ABC be X, of BCD be Y. Then $AX = BX = CX = BY = CY = DY = \frac{\sqrt{3}}{3}$ Let O be the midpoint of segment BC.

Since there does not exist a circle that would pass through all four points, for every such circle point A, B, C, D will either lie within or outside of the circle.

Case 1: If all 4 points would lie within a circle, then they would be equal distance from the centre of the circle, so it would lie on the bisectors of AB, BD, DC and CA, but since these four lines are not concurrent, such a circle does not exist.

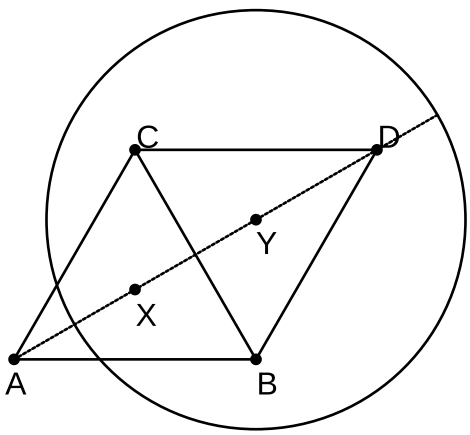
Case 2: If 3 points lie within the circle, 1 outside:

a) If this is point A: Then Y has to be the centre of this circle, let r be the radius of the circle, then

$$r - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3} - r \qquad (2)$$

must be true, so $r = \frac{\sqrt{3}}{2}$, there is 1 such circle.

Construction: We can construct the midpoint of AX, let this be F. The circle with centre Y, radius YF is distance $AF = FX$ from all the points since $YX = YC = YB = YD = \frac{\sqrt{3}}{3}$.



b) Similarly the is one circle when D lies on the outside, the others within.

c) If B lies outside the circle and A, C, D within: The centre of the circle must be the intersection of perpendicular bisector of segments AC and CD, but since this is B, it cannot lie outside, so in this case there is no such circle.

d) Similarly there is no such circle where C lies outside and the others within.

Case 3: If 2 points are within the circle and 2 outside:

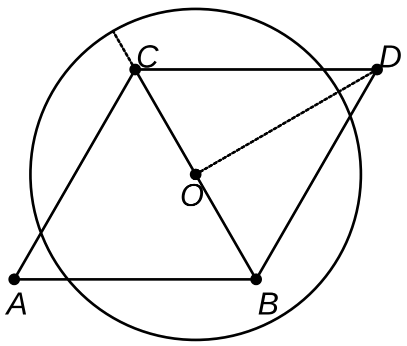
a) If B and C lie outside the circle, A and D inside: Let K be the centre of the circle, then $KC = KB$, $KA = KD$, and K also lies on the perpendicular bisectors of BC and AD, so the centre of the circle is O, it is distance 0.5 from points B and C, and distance $\frac{\sqrt{3}}{2}$ from points A and D, so such circle does not exist.

b) If A and D are outside the circle and B and C are within: similarly to the previous case the centre of circle is O and for the radius of the circle:

$$r - \frac{1}{2} = \frac{\sqrt{3}}{2} - r \tag{3}$$

so $r = \frac{\sqrt{3}+1}{4}$, in this case there is only one such circle.

Construction: First we construct the circle with center O through C. This intersects with AD in points M_1, M_2 (M_1 is closer to A). Then we construct the midpoint of AM_1, let this be G. The circle with centre O passing through G will be distance $GA = GM = 1$ from all 4 points.



c) If in the quadrilateral two neighbouring vertices are inside the circle, two are outside: let A and B be outside, C and D inside. Then the centre of the circle must lie on the perpendicular bisectors of AB and CD as well, but since these are parallel and not the same, they do not have a common point, so such a circle does not exist.

Case 4: If 3 points lie inside the circle, 1 outside:

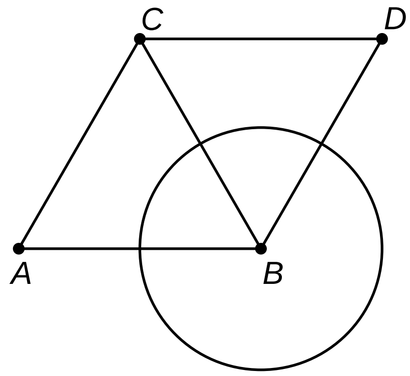
a) If A is inside, B, C, D are outside: In this case B, C and D are of the same distance from the centre of the circle, so this must be Y, on the other hand A is further away from this point than B, C, D, so such a circle cannot exist.

b) Similarly such a circle does not exist if D is inside and B, C and A are outside.

c) If B is inside, A, C and D outside: the centre of the circle lies on the perpendicular bisector of AC and CD, so it must be point B. Regarding the radius of the circle (r):

$$r = 1 - r \tag{4}$$

so $r = \frac{1}{2}$. This means that in this case there is 1 such circle. Construction: Let F the midpoint of side BC, the circle with centre B through F is of distance $\frac{1}{2}$ from every point.



d) If C lies on the inside and A, B, D outside, similarly we get that there is 1 such circle.

Case 5: If all 4 points are outside the circle, similarly to the first case, the centre of the circle must be of equal distance from every point, but since there is no such point, we do not get a solution here.

