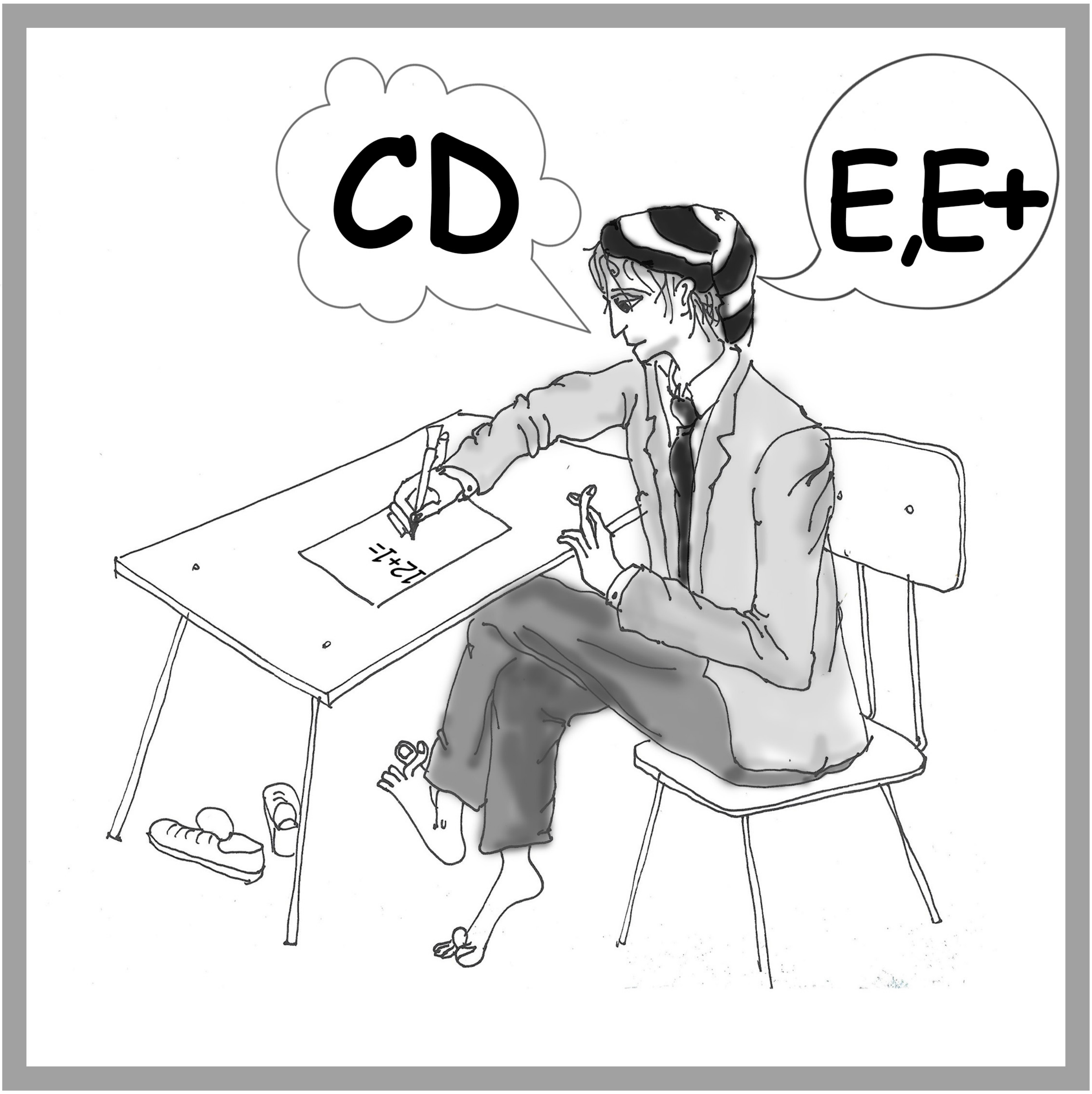
PROBLEMS, SOLUTIONS

DÜRER
2020
COMPETITION



Contents

Introduction – About the Dürer Competition

Plenty of mathematics contests are traditionally held in Hungary. From primary schoolers to university students, everybody can find a contest that fits their age and qualifications. These are mostly individual contests where the participants sit down in a room for a few hours, working on the problems quietly. However at the Dürer Competition, there are teams of 3 taking part. For the duration of the contest each team works together to solve the problems, so the contestants can experience the benefits of cooperative thinking. Our experience shows that the majority of students are happier and more relaxed than during an individual contest.

It is a very important goal for us to set interesting problems to show the beauty of mathematics and the joy of thinking to lots of students. We also wish to include as many original problems as possible. In each year, about 150 problems appear on the contest – of course not all can be original, but we invent most of the harder problems on our own.

It was the 13th competition, and the theme were superstitions. So we tried to omit the number 13 from everywhere including the logo.

At this point we definitely have to mention that the organising team traditionally consists of young people, mostly university students studying maths. This dates back to the early years of the competition, and ever since then, we can regularly welcome former competitors as new organisers. The success of the competition depends on this community, consisting of 30 to 70 people. Some of them have been organisers for 10 years already (and still take part enthusiastically, even alongside a full-time job) and some of them take important responsibilities as first-year undergraduates already.

This is the spirit in which we have been organising the contest for 12 years. The competition attracts more and more students and schools with each year. In the 2019–20 academic year, more than 700 Hungarian students competed in the high school maths categories.

This was the first year, we opened our two hardest categories for international competitors. In 7 locations (Tehran, Vienna, Innsbruck, Graz, Constanta, Maribor, Aachen) approximately 70 students took part in the regional round of the competition.

Primary school students can take part in our competition in the following two categories: 5^{th} and 6^{th} grade students compete in *category A* while 7^{th} and 8^{th} grade students compete in *category B*. The contest is regional: it is organised in 6 cities in northeastern Hungary, but is open to anyone provided that they travel to one of the locations. (The problems of these two categories are not included in this booklet.)

Four categories are available for high school students:

- **Category C** is open to 9^{th} and 10^{th} graders who have never previously qualified for the final of any national math contest.
- **Category D** is open to 9^{th} to 12^{th} graders who are a bit more experienced, but do not come from a school that is outstanding in handling mathematical talents.

