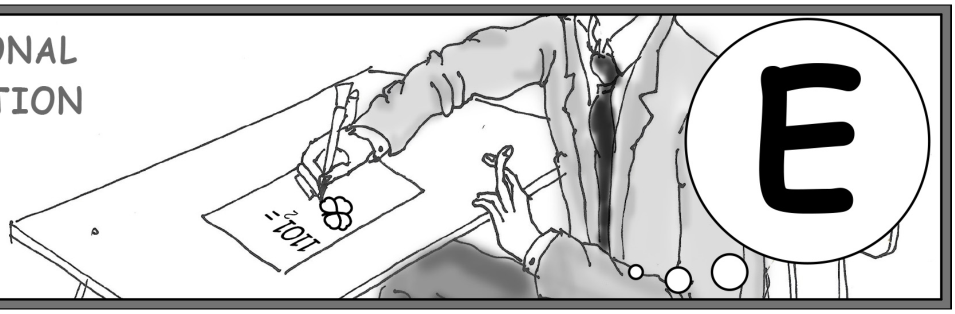
SOLUTIONS



E1. In maths class Albrecht had to compute $(a+2b-3)^2$. His result was a^2+4b^2-9. 'This is not correct' said his teacher, 'try substituting positive integers for a and b.' Albrecht did so, but his result proved to be correct. What numbers could he substitute?
a) Show a good substitution.
b) Give all the pairs that Albrecht could substitute and prove that there are no more.

Solution: a) A good substitution is $a=3$, $b=1$, since $2^2=9+4-9$.

b) If Albrecht has got a correct result, it means that $(a+2b-3)^2=a^2+4b^2-9$. This expands to:

$$a^2+4b^2+9+4ab-6a-12b=a^2+4b^2-9$$

After simplification:

$$2ab-3a-6b+9=0$$

Factorize the left side as:

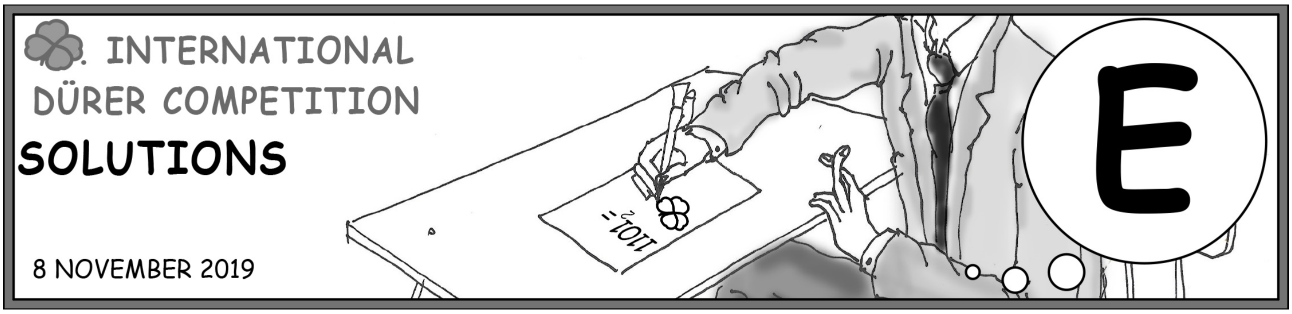
$$(2b-3)(a-3)=0$$

Now, since the product is zero, there are two cases.

First case: $2b-3=0$, from which $b=\frac{3}{2}$, this is not possible since b has to be an integer.

Second case: $a-3=0$, from which $a=3$.

So Albrecht gets the correct result if and only if $a=3$ and in this case he can substitute any positive integer for b.



E2. Initially we have a 2×2 table with at least one grain of wheat on each cell. In each step we may perform one of the following two kinds of moves:

(i) If there is at least one grain on every cell of a row, we can take away one grain from each cell in that row.
(ii) We can double the number of grains on each cell of an arbitrary column.

a) Show that it is possible to reach the empty table using the above moves, starting from the position on the right.

3	5
5	3

b) Show that it is possible to reach the empty table from any starting position.
c) Prove that the same is true for the 8×8 tables as well.

