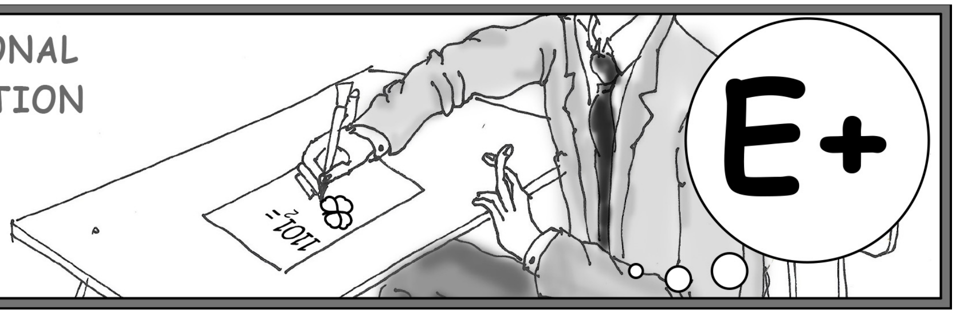
. INTERNATIONAL DÜRER COMPETITION

SOLUTIONS

8 NOVEMBER 2019

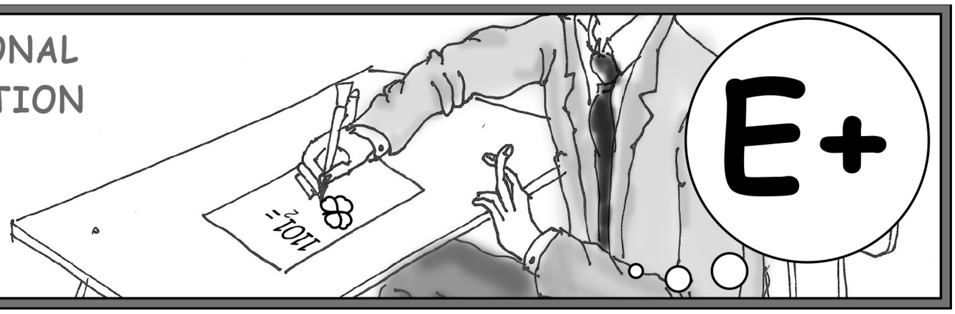


E+1. **a)** Is it possible that the sum of all the positive divisors of two different natural numbers are equal?
b) Is it possible that the product of all the positive divisors of two different natural numbers are equal?

Solution: a) Yes, for example the sum of divisors is 12 for 6 and 11 as well.

b) No, it is not possible that the product of the divisor of two different numbers are equal. First of all we prove that the product of the positive divisors of n is $n^{\frac{d(n)}{2}}$, where $d(n)$ denotes the number of divisors of n. The divisors can be arranged in pairs, so if $k \mid n$ then $\frac{n}{k} \mid n$ and $k \cdot \frac{n}{k} = n$. From this we get the formula easily.

After this we need to prove that if $n^{\frac{d(n)}{2}} = m^{\frac{d(m)}{2}}$ then $n = m$. So if any prime number divides n, then it divides m as well. Let us consider a prime p that divides n. Let k be the positive integer for which $p^k \mid n$ but $p^{k+1} \nmid n$, and similarly let l be the number for which $p^l \mid m$ but $p^{l+1} \nmid m$. In this case since $n^{\frac{d(n)}{2}} = m^{\frac{d(m)}{2}}$, therefore the exponent of p is equal on both sides, so $\frac{k \cdot d(n)}{2} = \frac{l \cdot d(m)}{2}$, this means that $\frac{k}{l} = \frac{d(m)}{d(n)}$. This is true for all prime numbers, so if $d(n) < d(m)$, then the exponent of prime p, where p divides n, is greater in n than in m, so $n > m$ and every divisor of m divides n as well, therefore the products of divisors cannot be equal. Similarly $d(n) > d(m)$ is not possible either, so for $d(n) = d(m)$ to hold, the exponent of every prime should be equal in n and in m, which is equivalent to $n = m$.