- **Category E** is open to 9^{th} to 12^{th} graders who already have good results from other contests, or come from a school outstanding in maths.

- **Category E$^+$** is designed for competitors who actively take part in olympiad training. In this category, most teams include some student who has taken part at an international olympiad (IMO, MEMO, EGMO, RMM, CMC), or is about to qualify for one in the same academic year.

We also organise the contest in physics (*category F*) and chemistry (*categories K* and *K*$^+$), but these are also omitted from this booklet.

For high schoolers (in categories C, D and E), the first round is an **online relay round** consisting of 9 problems. The answer to each question is an integer between 0 and 9999. Initially each team gets the 1^{st} question only. They have three attempts to submit an answer - if they get it right, they score a set number of points, and can proceed to the next question. Each wrong attempt to a question reduces the possible score by 1, and after 3 wrong attempts the team must move on to the next question without scoring.

The second round is a traditional **olympiad-style contest**, where detailed proofs have to be given. The teams have 3 hours to solve 5 problems. The contest can be sat in the whole country, at about 20 locations.

The final round takes place in Miskolc every year. For high schoolers (categories C, D, E, E$^+$, F, K, K$^+$) we organise it on a weekend in early February from Thursday to Sunday. The first competition day is Friday, with the students working on five **olympiad-style problems** and a **game**. If a team thinks that they have found the winning strategy for the game, they can challenge us. If they can defeat us twice in a row, they get the maximal score for the problem. If they lose, they can still challenge us two more times for a partial score. On Saturday we hold a **relay round** consisting of 16 questions. The rules are similar to the online round. Rankings are based on a combined score from the two competition days.

At the weekend of the final, the students and teachers can participate in many educational and recreational activities, such as lectures, games and discussions about universities.

The competition is expanded year after year. In 2020–21 we plan to add a game to the online round, and reach more international competitors.

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1 Problems

1.1 Online round

1.1.1 Category C

1. On a chessboard we wrote a 2 on every white square and a 3 on every black square. What is the sum of all numbers on the chessboard?

2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2

(Solution)

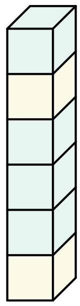
2. Aunt Theresa has 6 goats, 14 chickens and some cats in her yard. How many cats does she have if we know that the animals in her yard have a total of 100 legs?

(Solution)

3. Dürer drew a pentagon on one of his paintings. He took the first angle to be $90°$, then kept choosing larger and larger angles: the next ones were of sizes $100°$, $110°$ and $120°$. Find the fifth angle of the pentagon (in degrees).

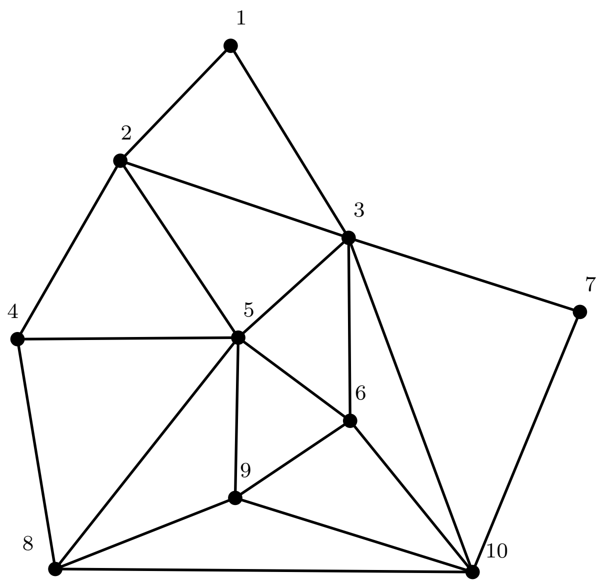
(Solution)

4. In the EKIA furniture store, towers of stacked cubes are used for decoration. Each tower consists of 4 green and 2 yellow cubes. How many ways are there to build such a tower?



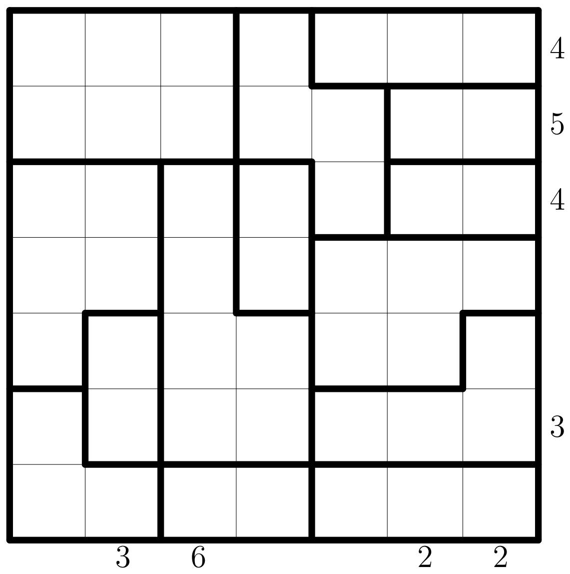
(Solution)

5. In the picture we can see the map of the big dark forest. Johnny started walking from the clearing numbered 10, and walked along each path precisely once. In which clearing did Johnny finish his walk?
(On the map, clearings are denoted by points, and paths by line segments.)



(Solution)

6. Each of the 49 small squares in the figure contains a light. The thick lines subdivide the lights into several groups. Within each group, either all lights are on or all are off. At the end of several rows and columns, the total number of lights turned on in that row/column is given. In total how many lights are on?



(Solution)

7. As it is well-known, Süsü, the dragon has one head. It is less known that his brothers have 3 heads each, and his father has 7 heads while his mother has 19. The average number of heads in the family is 6. However if we take the average without Süsü, we get 7 instead. How many brothers does Süsü have?

(Solution)

8. In a short story, each sentence ends with either a period, a question mark or an exclamation mark. Inside a sentence, there can be nothing but a comma.
Each sentence of the story contains one comma, and the first and last sentences both end with a question mark.
Between any two consecutive question marks there are precisely two periods, and the story contains a total of one exclamation mark.
If there are 6 question marks in the story, how many punctuation marks does it contain in total?
(Only commas, periods, question marks and exclamation marks count as punctuation marks.)