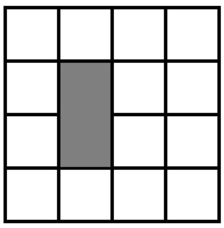
Now we have covered all the cases, this means that in total there are 5 such circles.

6. (Back to problem) Vincent has a winning strategy.
Note that at most 8 dominoes can be placed on the board. Therefore if Vincent manages to

place his 4th domino, and
make sure the table does not get covered completely,

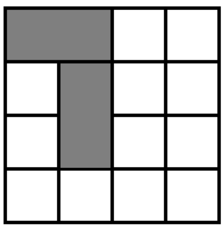
then he is guaranteed to win.

Let him start by covering two of the middle four cells of the board:

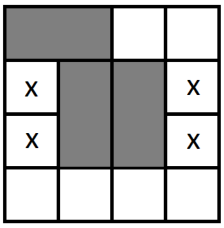


Then by symmetry Harry can choose from four different moves:

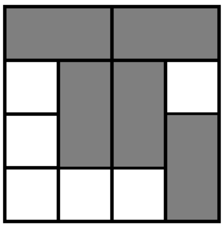
Case 1: Harry covers the first two cells of the first row.



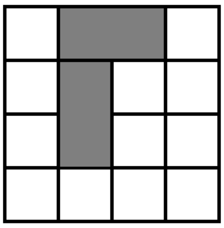
Then Vincent should cover the middle cells of the third column.



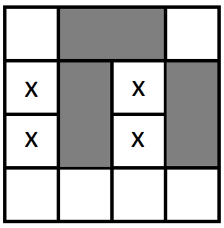
We can see that the cells marked by x belong to Vincent now, i.e. Harry cannot cover any of them. So Vincent can surely place four of his dominoes.
He also has to make sure that not all cells of the board get covered. He can do this by placing his third domino in the last column so that it intersects the top or bottom row (e.g. as in the figure). This way the board will be divided into two regions which both consist of an odd number of cells, so they cannot be covered completely by dominoes.



Case 2: Harry covers the middle cells of the first row.

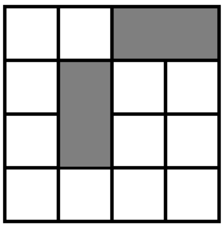


Then Vincent should place his second domino to the middle of the last column.

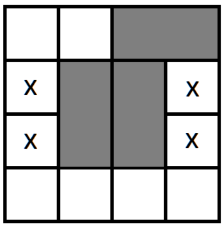


Now the four cells marked with x belong to Vincent, so he is guaranteed to have place for four if his dominoes. The top-right corner cannot be covered, so the board will not get completely covered. This means that Vincent wins.

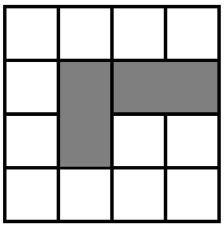
Case 3: Harry covers the last two cells of the first row.



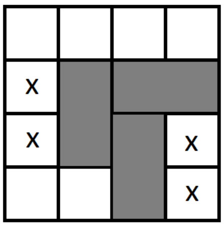
Vincent should place his second domino to the middle of the third column. This way we get the same setup as in Case 1 (up to a symmetry), so Vincent can win.



Case 4: Harry covers the last two cells of the second row.



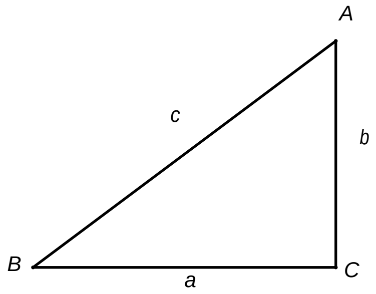
Vincent should cover the last two cells of column 3.



Then the cells marked with x belong to him, so he has guaranteed place for four of his dominoes. He can make sure not all of the board gets covered by placing his third domino to the top or bottom of the first column.

2.2.2 Category D

1. (Back to problem) Consider the following map of the roads:



The fork is at vertex C, Douglas gets to B along sides b and c. We know that ABC has a right angle at vertex C and also that $b + c = 99$ (in miles). By applying the Pythagorean theorem in triangle ABC and using the well-known identity:

$$a^2 + b^2 = c^2;$$
$$a^2 = c^2 - b^2 = (c + b)(c - b) = 99 \cdot (c - b).$$

Since the right hand side is divisible by 3, a must be divisible by 3, let $a = 3 \cdot k$ for some integer k. Then:

$$9k^2 = 99 \cdot (c - b);$$
$$k^2 = 11 \cdot (c - b).$$

This means that $11 \mid k$, let $k = 11 \cdot m$ for some integer m. Then:

$$121 \cdot m^2 = 11 \cdot (c - b);$$
$$11 \cdot m^2 = (c - b).$$

Since $b, c > 0$, it means that $c - b < c + b = 99$, so $c - b$ has two possible values, 11 and 44.

Case 1: If $c - b = 11$. Then $2c = c - b + c + b = 11 + 99$, so $c = 55$, $b = 44$, $m = 1$, here $a = 33 \cdot m = 33$.

Case 2: If $c - b = 44$. Then $2c = 44 + 99 = 143$, but this is a contradiction, since the lengths of all the roads are whole numbers of miles.

