



1. In maths class Albrecht had to compute $(a + 2b - 3)^2$. His result was $a^2 + 4b^2 - 9$. 'This is not correct' said his teacher, 'try substituting positive integers for a and b .' Albrecht did so, but his result proved to be correct. What numbers could he substitute?

a) Show a good substitution.

b) Give all the pairs that Albrecht could substitute and prove that there are no more.

2. Initially we have a 2×2 table with at least one grain of wheat on each cell. In each step we may perform one of the following two kinds of moves:

(i) If there is at least one grain on every cell of a row, we can take away one grain from each cell in that row.

(ii) We can double the number of grains on each cell of an arbitrary column.

a) Show that it is possible to reach the empty table using the above moves, starting from the position on the right.

3	5
5	3

b) Show that it is possible to reach the empty table from any starting position.

c) Prove that the same is true for the 8×8 tables as well.

3. a) Is it possible that the sum of all the positive divisors of two different natural numbers are equal?

b) Show that if the product of all the positive divisors of two natural numbers are equal, then the two numbers must be equal.

4. Let ABC be an acute triangle with side AB of length 1. Say we reflect the points A and B across the midpoints of BC and AC , respectively to obtain the points A' and B' . Assume that the orthocenters of triangles ABC , $A'BC$ and $B'AC$ form an equilateral triangle.

a) Prove that triangle ABC is isosceles.

b) What is the length of the altitude of ABC through C ?

5. We call a table of size $n \times n$ *self-describing* if each cell of the table contains the total number of even numbers in its row and column other than itself. How many *self-describing* tables of size

a) 3×3 exist?

b) 4×4 exist?

c) 5×5 exist?

Two tables are different if they differ in at least one cell.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.

Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XIII.Dürer Competition