E+2. How many ways can you fill a table of size $n \times n$ with integers such that each cell contains the total number of even numbers in its row and column other than itself?
Two tables are different if they differ in at least one cell.

First solution: Observe that it is enough to determine the parity of the elements of the table, as we can reconstruct any element if we know the parity of every element (we just count the number of even elements in its row and column). From now on we will denote odd elements as 1 and even ones as 0.

In this case we will prove that two adjacent columns are either the same in every entry (by row) or different in each one. From this it would follow that the same holds for any two columns and by symmetry for any two rows.

It is enough to prove the statement for subtables with 2 rows and columns (we define subtable as the intersection of some adjacent rows and columns). Let our table look like this:

A	C	E
B	D	F
G	H	

Where the letters denote the number of even numbers in their respective regions. Evidently the parity of an element in the subtable in top-left corner is the opposite of the parity of the letter corresponding to it, but it is still enough to prove that A, B and C, D are either the same or are opposite.

Now we can write the following congruences based on our 2×2 subtable:

$$A \equiv C + E + B + G \pmod 2$$

$$B \equiv D + F + A + G \pmod 2$$

$$C \equiv A + E + D + H \pmod 2$$

$$D \equiv B + F + C + H \pmod 2$$

The sum of these equations is:

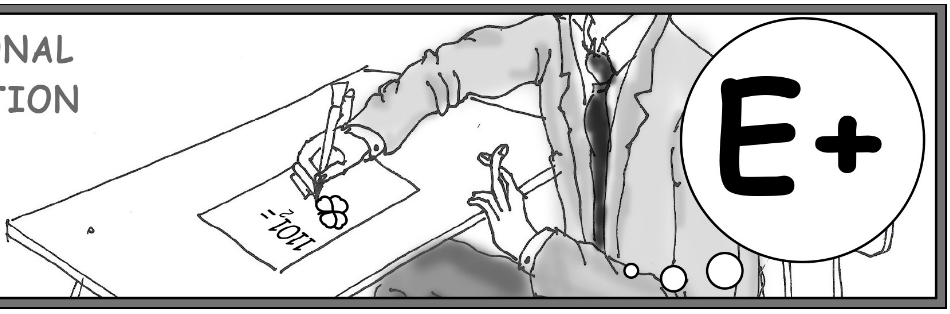
$$A + B + C + D \equiv 2A + 2B + 2C + 2D + 2E + 2F + 2G + 2H \equiv 0 \pmod 2$$

$$A + C \equiv B + D \pmod 2$$

Which means that if $A \equiv C \pmod 2$, then $B \equiv D \pmod 2$, and if $A \not\equiv C \pmod 2$, then $B \not\equiv D \pmod 2$, which is what we set out to prove.

We will introduce some notations: We call a table and its elements self-descriptive if they satisfy the properties required in the statement of this problem. We will denote the intersection of the i-th row and j-th column as (i, j), and the value of (i, j) as $f(i, j)$. We call an element good if the parity of the number of even values in its row and column (without itself) is the same as its parity. Let's call a table and its elements nice if every column is either the same as the first one or the opposite. Observe that we proved above that every self descriptive table is nice.

Now we will show that if n is even then there is only one self-descriptive table, which is the one where every entry is even (therefore $n - 2$).



Let us suppose that there is an odd element in a self-descriptive table. As our statement is invariant for the reordering of rows and columns we can suppose that $f(1,1) = 1$. According to the description of the problem this means that either the first row or column has an odd number of even values, thus there is at least one even number in the same row or column. Therefore we can suppose that $f(1,2) = 0$. Let x be the number of even values in the first column. As the table is nice and in the first row the first two elements are not the same, the first two columns must be the inverse of each other. Therefore the number of even numbers in the second column must be $n - x$. Let k be the number of even numbers in the first row excluding the first element.

1	0	k
x	$n-1-$ $-x$	

Let us write down the statement of the problem for the element $(1,1)$:

$$1 \equiv k + 1 + x \pmod 2$$

And for element $(1,2)$:

$$0 \equiv k + n - 1 - x \equiv k + x + 1 \pmod 2$$

Contradiction.