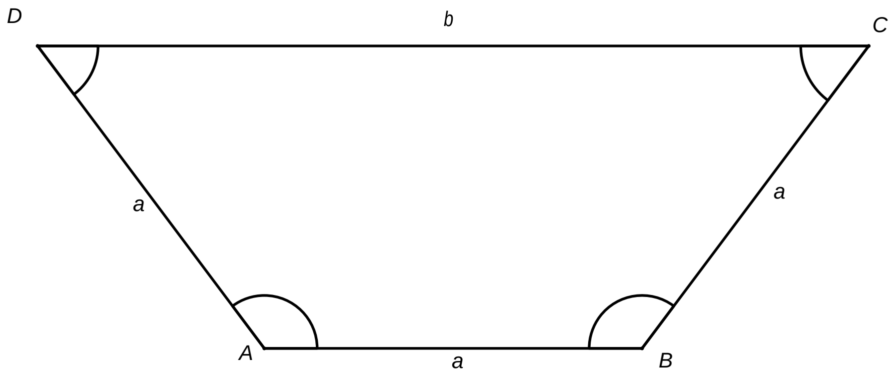
So Douglas would have only walked 33 miles if he had chosen the road on the left.

2. (Back to problem) For the solution, see Category C Problem 4.

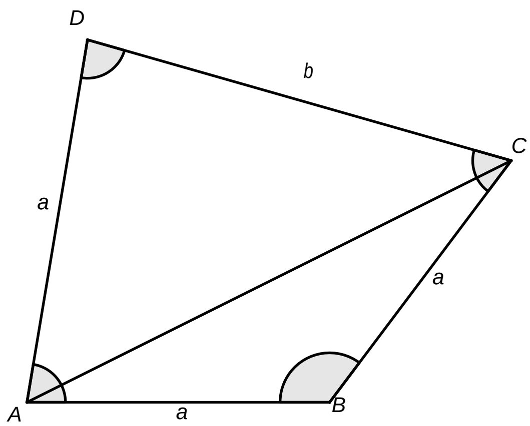
3. (Back to problem) a) There exists no such quadrilateral.

Assume there is a quadrilateral with sides of lengths a, a, a, b and angles , , , , where $a \neq b$ and $\neq$. There are two angles which are between two sides of the same length. We will consider two cases based on these angles.

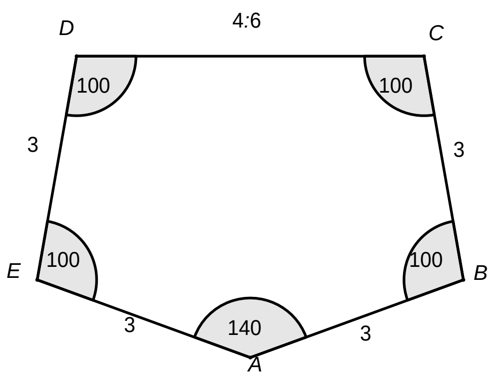
Case 1: If both angles are : by symmetry the quadrilateral is a trapezoid, therefore = , which is a contradiction.



Case 2: If one of the angles is , the other one is : Let us label the vertices as in the figure. We can see that triangle ABC is isosceles, i.e. $\angle CAB = \angle ACB$. Because of $\angle DAB = \angle BCD$ it means that $\angle CAD = \angle ACD$. So triangle CDA is also isosceles, therefore $a = b$ which is again a contradiction.



b) There exists such a pentagon, for example the following one:



4. (Back to problem) a) It is not possible.

Let us consider a colouring of the rationals with blue and red such that the sum of any two numbers of the same colour is the same colour as them.

Then all integer multiples nx of a rational number x are of the same colour as x. We can prove it by induction on n: It is true for $n = 1$. If nx is the same colour as x, then $(n+1)x = nx + x$ is also of the same colour.

Assume that there are numbers of both colours, let $\frac{p}{q}$ be blue and $\frac{r}{s}$ be red. Then pr is an integer multiple of both $\frac{p}{q}$ and $\frac{r}{s}$, so it has to be blue and red at the same time, which is a contradiction.

b) It is possible, we will show two examples.

First example: Let the numbers less than or equal to 1 be red, the numbers greater than 1 be blue. Then the product of two numbers of the same colour is also of that colour and we have coloured all positive rational numbers.

Second example: Let the exponent of 2 in a positive rational be defined as the exponent of 2 in the prime factorisation of the numerator minus the exponent of 2 in the prime factorisation of the denominator. (This is well-defined.) Note that the exponent of 2 in the product of two rationals is the sum of the exponents of 2 in the two numbers. Let a positive rational be blue if the exponent of 2 is 0 and let it be red if the exponent of 2 is positive. Then the product of two numbers of the same colour is also of that colour.