(Solution)

9. Ákos wants to have lunch at a restaurant. Today's menu is the following (where meat-free dishes are denoted by a "v"):

- Soups: fruit soup (v), tomato soup (v), beef soup.
- Main courses: beef stew, pork knuckles, breaded cheese (v), breaded meat, poppy-seed pasta (v).
- Desserts: ice cream sundae (v), fruit cake (v).

He wants to eat a soup, a main course and a dessert, but he does not believe in going vegetarian, so he definitely wants to eat some meat in at least one of the dishes. How many possible combinations of dishes can he order?

(Solution)

1.1.2 Category D

1. On a chessboard we wrote a 4 on every white square and a 6 on every black square. What is the sum of all numbers on the chessboard?

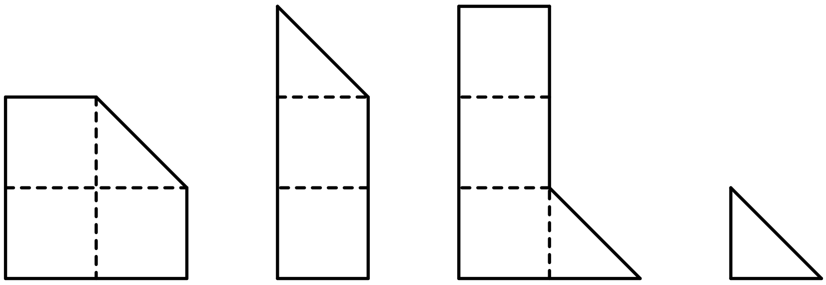
4	6	4	6	4	6	4	6
6	4	6	4	6	4	6	4
4	6	4	6	4	6	4	6
6	4	6	4	6	4	6	4
4	6	4	6	4	6	4	6
6	4	6	4	6	4	6	4
4	6	4	6	4	6	4	6
6	4	6	4	6	4	6	4

(Solution)

2. Even though Winnie the Pooh did not find Piglet at home, luckily he discovered Piglet's honey pots. He found pots of two sizes: 3 and 4 deciliters, and all of them were full of honey. Winnie then ate up all 13 deciliters of honey that he found. How many pots did he find?

(Solution)

3. We have cut out a rectangle from a sheet of grid paper, and then cut it into the four pieces shown in the figure. What was the perimeter of the rectangle in cm, if each small square of the grid paper has side length 1 cm?



(Solution)

4. In the picture we can see the map of the big dark forest. Johnny started walking from the clearing numbered 10, and walked along each path precisely once. In which clearing did Johnny finish his walk?
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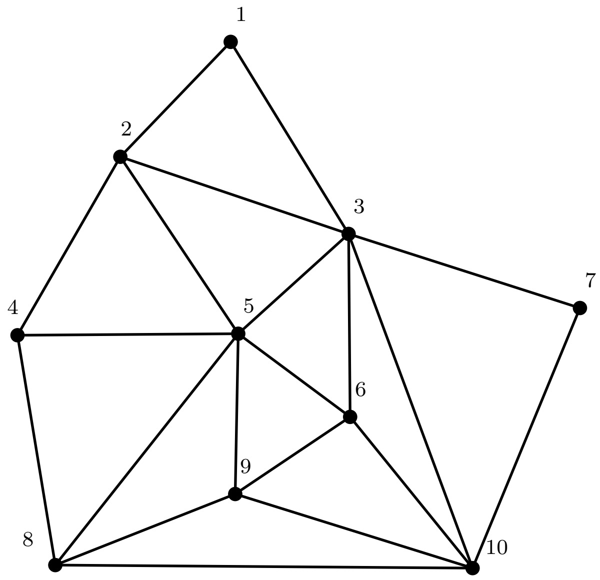


(Solution)

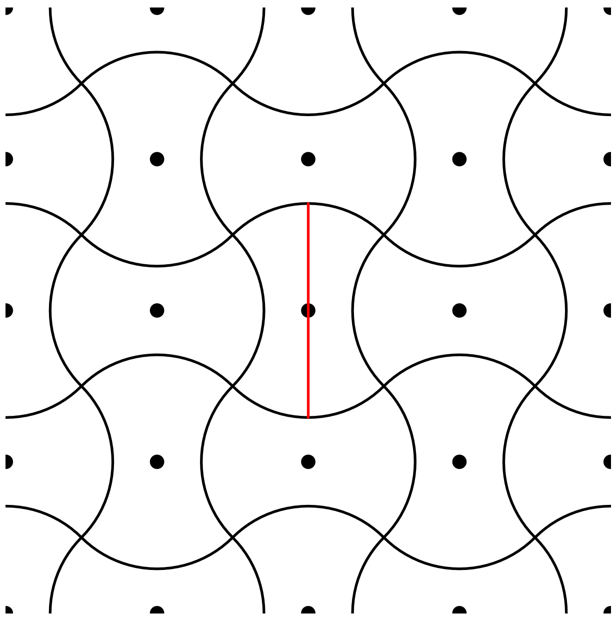
5. Ákos, Kartal and Kristóf went to the AKIE. As they got tired shopping, they decided to dine in the store's restaurant. Kristóf ordered 5 hotdogs and 8 meatballs, which cost him 1320 forints, while Kartal had 3 hotdogs and 7 meatballs for 990 forints. How many forints did Ákos have to pay, if he ordered 7 hotdogs and 9 meatballs?

(Solution)

6. In the sudden October heat wave, Dani wants to have an ice cream cone of 3 scoops. Ice cream is available in three flavours: vanilla, chocolate and lemon; and scoops are stacked on top of each other.
Dani does not like if chocolate is stacked directly on top of lemon. In how many different ways can he order his cone?
(Two cones are considered different even if they only differ in the order of scoops. It is possible for Dani to choose multiple scoops of the same flavour.)

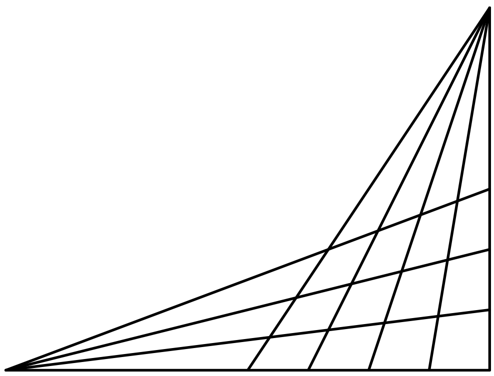
(Solution)

7. Each tile in this figure is bounded by arcs which are all quarter-circles, and the centre of each such quarter-circle is one of the marked points. Find the area of a tile in cm^2, if it is known that the height of a vertical tile (that is, the length of the segment marked in red) is 12 cm.