Solution:

a) First take away one grain from every cell in the first row, and then double the grains in the first column. Now we have 4 grains on both cells of the first row, so we can take them all away. In the second row we have 10 and 3 grains on the cells. Double the grains in every cell of the second column, and take away 2-2 grains from the cells in the second row. Doubling the grains in the second column leaves us with 4 grains on the cells in the second row, so we can take them all away, emptying the table.

b) Suppose the table looks like this:

a	b
c	d

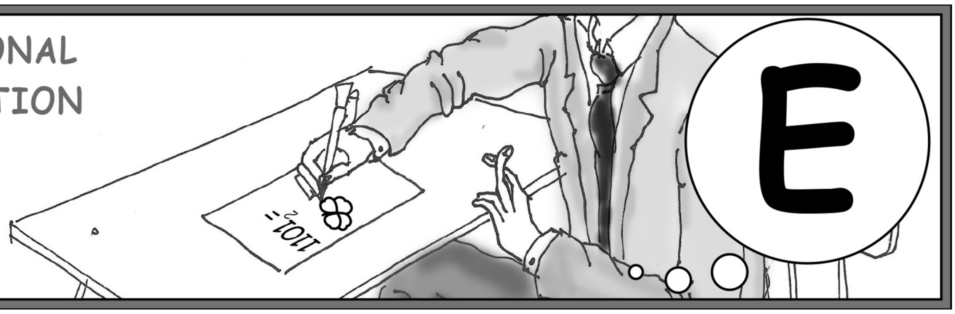
First we want to empty the first row. If $a = b$, take away a grains from every cell of the first row. If $a < b$, then keep doubling the first column until we reach $b \geq a' \geq b - a'$ (Here a' is the number of grains on the top-left cell). Then we take away $2a' - b$ grains from each cell of the first row, leaving us with $b - a'$ grains on the top left cell, and $2b - 2a'$ on the top right cell. Double the left column and take away $2b - 2a'$ grains from the first row's cells. This empties the top two cells. If $b < a$, we can do a similar algorithm.

These steps never decrease the number of grains in the second row, so after these steps we will have a positive number of grains on the other 2 cells of your table. We can use the same algorithm to empty these, and it never puts new grains on an empty cell, so thi way we can empty the whole table.

c) We will show that we can empty an arbitrary row from any starting table.

Take an arbitrary row, and look at the 2 cells in the row with the least amount of grains on them. Using the algorithm above we can make these equal (in example, we execute every step of the algorithm until we reach $2b - 2a'$ grains on both of these cells). Observe that none of the other cells in the table will become empty throughout this process. From now on, we consider these two columns the same, so that we will do the same steps on both of them. Then we have 7 columns that we consider differently, so we can use the same algorithm as above, leaving us with 6 differently considered columns. Repeat until all of the columns are considered the same. This means that every cell on the row that we took at the beginning has an equal number of grains. Then we take all of them away.

This way we can empty an arbitrary row, keeping the other cells in the table nonempty. So doing this for every row empties every cell, so we proved what we wanted.



E3. **a)** Is it possible that the sum of all the positive divisors of two different natural numbers are equal?
b) Show that if the product of all the positive divisors of two natural numbers are equal, then the two numbers must be equal.

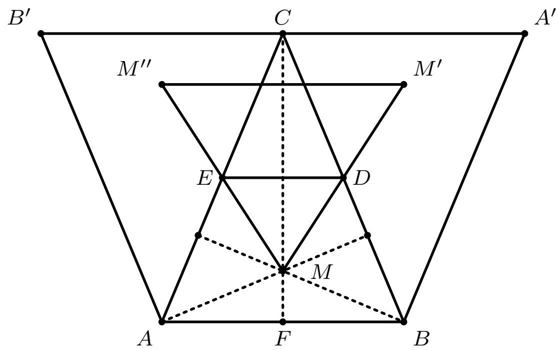
Solution: a) Yes, for example the sum of divisors is 12 for 6 and 11 as well.

b) First of all we prove that the product of the positive divisors of n is $n^{\frac{d(n)}{2}}$, where $d(n)$ denotes the number of divisors of n. The divisors can be arranged in pairs, so if $k \mid n$ then $\frac{n}{k} \mid n$ and $k \cdot \frac{n}{k} = n$. From this we get the formula easily.