5. (Back to problem) a) We will find pairs of fractions such that each pair adds up to an integer. Each pair will consist of a fraction $\frac{\text{odd}}{\text{odd}}$ and another one $\frac{\text{even}}{\text{even}}$, such that the first one is of form $\frac{8+n}{n}$ and the latter one is of form $\frac{m}{2n}$. The following pairs of fractions work:

$$\left(\frac{15}{7}+\frac{12}{14}\right)+\left(\frac{13}{5}+\frac{4}{10}\right)+\left(\frac{11}{3}+\frac{8}{6}\right)+\left(\frac{9}{1}+\frac{16}{2}\right)=3+3+5+17.$$

b) Analogously, we can find the following pairs of fractions:

$$\left(\frac{63}{31}+\frac{60}{62}\right); \left(\frac{61}{29}+\frac{52}{58}\right); \left(\frac{59}{27}+\frac{44}{54}\right); \left(\frac{57}{25}+\frac{36}{50}\right); \left(\frac{55}{23}+\frac{28}{46}\right); \left(\frac{53}{21}+\frac{20}{42}\right); \left(\frac{51}{19}+\frac{12}{38}\right); \left(\frac{49}{17}+\frac{4}{34}\right);$$

$$\left(\frac{47}{15}+\frac{56}{30}\right); \left(\frac{45}{13}+\frac{40}{26}\right); \left(\frac{43}{11}+\frac{24}{22}\right); \left(\frac{41}{9}+\frac{8}{18}\right); \left(\frac{39}{7}+\frac{48}{14}\right); \left(\frac{37}{5}+\frac{16}{10}\right); \left(\frac{35}{3}+\frac{32}{6}\right); \left(\frac{33}{1}+\frac{64}{2}\right)$$

Again each pair adds up to an integer, so the total sum is also an integer.

6. (Back to problem) For the solution, see Category C Problem 6.

2.2.3 Category E

1. (Back to problem) For the solution, see Category D Problem 5.

2. (Back to problem) Assume that the statement is not true, i.e. there exist a table which can be restored uniquely from the row and column sums, but it has no row full of 0's and no column full of 1's.

This is equivalent to the following statement: each row has a cell containing 0 and each column has a cell containing 1.

Let us consider a cell containing 0, call it C_1. There exists a cell in the row of C_1 containing 1, let C_2 be such a cell. Let C_3 be a cell in the column of C_2, which contains a 0. Proceeding similarly, sooner or later (in at most 65 steps) we will hit a cell already named. Let the first such cell be $C_j = C_i$ $(j > i)$.

Then changing the number in each of the cells C_i, $C_{i+1}, \ldots, C_{j-1}$ (from 0 to 1 and vice versa) we change the same number of 0's to 1 as 1's to 0 in each row and column. So we get a different table, with the same row and column sums, which is a contradiction.

3. (Back to problem) Let us consider two cases:

Case 1: $q = r$. In this case we are looking for the integer solutions of $p^q + p^q = 2p^q = n^2$. Since the left hand side is divisible by 2, so $2 \mid n^2$, which implies that $4 \mid n^2$. This also means that the left hand side is divisible by 4, which is only possible if $p = 2$. So $2 \cdot 2^q = 2^{q+1} = n^2$. Since in a perfect square all prime factors appear on even powers, so q must be odd. If $q = 2k+1$ an odd prime, then $2^q + 2^q = \left(2^{k+1}\right)^2$, which means that if $p = 2$, $q = r$ an odd prime, then $(p; q; r)$ is a solution.

Case 2: $q \neq r$. Without loss of generality we can assume that $q < r$, we will take this into consideration at the end.

This time we are looking for the integer solutions of $p^q(1 + p^{r-q}) = n^2$. Since $q < r$, $p \nmid 1 + p^{r-q}$, which means that the power of p in n^2 is q, which therefore has to be even, so $q = 2$. Then $p \mid n$, let $n = k \cdot p$ for some integer k. We get that

$$p^2(1 + p^{r-2}) = k^2p^2$$

$$1 + p^{r-2} = k^2$$

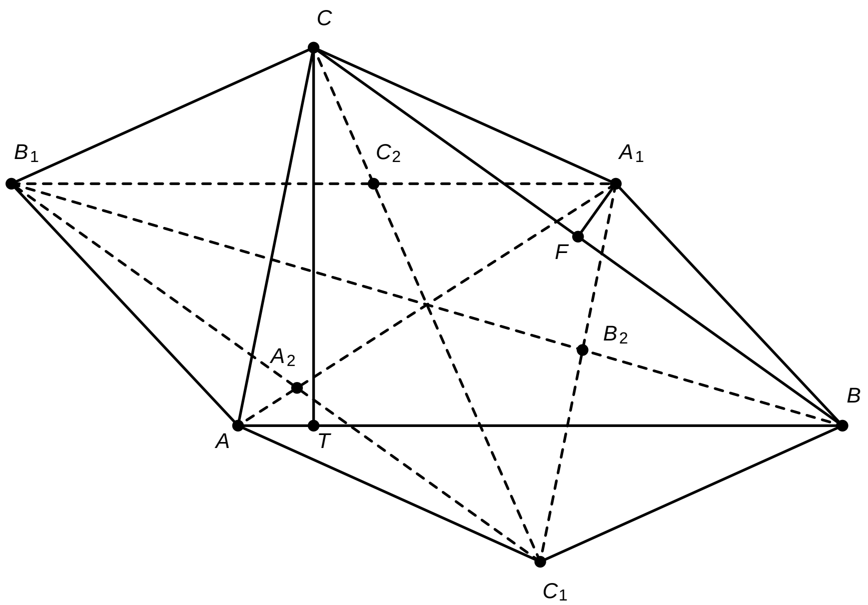
$$p^{r-2} = k^2 - 1 = (k + 1)(k - 1)$$

So $k+1$ and $k - 1$ are two powers of the same prime, which is only possible if these two numbers are 3 and 1, or 4 and 2 (this is easy to see since if $p \geq 5$, then the difference between any two powers is at least 4). In the former case $p = 3$, $r = 3$, $q = 2$, in the latter $p = 2$, $r = 5$, $q = 2$.