(Solution)

8. How many **convex** quadrilaterals are determined by the rays shown in the figure?
(Do not forget that we only consider convex quadrilaterals.)



(Solution)

9. In the following equation, identical letters represent identical digits and distinct letters represent distinct ones:

$$\overline{ERŐS} + \overline{ERDŐ} = \overline{DÜRER}$$

What four-digit number does the word $\overline{ÜRES}$ represent?

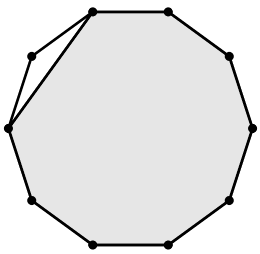
(Solution)

1.1.3 Category E

1. What is the sum of all two-digit positive integers divisible by 11?

(Solution)

2. One of the vertices of a regular decagon has been "cut off" (as can be seen in the figure). Find the smallest internal angle of the nonagon so obtained (in degrees).



(Solution)

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(Solution)

4. Ákos wants to have lunch at a restaurant. Today's menu is the following (where meat-free dishes are denoted by a "v"):

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He wants to eat a soup, a main course and a dessert, but he does not believe in going vegetarian, so he definitely wants to eat some meat in at least one of the dishes. How many possible combinations of dishes can he order?

(Solution)

5. How many **convex** quadrilaterals are determined by the rays shown in the figure?
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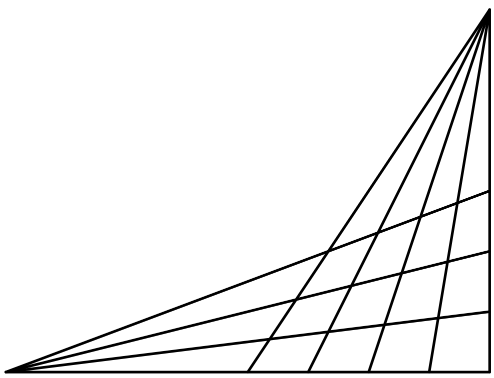
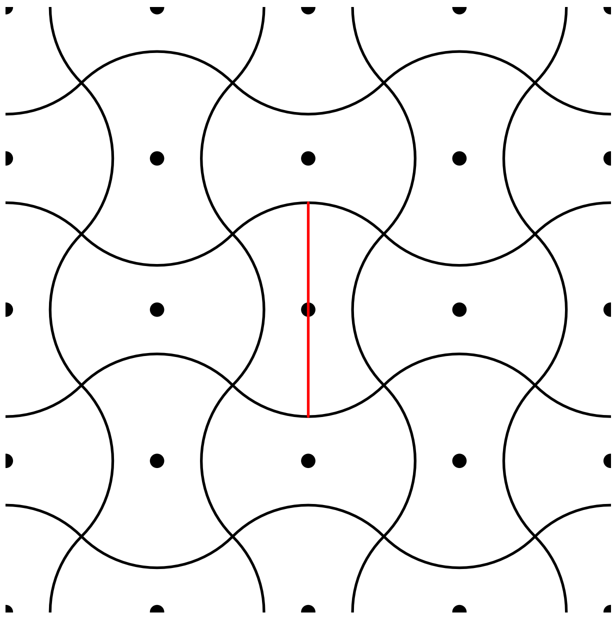


(Solution)

6. In the family of Süsü, the dragon, the average number of heads is 8. Since Süsü only has one head, his family kicked him out. This made the average number of heads in the family increase to 9. How many members remained in the family after Süsü has been kicked out?

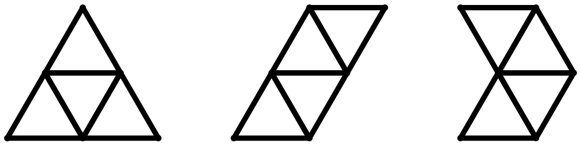
(Solution)

7. Each tile in this figure is bounded by arcs which are all quarter-circles, and the centre of each such quarter-circle is one of the marked points. Find the area of a tile in cm^2, if it is known that the height of a vertical tile (that is, the length of the segment marked in red) is 12 cm.



(Solution)

8. We create some tiles by joining together several equilateral triangles. By joining 4 triangles it is possible to make three kinds of tiles, as can be seen in the figure. How many different kinds of tiles can we make by joining 5 triangles? *(If two tiles are just rotations or reflections of each other, they are considered the same.)*



(Solution)

9. Kartal, Bálint, Gábor and Timi are playing with cards; each card contains a digit. They have placed 4 cards on the table (not necessarily distinct ones). Everyone thinks of a four-digit number that can be assembled from these four cards, and writes it down on a piece of paper. When they show each other their numbers, they observe that the sum of the three numbers of Kartal, Bálint and Gábor is equal to the number written by Timi. What is Timi's number, if it is known to be between 4210 and 4567?

(Solution)

1.2 Regional round

1.2.1 Category C

1. Dávid and Ákos are having the following conversation.

– Ákos: Last year I had three times as many points on the Dürer competition as you.
– Dávid: That is true, but I still scored more points than the number of Hungarian counties whose seats you can name.
– Ákos: Sure, but I know the seats of more than three times as many counties as you.
– Dávid: Yeah, but I know the seats of more counties than the number of goals you scored in our last PE lesson.
– Ákos: I still scored more than four times as many goals as you.
– Dávid: That is also true, but I also scored at least one goal in that PE lesson.

Find out how many counties' seats can Ákos name, if we know that the maximum score on last year's Dürer competition was 60 points. Justify your answer.

(Solution)

2. A natural number is called *unlucky* if it consists of different digits and the sum of its digits is 13.
a) How many 5-digit unlucky numbers are there?
b) How many of these are even?
c) How many of them are divisible by 3?

(Solution)

3. Albrecht enjoys constructing polygons in his notebook. He constructs some hexagons such that each of their interior angles have size 120° and each of their sides have integer length in centimetres. He labels each hexagon with the number of different side lengths it has. How many different labels can he obtain this way?
Give examples for as many possible values as you can.