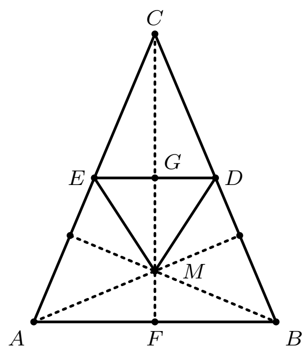
After this we need to prove that if $n^{\frac{d(n)}{2}} = m^{\frac{d(m)}{2}}$ then $n = m$. So if any prime number divides n, then it divides m as well. Let us consider a prime p that divides n. Let k be the positive integer for which $p^k \mid n$ but $p^{k+1} \nmid n$, and similarly let l be the number for which $p^l \mid m$ but $p^{l+1} \nmid m$. In this case since $n^{\frac{d(n)}{2}} = m^{\frac{d(m)}{2}}$, therefore the exponent of p is equal on both sides, so $\frac{k \cdot d(n)}{2} = \frac{l \cdot d(m)}{2}$, this means that $\frac{k}{l} = \frac{d(m)}{d(n)}$. This is true for all prime numbers, so if $d(n) < d(m)$, then the exponent of prime p, where p divides n, is greater in n than in m, so $n > m$ and every divisor of m divides n as well, therefore the products of divisors cannot be equal. Similarly $d(n) > d(m)$ is not possible either, so for $d(n) = d(m)$ to hold, the exponent of every prime should be equal in n and in m, which is equivalent to $n = m$.

E4. Let ABC be an acute triangle with side AB of length 1. Say we reflect the points A and B across the midpoints of BC and AC, respectively to obtain the points A' and B'. Assume that the orthocenters of triangles ABC, $A'BC$ and $B'AC$ form an equilateral triangle.
a) Prove that triangle ABC is isosceles.
b) What is the length of the altitude of ABC through C?

Solution: Let us use the notations from the figure. Let the midpoints of the sides of triangle ABC be D, E, F respectively, the orthocenters of ABC, $A'BC$ and $B'AC$ be M, M' and M'' respectively.



Since triangles ABC and $A'BC$ are mirror images of each other with respect to point D, this also holds for their orthocenters, so the midpoint of section MM' has to be D. Similarly the midpoint of section MM'' is point E. This means that segment DE is a midline in triangle $MM'M''$, so triangle DEM is also equilateral. From this follows that M point is on the perpendicular bisector of segment DE, which is also perpendicular to side AB, since DE is the midline corresponding to side AB in trangle ABC. So vertex C is also on the perpendicular bisector of DE, this means that triangle ABC is isosceles.



Now we know that triangle ABC is isosceles and triangle DEM is equilateral. Assume that point M lies between lines DE and AB. Let the midpoint of segment DE be G. Let us denote the length of the altitude of ABC through C (segment CF) be m. Since segment DE is midline, the length of segment CG is $\frac{m}{2}$. Triangle MED is equilateral, so the altitude over side DE (segment GM) is of length $\frac{\sqrt{3}}{2}DE = \frac{\sqrt{3}}{4}$. The angles of triangles AFC and MFA are pairwise the same, so they are similar, so

$$\frac{AF}{FC} = \frac{MF}{FA},$$

$$MF = \frac{AF}{FC} \cdot FA = \frac{\frac{1}{2}}{m} \cdot \frac{1}{2} = \frac{1}{4m}.$$

Since $CF = CG + GM + MF$ we get the following equation:

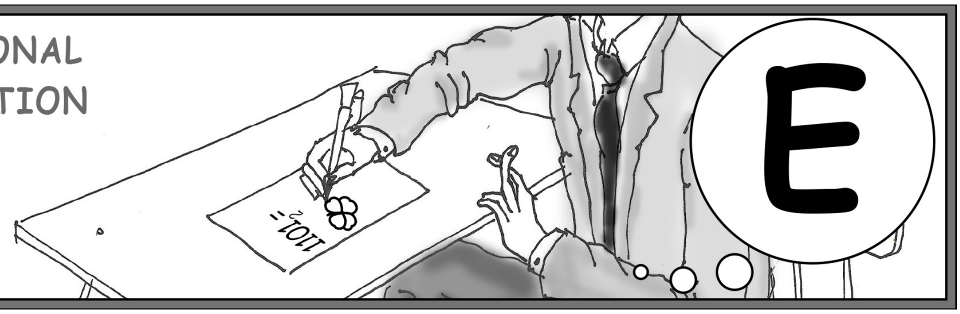
$$m = \frac{m}{2} + \frac{\sqrt{3}}{4} + \frac{1}{4m},$$

$$\frac{m}{2} - \frac{\sqrt{3}}{4} - \frac{1}{4m} = 0,$$

$$2m^2 - \sqrt{3}m - 1 = 0,$$

$$m_{1,2} = \frac{\sqrt{3} \pm \sqrt{11}}{4}.$$

Since the length of the altitude is positive, the length of the altitude of ABC through C is $m = \frac{\sqrt{3}+\sqrt{11}}{4}$.



E5. We call a table of size $n \times n$ *self-describing* if each cell of the table contains the total number of even numbers in its row and column other than itself. How many *self-describing* tables of size
a) 3×3 exist?
b) 4×4 exist?
c) 5×5 exist?
Two tables are different if they differ in at least one cell.

Solution: We will solve the problem generally for any n.

Observe that it is enough to determine the parity of the elements of the table, as we can reconstruct any element if we know the parity of every element (we just count the number of even elements in its row and column). From now on we will denote odd elements as 1 and even ones as 0.

In this case we will prove that two adjacent columns are either the same in every entry (by row) or different in each one. From this it would follow that the same holds for any two columns and by symmetry for any two rows.

It is enough to prove the statement for subtables with 2 rows and columns (we define subtable as the intersection of some adjacent rows and columns). Let our table look like this:

A	C	E
B	D	F
G	H	

Where the letters denote the number of even numbers in their respective regions. Evidently the parity of an element in the subtable in top-left corner is the opposite of the parity of the letter corresponding to it, but it is still enough to prove that A, B and C, D are either the same or are opposite.

Now we can write the following congruences based on our 2×2 subtable:

$$A \equiv C + E + B + G \pmod 2$$

$$B \equiv D + F + A + G \pmod 2$$

$$C \equiv A + E + D + H \pmod 2$$

$$D \equiv B + F + C + H \pmod 2$$

The sum of these equations is:

$$A + B + C + D \equiv 2A + 2B + 2C + 2D + 2E + 2F + 2G + 2H \equiv 0 \pmod 2$$

$$A + C \equiv B + D \pmod 2$$

Which means that if $A \equiv C \pmod 2$, then $B \equiv D \pmod 2$, and if $A \not\equiv C \pmod 2$, then $B \not\equiv D$ (mod 2), which is what we set out to prove.

We will introduce some notations: We call a table and its elements self-descriptive if they satisfy the properties required in the statement of this problem. We will denote the intersection of the i-th row and j-th column as (i, j), and the value of (i, j) as $f(i, j)$. We call an element good if the parity of the number of even values in its row and column (without itself) is the same as its parity. Let's call a

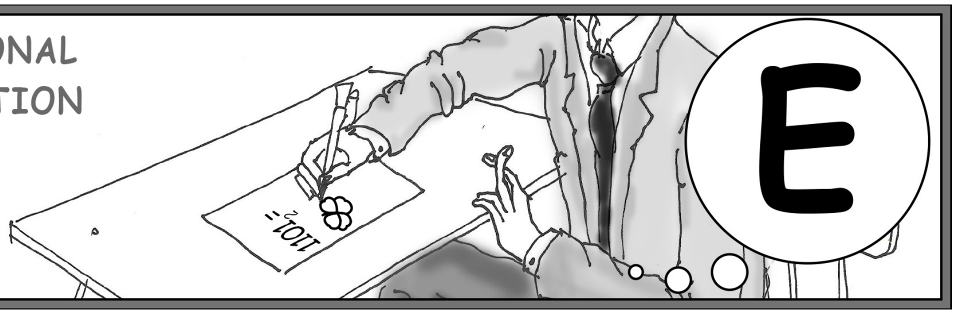


table and its elements nice if every column is either the same as the first one or the opposite. Observe that we proved above that every self descriptive table is nice.