In conclusion when $q \neq r$, then triples $(2; 2; 5)$, $(2; 5; 2)$, $(3; 2; 3)$ and $(3; 3; 2)$ are the only solutions. The perfect square in all the cases is 36.

Finally the only triples satisfying all conditions are $(2; 2; 5)$, $(2; 5; 2)$, $(3; 2; 3)$, $(3; 3; 2)$ and $(2; s; s)$ (where s is an odd prime).

4. (Back to problem)



First let us look at these fractions a little bit more:

$$\frac{AA_1}{A_2A_1} = \frac{AA_2 + A_2A_1}{A_2A_1} = 1 + \frac{AA_2}{A_2A_1} = 1 + \frac{T_{AC_1B_1}}{T_{A_1B_1C_1}}$$

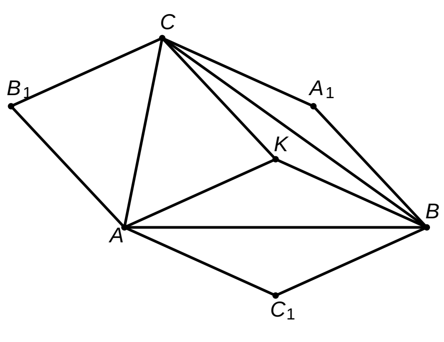
From here on our goal is to determine the areas of these triangles.

Let the foot of the altitude from C be T, the midpoint of side BC be F. Then triangles ATC and A_1FC are similar since their angles are pairwise equal. Let $a; b; c$ denote the sides of triangle ABC, t the area of ABC. Then

$$A_1C = CF \quad \frac{AC}{CT} = \frac{a}{2} \quad \frac{b}{\frac{2t}{c}} = \frac{abc}{4t}:$$

By symmetry we can conclude that all the sides of the hexagon $AC_1BA_1CB_1$ are of the same length.

With some angle chasing we get that the angles of the hexagon at vertices $A; B; C$ are 2 , 2 , 2 respectively, this means that the opposite angles of the hexagon are equal. From this follows that for example triangles AC_1B_1 and A_1BC are congruent since the length of 2 sides and the angle between them equal in the two triangles. This means that triangles ABC and $A_1B_1C_1$ are also congruent since their sides are pairwise equal.



Let the reflection of A_1 onto BC be K. Then triangles AKB and AC_1B and also AKC and AB_1C are congruent.

We are now able to determine the value of the expression:

$$\frac{AA_1}{A_2A_1}+\frac{BB_1}{B_2B_1}+\frac{CC_1}{C_2C_1}=3+\frac{T_{AC_1B_1}}{T_{A_1B_1C_1}}+\frac{T_{A_1CB_1}}{T_{A_1B_1C_1}}+\frac{T_{A_1C_1B}}{T_{A_1B_1C_1}}=3+\frac{T_{A_1BC}+T_{AB_1C}+T_{ABC_1}}{t}=$$

$$=3+\frac{T_{KBC}+T_{KAC}+T_{KAB}}{t}=3+1=4.$$

So the value of the expression is 4.

5. (Back to problem)

Lemma. *Let G be some graph. Then $\chi(G)+\chi(\bar{G})\le |V(G)|+1$, where $\chi(G)$ and $\chi(\bar{G})$ denote the chromatic numbers of G and $\bar{G}$ respectively and $|V(G)|$ is the number of vertices of G. (The chromatic number of a graph is the smallest number of colors needed to color the vertices of the graph such that no two adjacent vertices are of the same color.)*

Proof. By induction on the number of vertices of G.

The statement is true for a graph of one vertex.

Assume that the statement is true for graphs of n vertices. Now let us consider a graph G' of $n+1$ vertices. Let v be a vertex of G' and let G be the graph obtained from G' by removing vertex v and the edges ending in v. We know that $\chi(G)+\chi(\bar{G})\le V(G)+1$. Also: $\chi(G')$ is $\chi(G)$ or $\chi(G)+1$, while $\chi(\bar{G'})$ is $\chi(\bar{G})$ or $\chi(\bar{G})+1$ (adding a new vertex cannot decrease the chromatic number; taking a good coloring and introducing a new color for v works).

If G satisfies $\chi(G)+\chi(\bar{G})\le |V(G)|$, then from the aboves $\chi(G')+\chi(\bar{G'})\le |V(G)|+2=|V(G')|+1$, as required.

If $\chi(G)+\chi(\bar{G})=|V(G)|+1$: The degree (i.e. number of neighbouring vertices) of v in G' and the degree of v in $\bar{G'}$ add up to $V(G)$. Therefore the degree of v in G' is smaller than $\chi(G)$ or the degree of v in $\bar{G'}$ is smaller than $\chi(\bar{G})$. We can assume the first case holds. Let us consider a coloring of G with $\chi(G)$ different colors such that no two adjacent vertices are of the same color. Then there is some color such that v is not adjacent (in G') to any vertex of that color. We can choose this to be the color of v, therefore $\chi(G')=\chi(G)$. So $\chi(G')+\chi(\bar{G'})\le\chi(G)+\chi(\bar{G})+1=|V(G)|+2=V(G')+1$. □

Let G be the acquaintance graph of the couples (i.e. the vertices correspond to the couples and two vertices are connected if and only if the corresponding couples know each others). Then the least number of saunas needed for the women is $\chi(G)$, while the least number of saunas needed for the men is $\chi(\bar{G})$. From the lemma we know that the total number of saunas needed is $\chi(G)+\chi(\bar{G})\le |V(G)|+1$. So for $|V(G)|=2018$, i.e. 2018 couples, having 2019 saunas is certainly enough. If there are 2019 couples and no two of them know each other, they need 2020 saunas (2019 for women and one for men).