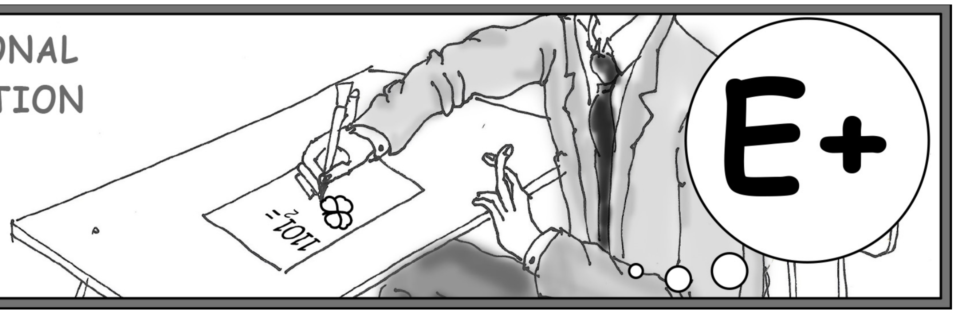
In the case where n is odd we will prove that there are 2^{2n-2} different self-descriptive tables.

Let us choose the elements of first row except for the last element and the entire first column. These are 2^{2n-2} cases. We will prove that we can complete the table exactly one way for every choice. As $(1,1)$ must be a good element, it determines the value of $(1,n)$. As the table must be nice and we know the first column and the first element in every column we can determine the value of any element. Now we only need to prove that every such table is self-descriptive. This requires that every element is good.

We know that $(1,1)$ is good. Let us prove that the first row and column is good, or equivalently that $(1,2)$ is good. If $f(1,1) = f(1,2)$ then the entire second column is the same as the first one thus the number of even numbers is the same in their columns, and they are in the same row, and their value is the same, thus the sum corresponding to them must be the same as well. If $f(1,1) \neq f(1,2)$ then let x-be the number of even numbers in the first column except for $(1,1)$, as $f(1,1) \neq f(1,2)$ the second row without $(1,2)$ must have $n-1-x$ even entries, which as $n-1$ is even means that the sum in the column of $(1,1)$ and $(1,2)$ is the same (without the first row), but the sum in their row without themselves is different, as they are in the same row, but have different values. Therefore the sum will differ by one between $(1,1)$ and $(1,2)$ which is what we wanted to prove. (And from that we have also proved that the entire first row is good).

We can use the same method to prove that any (i,j) is good from the fact that $(1,j)$ is good. Therefore the table is self-descriptive.

In conlusion: if n is even the answer is 1, if it is odd there are 2^{2n-2} different tables.



Second solution: As in the previous solution we only need to determine the parity of each element, therefore we will count modulo 2 in the proof. Let $t_{i,j}$ be the value corresponding to the intersection of row i and column j modulo 2. Let s_i and o_i respectively be the sum of elements in a row or column modulo 2. Then

$$t_{i,j} \equiv s_i + o_j \tag{1}$$

.

So if we know all s_i and o_i for every i, then we can detemine every value in the table. We can express s_i from $t_{i,j}$ modulo 2, as we counted $t_{i,j}$ if $t_{i,j} \equiv 0$. Formally:

$$s_i \equiv \sum_{j=1}^{n} (1 + t_{i,j})$$

We can write this using (1) as:

$$s_i \equiv \sum_{j=1}^{n} (1 + t_{i,j}) \equiv \sum_{j=1}^{n} (1 + s_i + o_j) \equiv n + ns_i + \sum_{j=1}^{n} o_j \tag{2}$$

Similarly for o_i:

$$o_i \equiv \sum_{j=1}^{n} (1 + t_{j,i}) \equiv \sum_{j=1}^{n} (1 + s_j + o_i) \equiv n + no_i + \sum_{j=1}^{n} s_j$$

If n is even then the last two equations:

$$s_i \equiv \sum_{j=1}^{n} o_j \equiv 0$$

$$o_i \equiv \sum_{j=1}^{n} s_j \equiv 0$$

As the sums on the right sides are independent from i, therefore all of s_i and o_j are equal. But in this case we added the same number to itself even times thus the value is 0. Therefore $t_{i,j} = 0$ for every i, j. If n is even then this is the only solution.