(Solution)

4. In a class of 26 students there is one student on duty each week. They choose this student according to the following procedure: they look up the numbers of the previous two students on duty in the register book (alphabetical order) and add them up. If the result is at most 26, the student with that number will be on duty, otherwise they choose the student with number 26 less than the sum. *E.g. if the sum is* 28*, they choose the student number* 2.
a) Is it possible that Smith, Miller and Lenger will be on duty on three consecutive weeks, in this order?
b) Is it possible that Lenger, Smith and Miller will be on duty on three consecutive weeks, in this order?
If an order is possible, give the number of each of the three students in the register book. If not, justify why that order is not possible.

(Solution)

5. Initially we have a 2×2 table with at least one grain of wheat on each cell. In each step we may perform one of the following two kinds of moves:

(i) If there is at least one grain on every cell of a row, we can take away one grain from each cell in that row.

(ii) We can double the number of grains on each cell of an arbitrary column.

a) Show that it is possible to reach the empty table using the above moves, starting from the position on the right.

3	5
5	3

b) Show that it is possible to reach the empty table from any starting position.
c) Prove that the same is true for the 8×8 tables as well.

(Solution)

1.2.2 Category D

1. A natural number is called *absolutely unlucky* if each of its digits is 1 or 3 and the sum of its digits is 13. How many absolutely unlucky numbers are there in total?

(Solution)

2. On the Island of Oxys there live some rabbits and some foxes. In spring the number of rabbits doubles, in autumn each fox eats one rabbit and then the number of foxes also doubles. If a fox cannot get enough food (i.e. there are not enough rabbits on the island), it passes away. Now it is winter and there are 24 rabbits and 2 foxes on the island. What will be the number of rabbits on the island
a) 2 years from now?
b) 10 years from now?
c) Show that after some time all animals on the island will go extinct and find the time it will happen.

(Solution)

3. Find all solutions to the equation

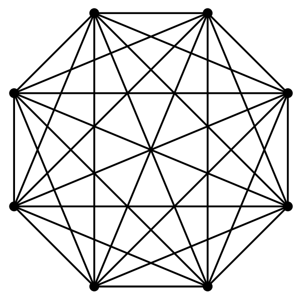
$$p^q + q^p = r$$

where p, q, r are positive prime numbers.

(Solution)

4. The figure shows the sides and diagonals of a regular octagon. We measured all the arising angles. What values did we get?

Remark: Two intersecting line segments form an angle between $0°$ *and* $180°$. *We consider line segments meeting at a vertex of the octagon to be intersecting.*



(Solution)

5. **a)** Prove that there exist 6 integers such that taking their pairwise sums we get different numbers, and writing down these sums in increasing order the difference of any two consecutive numbers is at most 2.
b) Prove that there are no 7 numbers with the above property.

(Solution)

1.2.3 Category E

1. In maths class Albrecht had to compute $(a + 2b - 3)^2$. His result was $a^2 + 4b^2 - 9$. 'This is not correct' said his teacher, 'try substituting positive integers for a and b.' Albrecht did so, but his result proved to be correct. What numbers could he substitute?
a) Show a good substitution.
b) Give all the pairs that Albrecht could substitute and prove that there are no more.

(Solution)

2. Initially we have a 2×2 table with at least one grain of wheat on each cell. In each step we may perform one of the following two kinds of moves:
(i) If there is at least one grain on every cell of a row, we can take away one grain from each cell in that row.
(ii) We can double the number of grains on each cell of an arbitrary column.
a) Show that it is possible to reach the empty table using the above moves, starting from the position on the right.

3	5
5	3

b) Show that it is possible to reach the empty table from any starting position.
c) Prove that the same is true for the 8×8 tables as well.

(Solution)

3. **a)** Is it possible that the sum of all the positive divisors of two different natural numbers are equal?
b) Show that if the product of all the positive divisors of two natural numbers are equal, then the two numbers must be equal.

(Solution)

4. Let ABC be an acute triangle with side AB of length 1. Say we reflect the points A and B across the midpoints of BC and AC, respectively to obtain the points A' and B'. Assume that the orthocenters of triangles ABC, $A'BC$ and $B'AC$ form an equilateral triangle.
a) Prove that triangle ABC is isosceles.
b) What is the length of the altitude of ABC through C?

(Solution)

5. We call a table of size $n \times n$ *self-describing* if each cell of the table contains the total number of even numbers in its row and column other than itself. How many *self-describing* tables of size
a) 3×3 exist?
b) 4×4 exist?
c) 5×5 exist?
Two tables are different if they differ in at least one cell.

(Solution)

1.2.4 Category E$^+$

1. a) Is it possible that the sum of all the positive divisors of two different natural numbers are equal?
b) Is it possible that the product of all the positive divisors of two different natural numbers are equal?

(Solution)

2. How many ways can you fill a table of size $n \times n$ with integers such that each cell contains the total number of even numbers in its row and column other than itself?
Two tables are different if they differ in at least one cell.

(Solution)

3. At least how many non-zero real numbers do we have to select such that every one of them can be written as a sum of 2019 other selected numbers and
a) the selected numbers are not necessarily different?
b) the selected numbers are pairwise different?

(Solution)

4. Suppose that you are given the foot of the altitude from vertex A of a scalene triangle ABC, the midpoint of the arc with endpoints B and C, not containing A of the circumscribed circle of ABC, and also a third point P. Construct the triangle from these three points if P is the
a) orthocenter
b) centroid
c) incenter
of the triangle.

(Solution)

5. Let p be prime and $k > 1$ be a divisor of $p - 1$. Show that if a polynomial of degree k with integer coefficients attains every possible value modulo p (that is, $0, 1, \ldots, p-1$) at integer inputs then its leading coefficient must be divisible by p.
Note: the leading coefficient of a polynomial of degree d is the coefficient of the x^d *term.*

(Solution)

1.3 Final round – day 1

1.3.1 Category C

1. Timi, Luca, Lilla and Dani want to get from Miskolc to Szikszó, which is 20 kilometers away, on foot. At the same time, Zsófi and Gábor wish to complete the same route the other way around. Timi departs from Miskolc at 8.00, Luca at 9.00, Lilla at 10.00 and Dani at 11.00. From Szikszó Zsófi departs at 8.00, Gábor at 9.00. All of them walk at a speed of 5 km/h. Marvin, who is loved by everyone, is more than happy to accompany anyone for a while, but whenever they meet someone else, Marvin changes companions. If Marvin is in Miskolc right now, who could he set off with to end up in Szikszó?