So the maximal number of couples who can certainly use the saunas simultaneously is 2018.

6. (Back to problem) Let us label each possible state of the game as a *winning state* or a *losing state*. A winning state is such that the player on turn is able to win from there (no matter how the opponent plays), while a losing state is such that no matter what the player on turn does, the opponent is able to win (if she plays optimally).

Lemma. *Let us consider a state of a game and let the smallest pile consist of k disks. The state is a winning state if and only if the number of piles of size k is odd.*

Proof. By induction on the total number of discs.

If there is only one disc, then $k = 1$, there is one pile of size k, and the player on turn can win by removing the only disc. So the statement is true.

Now assume that there is an odd number of piles of size k and the statement is true for any smaller number of discs.

If there is only one pile, it is a winning state, as the player on turn can remove all the discs.

If there are at least two piles, we can show that no matter how the opponent chooses two of them, the player on turn can obtain a losing state.

Case 1: Both piles chosen are of size k: Let us remove one of them completely. Then the smallest pile size is still k, but the number of piles of size k is reduced by one, i.e. it is even now. It is a losing state by induction hypothesis.

Case 2: One of the piles chosen is of size $> k$: Let us remove all but k discs from it. Then the number of piles of size k is increased by one, i.e. it becomes even. It is a losing state by induction hypothesis.

Now assume that there is an even number of piles of size k and the statement is true for any smaller number of discs.

If the opponent chooses two piles of size k, the player on turn will certainly obtain a winning state:

If he removes one of the piles completely, the number of piles of size k is decreased by one, so it becomes odd. This is a winning state by induction hypothesis.

If he removes some, but not all discs from one pile, then this will become the smallest pile, so there will be only one smallest pile. This is a winning state by induction hypothesis. □

So we can win the game with the following strategy: Check whether the starting state is a winning state and go first if and only if it is. Then at each of our turns obtain a losing state (as above), and at each of the opponent's turns choose two of the smallest piles.

2.2.4 Category E$^+$

1. (Back to problem) We will prove by induction: if $n = 0$, then $0 \quad 0$ obviously holds. The inductive step from $n \quad 1$ to n. We want to prove that

$$\Bigl(\sum_{i=0} a_i\Bigr)^2 \qquad \sum_{i=0} a_i^3 \tag{1}$$

By induction we know that

$$\Bigl(\sum_{i=0}^{n\;1} a_i\Bigr)^2 \qquad \sum_{i=0}^{n\;1} a_i^3 \tag{2}$$

To prove (1), it suffices to show by (2) that

$$a_n^2 + 2a_n \sum_{i=0}^{n-1} a_i \le a_n^3$$

If $a_n = 0$, then both sides equal 0. Now let us assume that $a_n \neq 0$.

Simplifying

$$a_n + 2\sum_{i=0}^{n-1} a_i \le a_n^2$$

$$\sum_{i=0}^{n-1} a_i \le \frac{a_n(a_n - 1)}{2}$$

By the conditions of the problem

$$\frac{a_n(a_n - 1)}{2} \ge \frac{(a_{n-1} + 1)a_{n-1}}{2}$$

This means that it is enough to show that

$$\sum_{i=0}^{n-1} a_i \le \frac{(a_{n-1} + 1)a_{n-1}}{2}$$

Let $a_{n-1} = k + x$, where k is and integer and $x < 1$. Then

$$\sum_{i=0}^{n-1} a_i \le (k - 1 + x) + (k - 2 + x) + \ldots + x + 0 + \ldots + 0 = \frac{k(k+1)}{2} + (k+1)x$$

$$\frac{(a_{n-1} + 1)a_{n-1}}{2} = \frac{(k + x + 1)(k + x)}{2} = \frac{k(k+1)}{2} + \frac{(2k+1)x}{2} + \frac{x^2}{2}$$

So

$$\sum_{i=0}^{n-1} a_i \le \frac{k(k+1)}{2} + (k+1)x \le \frac{k(k+1)}{2} + \frac{(2k+1)x}{2} + \frac{x^2}{2};$$

since $\frac{x^2}{2} \le \frac{x}{2}$, because $x < 1$.

2. (Back to problem) Let us denote the length of the sides of the triangle by a, b, c, its area by T, and the radius of its circumcircle by R. The well-known indetity of the circumradius and the area:

$$4T = \frac{abc}{R}.$$

From this immediately follows that $4T$ is rational. In addition by Heron's formula

$$4T = \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}.$$

So $4T$ is rational, moreover it is the square root of an integer, which means that $4T$ has to be an integer. Then at least one of the side lengths has to be divisible by R, since R is a prime. Let this be side a. Let us also assume that the triangle is not right-angled. Then

$$0 < a, b, c < 2R,$$

so $a = R$. Let the angle at vertex A be α. Then $a = 2R\sin\alpha$, this means that $\sin\alpha = \frac{1}{2}$. Then

$$\frac{b^2 + c^2 - a^2}{2bc} = \cos\alpha = \pm\frac{\sqrt{3}}{2}.$$

Since the left hand side is rational and the right hand side is irrational, we have arrived to a contradiction, so the triangle must be right.