If n is odd then from equation (2):

$$1 \equiv -n + (1 - n)s_i \equiv \sum_{j=1}^{n} o_j \tag{3}$$

An alike:

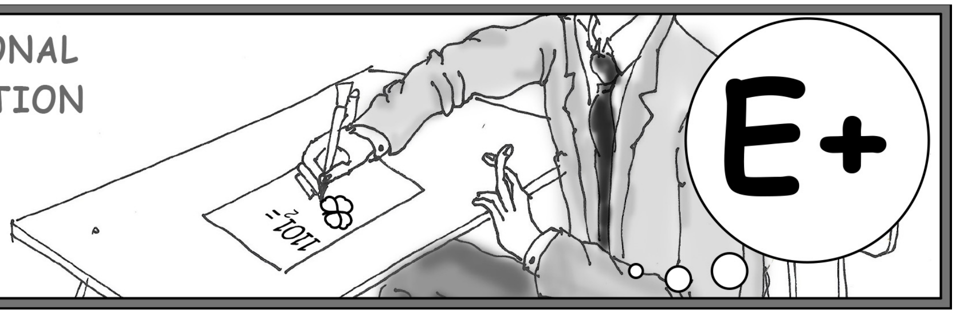
$$1 \equiv \sum_{j=1}^{n} s_j \tag{3}$$

Let us choose s_i and o_i for every $i < n$. We will prove that this determines one unique solution. s_n and o_n are determined by the congruences (3), and from that (1) determines every $t_{i,j}$. For every choice of o_i and s_j we get a different table, so we only need to prove that every such solution is self-descriptive. This is true as:

$$t_{k,l} \equiv s_k + o_l \equiv 1 - \sum_{\substack{i=1 \\ i \neq k}}^{n} s_i + 1 - \sum_{\substack{j=1 \\ j \neq l}}^{n} s_i + (n-1)(s_k + o_l) \equiv \sum_{\substack{i=1 \\ i \neq k}}^{n} (s_i + o_l) + \sum_{\substack{j=1 \\ j \neq l}}^{n} (s_k + o_j) \equiv \sum_{\substack{i=1 \\ i \neq k}}^{n} t_{i,l} + \sum_{\substack{j=1 \\ j \neq l}}^{n} t_{k,j}$$

Which is what we wanted to prove.

In conlusion: if n is even the answer is 1, if it is odd there are 2^{2n-2} different tables.



E+3. At least how many non-zero real numbers do we have to select such that every one of them can be written as a sum of 2019 other selected numbers and
a) the selected numbers are not necessarily different?
b) the selected numbers are pairwise different?

Solution: Instead of solving the problem for 2019, we generalise it and solve for n, where $n > 3$ odd integer. For even n and $n = 3$ we can get a solution using the same ideas.

a) Answer: $n + 3$.

It is clear that we need at least $n + 1$ numbers. First we prove that $n + 1$ and $n + 2$ is not enough, then we show that there is a construction for $n + 3$.

Suppose that we have $n+1$ numbers that satisfy the condition and let them be $a_1 \le a_2 \le ... \le a_{n+1}$. Then $a_1 = a_2 + a_3 + ... + a_{n+1}$ and $a_{n+1} = a_1 + a_2 + ... + a_n$. Subtracting the second equality from the first one: $a_1 - a_{n+1} = a_{n+1} - a_1$, so $a_1 = a_{n+1}$, which means that $a_i = a_j$ for all i, j. Substituting back to the first equation, $a_1 = a_2 + a_3 + ... + a_{n+1} = n \cdot a_1$, which is not possible as $n > 1$ and $a_1 \ne 0$. Suppose that we have $n + 2$ numbers that satisfy the condition and let them be $a_1 \le a_2 \le ... \le a_{n+2}$. Then a_1 is equal to a sum of n other a_i, so $a_1 \ge a_2 + a_3 + ... + a_{n+1}$. Similarly, $a_{n+2} \le a_2 + a_3 + ... + a_{n+1}$, which means that $a_1 \ge a_{n+2}$, so $a_1 = a_{n+2}$. As seen above, it is not possible, so $n + 2$ numbers are not enough.