(Solution)

2. Is it possible to place a queen, a rook, a bishop and a knight on the top left quarter of the chessboard in such a way that each piece could capture the same number of pieces, and each piece could be captured by **a)** 0, **b)** 1, **c)** 2 pieces?

(Solution)

3. We would like to write the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ into the following table in such a way that we use each of them exactly once.
a) Is it possible to do it so that the sum of numbers is divisible by 3 in each row, column and diagonal?
b) Is it possible to do it so that each sum is divisible by 5?
c) Is it possible to do it so that each sum is divisible by 7?
c) Is it possible to do it so that each sum is divisible by 9?

(Solution)

4. Albrecht likes to draw hexagons with all sides having equal length. He calls an angle of such a hexagon nice if it is exactly $120°$. He writes the number of its nice angles inside each hexagon. How many different numbers could Albrecht write inside the hexagons? Show examples for as many values as possible and give a reasoning why others cannot appear.
Albrecht can also draw concave hexagons.

(Solution)

5. We are given a map divided into 13×13 fields. It is also known that at one of the fields a tank of the enemy is stationed, which we must destroy. To achieve this we need to hit it twice with shots aimed at the centre of some field. When the tank gets hit it gets moved to a neighbouring field out of precaution. At least how many shots must we fire, so that the tank gets destroyed certainly?
We can neither see the tank, nor get any other feedback regarding its position.

(Solution)

6. Game: Two players play a game of ordinary tic-tac-toe on a 3×3 board with red and blue disks. That is, if there are three disks of the same colour in a row, column or diagonal, then the person placing that colour wins. In case no one wins after the placement of the first 9 disks, the subsequent player colours one of the opponent's already placed disks purple. Now whoever first creates three purple disks in a row, column or diagonal, wins.
Defeat the organisers in this game twice in a row! You can decide whether you want to go first or second.

(Solution)

1.3.2 Category D

1. We would like to write the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ into the following table in such a way that we use each of them exactly once.
a) Is it possible to do it so that the sum of numbers is divisible by 3 in each row, column and diagonal?
b) Is it possible to do it so that each sum is divisible by 5?
c) Is it possible to do it so that each sum is divisible by 7?
c) Is it possible to do it so that each sum is divisible by 9?

(Solution)

2. Let ABC be an acute triangle where $AC > BC$. Let T denote the foot of the altitude from vertex C, denote the circumcentre of the triangle by O. Show that quadrilaterals $ATOC$ and $BTOC$ have equal area.

(Solution)

3. An $n \times n$ table has one disk on each field of its lower-left 3×3 subtable. In one step we can reflect any disk to any other disk in its row or coloumn. Can we reach the state where the disks are on the fields of the upper-right 3×3 subtable, one disk on each field,
a) if $n = 9$?
b) if $n = 8$?

(Solution)

4. Albrecht likes to draw hexagons with all sides having equal length. He calls an angle of such a hexagon nice if it is exactly $120°$. He writes the number of its nice angles inside each hexagon. How many different numbers could Albrecht write inside the hexagons? Show examples for as many values as possible and give a reasoning why others cannot appear.
Albrecht can also draw concave hexagons.

(Solution)

5. Let n be a positive integer. Prove that $2^{2^n} + 2^{2^{n-1}} + 1$ has at least n different positive prime divisors.

(Solution)

6. **Game:** Károly and Dezső wish to count up to m and play the following game in the meantime: they start from 0 and the two players can add a positive number less than 13 to the previous number, taking turns. However because of their superstition, if one of them added x, then the other one in the next step cannot add $13-x$. Whoever reaches (or surpasses) m first, loses.
Defeat the organisers in this game twice in a row! A starting position will be given and then you can decide whether you want to go first or second.

(Solution)

1.3.3 Category E

1. Let ABC be an acute triangle where $AC > BC$. Let T denote the foot of the altitude from vertex C, denote the circumcentre of the triangle by O. Show that quadrilaterals $ATOC$ and $BTOC$ have equal area.

(Solution)

2. We are given a map divided into 13×13 fields. It is also known that at one of the fields a tank of the enemy is stationed, which we must destroy. To achieve this we need to hit it twice with shots aimed at the centre of some field. When the tank gets hit it gets moved to a neighbouring field out of precaution. At least how many shots must we fire, so that the tank gets destroyed certainly?
We can neither see the tank, nor get any other feedback regarding its position.

(Solution)

3. Is it possible for the least common multiple of five consecutive positive integers to be a perfect square?

(Solution)

4. Endre wrote n (not necessarily distinct) integers on a paper. Then for each of the 2^n subsets, Kelemen wrote their sum on the blackboard.
a) For which values of n is it possible that two different n-tuples give the same numbers on the blackboard? **b)** Prove that if Endre only wrote positive integers on the paper and Ferenc only sees the numbers on the blackboard, then he can determine which integers are on the paper.

(Solution)

5. Let $H = \{-2019, -2018, ..., -1, 0, 1, 2, ..., 2020\}$. Describe all functions $f : H \to H$ for which
a) $x = f(x) - f(f(x))$ holds for every $x \in H$.
b) $x = f(x) + f(f(x)) - f(f(f(x)))$ holds for every $x \in H$.

(Solution)

6. Game: Károly and Dezső wish to count up to m and play the following game in the meantime: they start from 0 and the two players can add a positive number less than 13 to the previous number, taking turns. However because of their superstition, if one of them added x, then the other one in the next step cannot add $13 - x$. Whoever reaches (or surpasses) m first, loses.
Defeat the organisers in this game twice in a row! A starting position will be given and then you can decide whether you want to go first or second.

(Solution)

1.3.4 Category E$^+$

1. Consider the sequence defined by the following recursion: let $a_1 = 2$ and for every $n > 1$ let $a_n = 2^{a_{n-1}}$. Now let b_n denote the remainder of a_n when divided by 2020. Prove that eventually b_n will be constant.