Such a triangle indeed exists, let us consider for example the one with sides $6, 8, 10$, the radius of its circumcircle is 5.

3. (Back to problem) It is known that k and n are not coprime if and only if $n - k$ and n are not coprime. If n is odd, then this way we have formed pairs of all the terms in the addition. The sum of each pair is n, so $n \mid f(n)$. If n is even, then $\frac{n}{2}$ will not have a pair, so we only know that $n \mid 2f(n)$.

Now let us assume that $f(n) = f(n + p)$. Since n and p are coprimes and $n \mid 2f(n)$, and also $n + p \mid 2f(n)$, this means that $n(n + p) \mid 2f(n)$, but $2f(n)$ is smaller than the double of the sum of the integers from 1 to n, we get that $n(n + p) \le n(n + 1)$, which is a contradiction. With this the proof is complete.

4. (Back to problem) We will prove a more general statement: Let us have a set of size $nk + n - 1$. If we partition its subsets of size k into two groups, then there exist n disjoint subsets, which are in the same group.

We will prove it by induction on n.

If $n = 1$, the statement is trivially true. (Partitioning the size k subsets of a set of size k into two groups, there will be a group containing a subset of size k.)

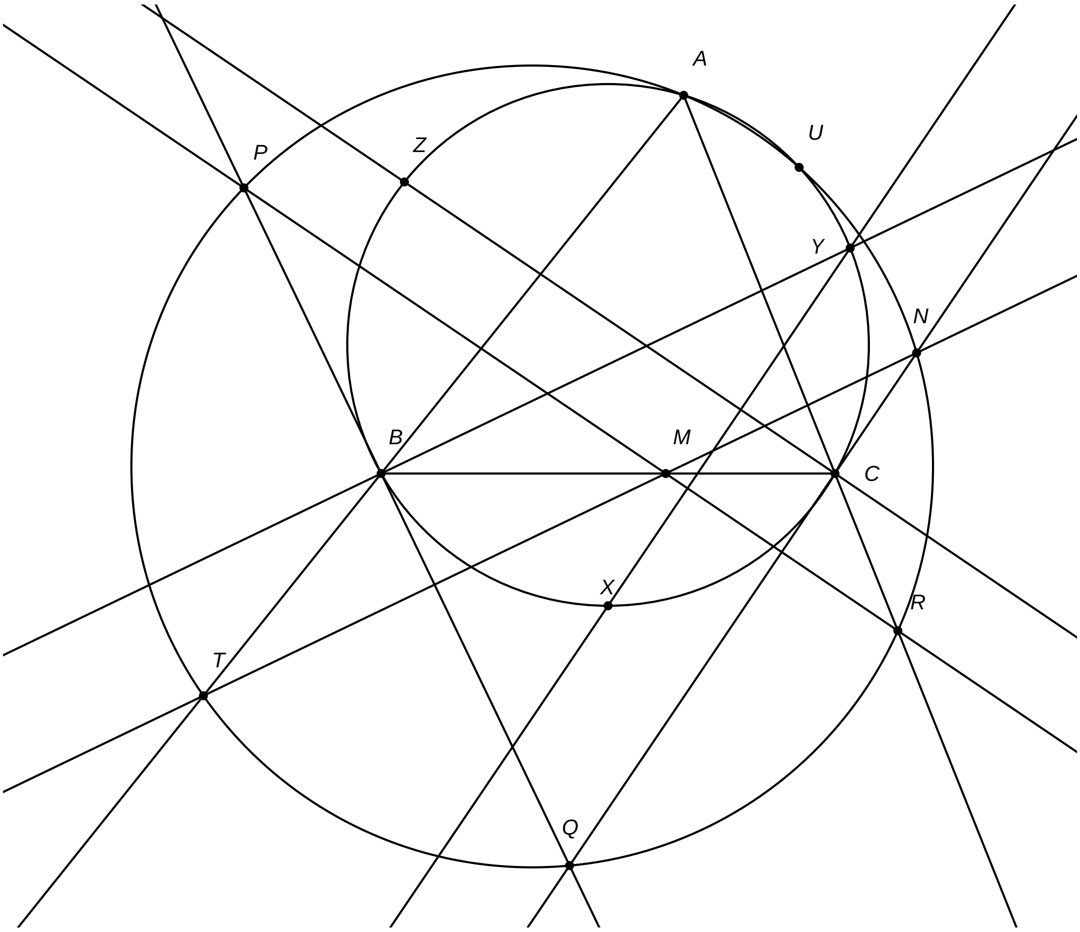
If we know that the statement is true for $n = N$: Now let $n = N + 1$.

If there exist no subsets A, B of size k such that they are in different groups and they have $k - 1$ common elements, then we can see that all subsets of size k must be in the same group, so the statement is true.

If A and B are subsets of size k such that they are in different groups and they have $k - 1$ common elements: $|A \cup B| = k + 1$, thus there are $(N + 1)k + N - (k + 1) = Nk + N - 1$ elements which are not in A or B. By induction hypothesis there exist N disjoint subsets of size k which are in the same group and only contain these elements. We can add either A or B to get $N + 1$ disjoint subsets of size k which are in the same group. So the statement is true for $n = N + 1$ as well.