It is enough to write down $n + 3$ numbers: Write down $\frac{n+3}{2}$ (-1)-s and $\frac{n+3}{2}$ 1-s. Then every (-1) can be written as a sum of $\frac{n+1}{2}$ (-1)-s and $\frac{n-1}{2}$ 1-s, and similarly every 1 is the sum of $\frac{n+1}{2}$ 1-s and $\frac{n-1}{2}$ (-1)-s. So this is a good construction for $n + 3$ numbers.

b) Answer: $n + 4$.

Because of the above, we need at least $n + 3$ numbers.

First we prove that $n + 3$ numbers are not enough by contradiction: Suppose that we have $n + 3$ numbers satisfying the condition, and let the numbers be $a_1 < a_2 < ... < a_{n+3}$. Then a_1 can be written as a sum of n other numbers, so $a_1 \ge a_2 + a_3 + ... + a_{n+1}$. Similarly $a_{n+3} \le a_3 + a_4 + ... + a_{n+2}$. Subtracting the first inequality from the second one $a_{n+3} - a_1 \le a_{n+2} - a_2$, but this is not possible, since $a_1 < a_2$ and $a_{n+2} < a_{n+3}$.

Now we show that there is a solution with $n+4$ numbers. Write n as $2k+1$, where $k > 1$ is an integer. Then our construction for $n + 4 = 2k + 5$ numbers is: $-(k + 3),\ -(k + 2),\ \dots,\ -2,\ 1,\ 2,\ \dots,\ (k + 3)$.

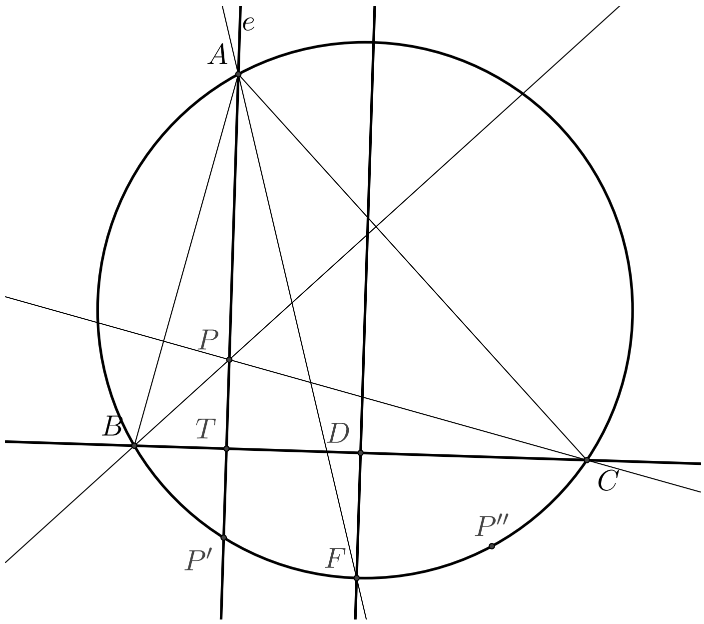
The sum of these numbers is 1. When we want to write up l as a sum of $n = 2k + 1$ other numbers, then the 3 numbers that are not in the sum and are not l, sum to $1 - 2l$. The converse is also true, because if we find 3 numbers not equal to l, which sum to $1 - 2l$, then the remaining $2k + 1$ numbers sum to l. Because of this, it is enough to show that the following is true for our numbers: for any l from the numbers there are 3 other numbers, which sum to $1 - 2l$.