(Solution)

2. Let $H = \{-2019, -2018, ..., -1, 0, 1, 2, ..., 2020\}$. Describe all functions $f : H \to H$ for which
a) $x = f(x) + f(f(x)) - f(f(f(x)))$ holds for every $x \in H$.
b) $x = f(x) + 2f(f(x)) - 3f(f(f(x)))$ holds for every $x \in H$.

(Solution)

3. In the plane, construct as many lines in general position as possible, with any two of them intersecting in a point with integer coordinates.

(Solution)

4. Let ABC be a scalene triangle and its incentre I. Denote by F_A the intersection of the line BC and the perpendicular to the angle bisector at A through I. Let us define points F_B and F_C in a similar manner. Prove that points F_A, F_B and F_C are collinear.

(Solution)

5. Prove that the number of orientations of a connected 3-regular graph on $2n$ vertices where the number of vertices with indegree 0 and outdegree 0 are equal, is exactly $2^{n+1}\binom{2n}{n}$.

(Solution)

6. **Game:** At the beginning of the game the organisers place 4 piles of paper disks onto the table. The player who is in turn takes away a pile, then divides one of the remaining piles into two nonempty piles. Whoever is unable to move, loses.
Defeat the organisers in this game twice in a row! A starting position will be given and then you can decide whether you want to go first or second.

(Solution)

1.4 Final round – day 2

1.4.1 Category C

C-1. In a hotel, the rooms are numbered from 2 to 27 however, because of the well-known superstition, no room number can contain a digit that appears in the number 13. How many rooms are there in the hotel?

(3 points)

C-2. Picur wanted to make a strangely shaped piece of chocolate for her friend, Arthur Dumpling. She planned to make a big chocolate cube made out of 27 small cubes. However, when she started building the big cube, he noticed that she only has 26 small chocolate cubes, so she could not place a small cube in the middle of one of the faces of the big cube. Arthur still received this gift very happily, and decided that he will eat it for as many days as there are vertices, edges and faces of this strangely shaped gift in total. For how many days will he eat the gift?

(3 points)

C-3. Grandma has four grandsons: Andrew, Billy, Charles and Daniel. One afternoon, Grandma baked a cake for her grandsons. They sliced up the cake in the following way: first they cut it in half, then they cut all the resulting pieces in half, and they repeated this a few times. It is known that Andrew ate three slices. Charles ate twice as many slices as Billy. Daniel ate twice as many slices as Charles, and one slice remained for Grandma. How many slices did Charles eat, if they sliced up the cake into the smallest possible number of slices such that the conditions above are fulfilled, everybody ate at least one slice from the cake and together they ate up all of the cake?

(3 points)

C-4. Andrew has written down 4 positive integers whose sum and product are the same. What is the sum of squares of the 4 numbers?

(3 points)

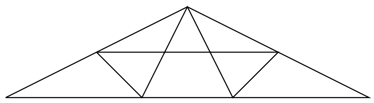
C-5. How many ways are there to tile a 3×3 square with 4 dominoes of size 1×2 and 1 domino of size 1×1?

Tilings that can be obtained from each other by rotating the square are considered different. Dominoes of the same size are completely identical.

(4 points)

C-6. How many triangles are there in the figure?

(4 points)



C-7. In a movie theatre there are 4 VIP chairs labelled from 1 to 4. We call a few consecutive vacant chairs a block. In the online VIP seat reservation process the guest can choose in which block she wants to sit, and that if she wants to choose the first, last or middle seat (in case of a block of size even this means the middle chair with the smaller number). At some screening all VIP seats were reserved. In how many ways the reservation order could have happened?

For instance, if the seat 2 *is reserved, then there are two blocks, one of them consists of the seat* 1*, the other one consists of the seats* 3 *and* 4*. Two reservation orders are different if there is a seat that was reserved in a different moment in the two orders.*

(4 points)

C-8. A pirate captain and his assistant have successfully stolen 96 gold coins. In the pirate laws it is written that the captain has to receive more coins than the assistant, but as the captain is selfless, he divides the coins such that the number of coins received by both of them have the same sum of digits. How many coins does the assistant get?

(4 points)

C-9. 4 horses ran a race . Before the race, everybody has made a guess on the finishing order of the 4 horses. There were no ties in the race, and at the end everybody got 2 dollars for each horse whose position they predicted correctly, and 1 dollar for each horse whose position they predicted wrongly by 1. In total there were 23 guessers. All of their guesses were different, and none of them predicted the full result correctly. How many dollars were paid to the guessers in total?

(5 points)

C-10. What is the largest possible k such that the product of all digits of a 4200-digit integer can be divisible by 12^k, where the number cannot contain a digit 0?

(5 points)

C-11. There are red and blue balls in a bag, 729 in total . In one round we do the following: we draw the balls from the bag three by three. For every three balls drawn, we place one of the balls belonging to the majority colour (out of the 3 drawn) down on the table and we throw away the other two balls. After the bag is emptied, we put all balls on the table back into the bag, and continue with the next round. After six rounds, a single ball remains on the table, which is red.

What is the maximal number of blue balls that might have been in the bag at the very beginning?

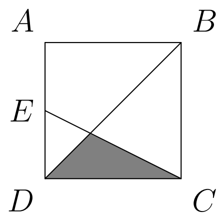
(5 points)

C-12. We have written 25! as a product of distinct integers. What is the maximal number of factors in the product?

(5 points)

C-13. $ABCD$ is a square of side length 108 cm , and E is the midpoint of side AD. Find the area of the grey region in square centimeters.

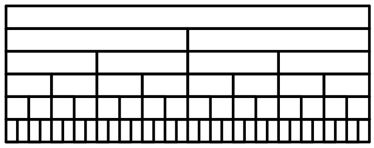
(6 points)



C-14. What is the smallest integer k greater than 1 such that $k \times 9997$ contains odd digits only?

(6 points)

C-15. Soma has a tower of 63 bricks , consisting of 6 levels. On the k-th level from the top, there are 2^{k-1} bricks (where $k = 1, 2, 3, 4, 5, 6$), and every brick which is not on the lowest level lies on precisely 2 smaller bricks (which lie one level below) - see the figure. Soma takes away 7 bricks from the tower, one by one. He can only remove a brick if there is no brick lying on it. In how many ways can he do this, if the order of removals is considered as well?