Substituting $n = k = 7$ we get the statement of the problem.

5. (Back to problem)



Let the outer angle bisectors at vertices B and C meet at point Q, the lines AB and MN at point T, the lines AC and MP at point R.

Then we can notice that $ATQR$ is a cyclic quadrilateral since the perpendicular bisector of segment MT is BQ, and the perpendicular bisector of segment MR is CQ so Q is the circumcenter of the triangle TMR, so $\angle TMR = 90 \ + \frac{}{2}$ so the $\angle TQR$ central angle is 180 .

Points X and Q lie on the inner bisector of the angle $\angle BAC$, so $\frac{}{2} = \angle TAQ = \angle TNQ$ holds so points N and P lies on the circle $ATQR$.

Let U denote the second intersection point of circles $ATQR$ and ABC. We will show that lines PZ, NY and MX all go through this point.
By simple angle chasing we can see that lines XY and QN are parallel so $\angle AUY = 180 \quad \angle AXY = 180 \quad \angle AQN = \angle AUN$ so $U; Y; N$ are collinear. We prove the collinearity of $U; Z; P$ in a similar manner.

Using these informations we obtain $\angle UNM = \angle UYB = \angle UCB = \angle UCM$, so the quadrilateral $UNCM$ is cyclic. This further implies $\angle MUC = \angle MNC = \angle BYX = \angle BUX = \angle XUC$, so $X; M; U$ are collinear. And with this we have proven the statement.

6. (Back to problem) For the solution, see Category E Problem 6.

2.3 Final round – day 2

2.3.1 Category C

#	ANS	Problem	P
C-1	40	Simple John has a square cabbage	3p
C-2	630	What is the total number of minutes	3p
C-3	558	What is the smallest positive integer	3p
C-4	9	Bob wrote down all positive integers	3p
C-5	9	What is the maximum possible	4p
C-6	7	Let $A_1A_2A_3A_4A_5A_6$ be a regular	4p
C-7	91	A bicycle race consists of	4p
C-8	34	The following 100 statements	4p
C-9	1963	What remainder do we get if we divide	5p
C-10	60	In Miskolc there are two tram lines:	5p
C-11	1001	We call a non-negative integer *nice*	5p
C-12	36	Let O be the incenter	5p
C-13	11	We write down all natural numbers	6p
C-14	1	We choose a point on each side of a	6p
C-15	21	How many ways are there to arrange	6p
C-16	392	How many ways are there to go	6p

2.3.2 Category D

#	ANS	Problem	P
D-1	9	What is the maximum possible	3p
D-2	7	Let $A_1A_2A_3A_4A_5A_6$ be a regular	3p
D-3	91	A bicycle race consists of	3p
D-4	34	The following 100 statements	3p
D-5	1963	What remainder do we get if we divide	4p
D-6	36	Let O be the incenter	4p
D-7	1001	We call a non-negative integer *nice*	4p
D-8	60	In Miskolc there are two tram lines:	4p
D-9	11	We write down all natural numbers	5p
D-10	81	Find the smallest multiple of 81	5p
D-11	1	We choose a point on each side of a	5p
D-12	56	A chess piece is placed on	5p
D-13	244	We call a positive integer n	6p
D-14	7282	The sum of all positive integers	6p
D-15	108	In an isosceles, obtuse-angled	6p
D-16	21	How many ways are there to arrange	6p

2.3.3 Category E

#	ANS	Problem	P
E-1	36	Find the number of non-isosceles	3p
E-2	4	Anne multiplies each two-digit	3p
E-3	1010	On a piece of paper we have 2019	3p
E-4	60	In Miskolc there are two tram lines:	3p
E-5	1010	We want to write down as many	4p
E-6	81	Find the smallest multiple of 81	4p
E-7	1	We choose a point on each side of a	4p
E-8	56	A chess piece is placed on	4p
E-9	7	A cube has been divided into 27	5p
E-10	108	In an isosceles, obtuse-angled	5p
E-11	480	What is the smallest possible value	5p
E-12	987	How many ways are there to arrange	5p
E-13	4	Let $k > 1$ be a positive integer	6p
E-14	6	Seven classmates are comparing their	6p
E-15	102	ABC is an isosceles triangle	6p
E-16	2048	How many ways are there to paint	6p

2.3.4 Category E$^+$

#	ANS	Problem	P
E$^+$-1	1010	We want to write down as many	3p
E$^+$-2	81	Find the smallest multiple of 81	3p
E$^+$-3	25	Let P be an interior point	3p
E$^+$-4	7	A cube has been divided into 27	3p
E$^+$-5	987	How many permutations s does the set	4p
E$^+$-6	1	We choose a point on each side of a	4p
E$^+$-7	63	Find the smallest positive integer	4p
E$^+$-8	9376	Let N be a positive integer	4p
E$^+$-9	2048	How many ways are there to paint	5p
E$^+$-10	480	What is the smallest possible value	5p
E$^+$-11	504	What is the smallest N for which	5p
E$^+$-12	6361	P and Q are two different non-constant	5p
E$^+$-13	2597	There are 12 chairs arranged	6p
E$^+$-14	6248	Let S be the set of all	6p
E$^+$-15	16	The positive integer m and non-negative	6p
E$^+$-16	389	Triangle ABC has side lengths	6p