Now we will show that if n is even then there is only one self-descriptive table, which is the one where every entry is even (therefore $n-2$).

Let us suppose that there is an odd element in a self-descriptive table. As our statement is invariant for the reordering of rows and columns we can suppose that $f(1,1)=1$. According to the description of the problem this means that either the first row or column has an odd number of even values, thus there is at least one even number in the same row or column. Therefore we can suppose that $f(1,2)=0$. Let x be the number of even values in the first column. As the table is nice and in the first row the first two elements are not the same, the first two columns must be the inverse of each other. Therefore the number of even numbers in the second column must be $n-x$. Let k be the number of even numbers in the first row excluding the first element.

1	0	k
x	$n-1-$ $-x$	

Let us write down the statement of the problem for the element $(1,1)$:

$$1 \equiv k+1+x \pmod 2$$

And for element $(1,2)$:

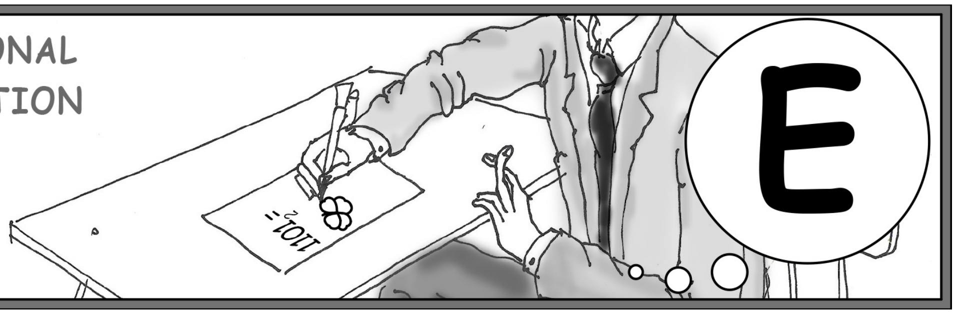
$$0 \equiv k+n-1-x \equiv k+x+1 \pmod 2$$

Contradiction.

In the case where n is odd we will prove that there are 2^{2n-2} different self-descriptive tables.

Let us choose the elements of first row except for the last element and the entire first column. These are 2^{2n-2} cases. We will prove that we can complete the table exactly one way for every choice. As $(1,1)$ must be a good element, it determines the value of $(1,n)$. As the table must be nice and we know the first column and the first element in every column we can determine the value of any element. Now we only need to prove that every such table is self-descriptive. This requires that every element is good.

We know that $(1,1)$ is good. Let us prove that the first row and column is good, or equivalently that $(1,2)$ is good. If $f(1,1)=f(1,2)$ then the entire second column is the same as the first one thus the number of even numbers is the same in their columns, and they are in the same row, and their value is the same, thus the sum corresponding to them must be the same as well. If $f(1,1)\neq f(1,2)$ then let x-be the number of even numbers in the first column except for $(1,1)$, as $f(1,1)\neq f(1,2)$ the



second row without $(1,2)$ must have $n-1-x$ even entries, which as $n-1$ is even means that the sum in the column of $(1,1)$ and $(1,2)$ is the same (without the first row), but the sum in their row without themselves is different, as they are in the same row, but have different values. Therefore the sum will differ by one between $(1,1)$ and $(1,2)$ which is what we wanted to prove. (And from that we have also proved that the entire first row is good).

We can use the same method to prove that any (i,j) is good from the fact that $(1,j)$ is good. Therefore the table is self-descriptive.

In conlusion: if n is even the answer is 1, if it is odd there are 2^{2n-2} different tables. So the answers: **a)** $2^4 = 16$, **b)** 1, **c)** $2^8 = 256$.