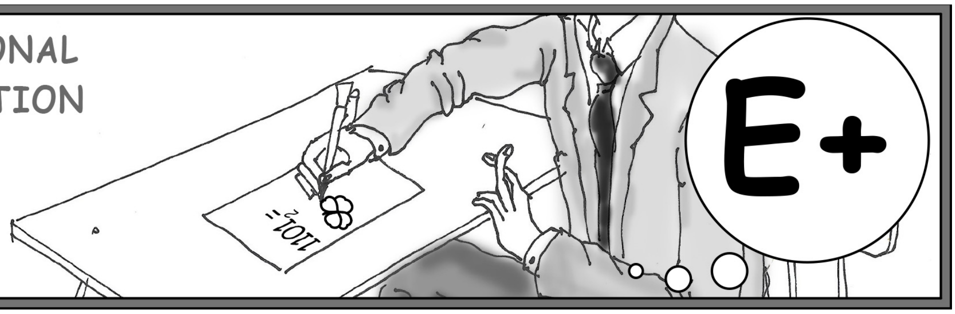
For $l = -k - 3$: $2,\ (k + 2),\ (k + 3)$ sum to $1 - 2l$.
$3 \le -l \le (k + 2)$: $1,\ (-l - 1),\ (-l + 1)$ sum to $1 - 2l$.
$l = -2$: $-3, 3, 5$ sum to $1 - 2l$ (only if $5 \le (k + 3)$, so $k \ge 2$).
$l = 1$: $4, -2, -3$ sum to $1 - 2l$.
$l = 2$: $3, -2, -4$ sum to $1 - 2l$.
$3 \le l \le (k + 2)$: $1,\ (-l - 1),\ (-l + 1)$ sum to $1 - 2l$.
$l = k + 3$: $-2,\ -k - 1,\ -k - 2$ sum to $1 - 2l$ (only if $k \ge 2$, so $-k - 1 \le -2$).
So our construction satisfies the condition.

E+4. Suppose that you are given the foot of the altitude from vertex A of a scalene triangle ABC, the midpoint of the arc with endpoints B and C, not containing A of the circumscribed circle of ABC, and also a third point P. Construct the triangle from these three points if P is the
a) orthocenter
b) centroid
c) incenter
of the triangle.

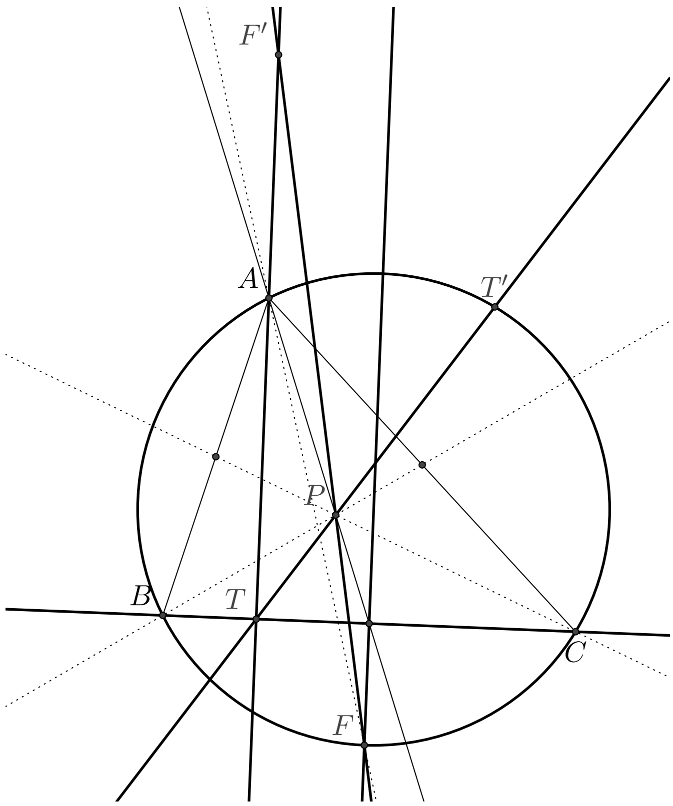
Solution: Throughout the solution let T denote the given foot of the altitude and let F denote the given midpoint of the arc.

a) First construct line PT, let this be e. Let the intersection of the line through F parallel with e and the line through T perpendicular to e be D. Then point D is the midpoint of side BC. Reflect P by T and D, let these points be P' and P'' respectively. It is known that P' and P'' lie on the circumscribed circle of ABC, this can be verified by some angle chasing. P', F and P'' are pairwise different since ABC is scalene, so by constructing the circle $P'FP''$ we get the circumscribed circle of ABC, intersecting it with lines PT and TD we get points A, P', B and C.



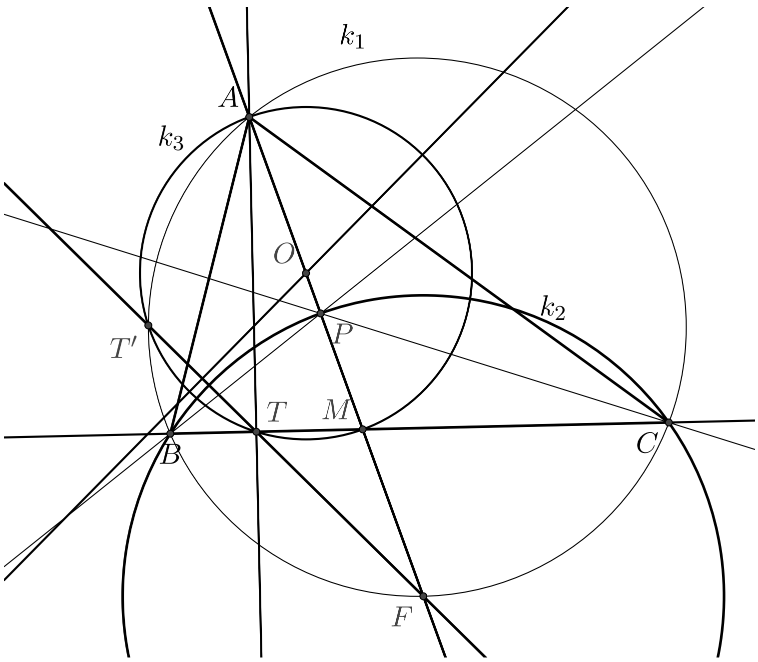


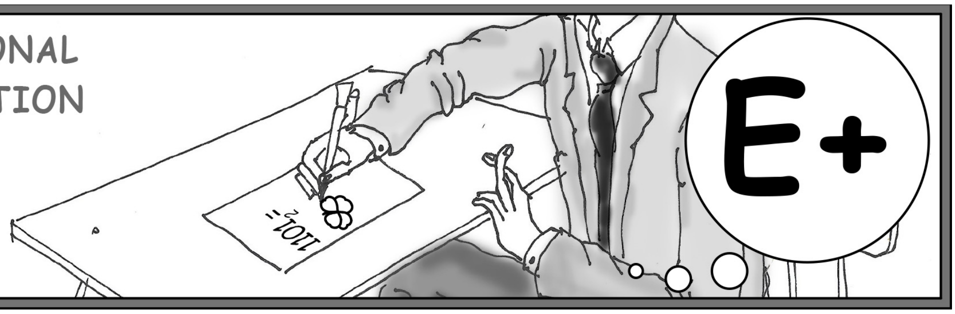
b) We will use the following fact multiple times: the centroid divides the medians in the ratio of $2:1$. It follows from this fact that point T', which we get by enlarging point T from P with scale factor -2, lies on the circumcircle of ABC and coincides with the the reflection of point A by the perpendicular bisector of side BC. It also follows that F', which we get by enlarging point F from point P with scale factor -2, lies on line AT. Now let us start the construction. We can construct points F' and T'. Consider the line that is the enlargement of line TF' from P with scale factor $-1/2$. This will be the perpendicular bisector of side BC; by reflecting T' by this line, we get A. Since triangle ABC is scalene, this means that $A \neq T'$ so we can construct the circumscribed circle of ABC, knowing its three different points. By intersecting this circle with the line perpendicular to TF' through T, we get B and C.



c) A few remarks. Let the circumcircle of triangle ABC be k_1. It is known that points B, P and C lie on a circle with center F let this be k_2 (this can be verified by angle chasing). Let the intersection of lines AF and BC be M. Let the circle with diameter AM be k_3, denote its center with O. Since T is the foot of the altitude through A, triangle ATM is right, this means that T lies on k_3. Let T' denote the second intersection of line FT with k_3. Since the inversion with respect to k_2 maps k_1 to line BC, so the image of pont M is point A, so the image of k_3 is itself, it means that the image of T is T'.

The construction: (we are using the same notations as in the remarks). First construct k_2 (knowing its center and one point on the perimeter). Construct T', the image of T in the inversion with respect to k_2. Now the intersection of line FP and the perpendicular bisector of TT' is O, so we can construct k_3. The intersection of k_3 and line FP farther from F will be A. The intersections of k_2 and the line through T perpendicular to AT will be points B and C.





E+5. Let p be prime and $k > 1$ be a divisor of $p-1$. Show that if a polynomial of degree k with integer coefficients attains every possible value modulo p (that is, $0, 1, \ldots, p-1$) at integer inputs then its leading coefficient must be divisible by p.
Note: the leading coefficient of a polynomial of degree d is the coefficient of the x^d term.

Solution: Throughout the proof we will compute modulo p, every congruence is modulo p.
Let $Q(x)$ be the polynomial. Assume that $k \mid p-1$, $k > 1$ and Q attains every possible value modulo p. Since $Q(x) \equiv Q(x+p)$, the polynomial attains different values at $x = 0, 1, 2, \ldots, p-1$.
Lemma:

$$\sum_{j=0}^{p-1} j^t \equiv \begin{cases} -1 & \text{if } t = p-1 \\ 0 & \text{if } t = 0, 1 \ldots p-2 \end{cases}$$

Proof: It is known that there exists a primitive root g modulo p. If $t \neq p-1$, by the sum of the geometric series:

$$\sum_{j=0}^{p-1} j^t \equiv \sum_{j=0}^{p-2} g^{jt} \equiv \frac{g^{(p-1)t} - 1}{g^t - 1} \equiv 0$$

If $t = p-1$ then $x^t \equiv 1$ when $x \not\equiv 0$, by Fermat's little theorem. So

$$\sum_{j=0}^{p-1} x^{p-1} \equiv p - 1 \equiv -1.$$

With this we have proven the lemma.

Now suppose indirectly that the leading coefficient of Q is not divisible by p. Let $c = \frac{p-1}{k}$ and let $R(x) = Q^c(x) = \sum_{i=0}^{p-1} b_i x^i$. Firstly we can see that in R the coefficient b_{p-1} is not 0 modulo p since the leading coefficient of Q is nonzero and $b_{p-1} = a_k^c$.

Let us consider the following sum:

$$\sum_{j=0}^{p-1} R(j)$$

Since $k > 1$ and k is a divisor of $p-1$, c is a positive integer smaller than $p-1$. By our hypothesis $Q(x)$ attains every value modulo p, in other words $Q(0), Q(1), \ldots Q(p-1)$ is a permutation of $0, 1 \ldots p-1$, so the sum can we written the following way:

$$\sum_{j=0}^{p-1} R(j) = \sum_{j=0}^{p-1} Q^c(j) \equiv \sum_{j=0}^{p-1} j^c \equiv 0$$

The last congruence follows from our lemma. And writing the sum in a different way:

$$\sum_{j=0}^{p-1} R(j) = \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} b_i j^i = \sum_{i=0}^{p-1} b_i \sum_{j=0}^{p-1} j^i \equiv -b_{p-1}$$

Now we used the lemma again. So $0 \equiv -b_{p-1}$, which is a contradiction since we have shown already that b_{p-1} is nonzero.

So the leading coefficient of Q must be divisible by p.