(6 points)

C-16. Positive integers a, b and c are all less than 2020. We know that a divides $b + c$, b divides $a + c$ and c divides $a + b$. How many such ordered triples (a, b, c) are there?

Note: In an ordered triple, the order of the numbers matters, so the ordered triple $(0, 1, 2)$ *is not the same as the ordered triple* $(2, 0, 1)$.

(6 points)

1.4.2 Category D

D-1. In a hotel, the rooms are numbered from 2 to 27 however, because of the well-known superstition, no room number can contain a digit that appears in the number 13. How many rooms are there in the hotel?

(3 points)

D-2. How many ways are there to tile a 3×3 square with 4 dominoes of size 1×2 and 1 domino of size 1×1?

Tilings that can be obtained from each other by rotating the square are considered different. Dominoes of the same size are completely identical.

(3 points)

D-3. What number should we put in place of the question mark such that the following statement becomes true?

$$11001_? = 54001_{10}$$

A number written in the subscript means which base the number is in.

(3 points)

D-4. When returning from her holidays, Picur brought a strangely shaped piece of chocolate for his friend, Arthur Dumpling, that he has never seen before: the chocolate was a cube of size 3 dm that contained two holes, one of them was a cube of size 1 dm in the middle of one of its faces of the big cube, the other one was at a corner of the opposite face. Arthur decided that he will eat it for as many days as there are vertices, edges and faces of this strangely shaped gift in total. For how many days will he eat the gift?

(3 points)

D-5. A pirate captain and his assistant have successfully stolen 96 gold coins. In the pirate laws it is written that the captain has to receive more coins than the assistant, but as the captain is selfless, he divides the coins such that the number of coins received by both of them have the same sum of digits. How many coins does the assistant get?

(4 points)

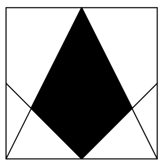
D-6. We have a positive integer n, whose sum of digits is 100 . If the sum of digits of $44n$ is 800 then what is the sum of digits of $3n$?

(4 points)

D-7. The hexagon $ABCDEF$ has all angles equal . We know that four consecutive sides of the hexagon have lengths 7, 6, 3 and 5 in this order. What is the sum of the lengths of the two remaining sides?

(4 points)

D-8. The square in the diagram has side length 78 . Some line segments have been drawn, with all endpoints being vertices or midpoints of sides of the square. Find the area of the black region.



(4 points)

D-9. We have written 25! as a product of distinct integers. What is the maximal number of factors in the product?

(5 points)

D-10. 6 horses ran a race . Before the race, everybody has made a guess on the finishing order of the 6 horses. There were no ties in the race, and at the end, everybody got 2 dollars for each horse whose position they predicted correctly, and 1 dollar for each horse whose position they predicted wrongly by 1. In total there were 719 guessers. All of their guesses were different, and none of them predicted the full result correctly. How many dollars were paid to the guessers in total?

(5 points)

D-11. What is the smallest integer k greater than 1 such that $k \times 9997$ contains odd digits only?

(5 points)

D-12. Santa Claus plays a guessing game with Marvin before giving him his present. He hides the present behind one of 100 doors, numbered from 1 to 100. Marvin can point at a door, and then Santa Claus will reply with one of the following words:

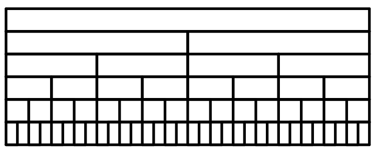
- "hot" if the present lies behind the guessed door,
- "warm" if the guess is not exact but the number of the guessed door differs from that of the present's door by at most 5,
- "cold" if the numbers of the two doors differ by more than 5.

At least how many such guesses does Marvin need, so that he can be certain about where his present is?

Marvin does not necessarily need to make a "hot" guess, just to know the correct door with 100% certainty.

(5 points)

D-13. Soma has a tower of 63 bricks , consisting of 6 levels. On the k-th level from the top, there are 2^{k-1} bricks (where $k = 1, 2, 3, 4, 5, 6$), and every brick which is not on the lowest level lies on precisely 2 smaller bricks (which lie one level below) - see the figure. Soma takes away 7 bricks from the tower, one by one. He can only remove a brick if there is no brick lying on it. In how many ways can he do this, if the order of removals is considered as well?



(6 points)

D-14. Positive integers a, b and c are all less than 2020. We know that a divides $b + c$, b divides $a + c$ and c divides $a + b$. How many such ordered triples (a, b, c) are there?

Note: In an ordered triple, the order of the numbers matters, so the ordered triple $(0, 1, 2)$ *is not the same as the ordered triple* $(2, 0, 1)$.

(6 points)

D-15. An integer n is called k-nice if it is divisible by k and the number of its positive divisors is k. Let S be the sum of all 20-nice positive integers which are not greater than 2020. What is $\frac{S}{20}$?

(6 points)

D-16. In a movie theatre there are 6 VIP chairs labelled from 1 to 6. We call a few consecutive vacant chairs a block. In the online VIP seat reservation process the reservation of a seat consists of two steps: in the first step we choose the block, in the second step we reserve the first, last or middle seat (in case of a block of size even this means the middle chair with the smaller number) of that block. (In the second step the online system offers the three possibilities even though they might mean the same seat.) Benedek reserved all seats at some screeining. In how many ways could he do it if we distinguish two reservation if there were a step when Benedek chose a different option?

For instance, if the seats 1 *and* 6 *are reserved, then there are two blocks, the first one consists of the seat* 1*, the second block consists of the seats* $3, 4$ *and* 5*. Two reservation orders are different if there is a chair that was reserved in a different step, or there is a chair that was reserved with different option (first, last or middle). So if there were* 2 *VIP chairs, then the answer would have been* 9*.*

(6 points)

1.4.3 Category E

E-1. How many ways are there to tile a 3×3 square with 4 dominoes of size 1×2 and 1 domino of size 1×1?

Tilings that can be obtained from each other by rotating the square are considered different. Dominoes of the same size are completely identical.

(3 points)

E-2. What number should we put in place of the question mark such that the following statement becomes true?

$$11001_{?} = 54001_{10}$$

A number written in the subscript means which base the number is in.

(3 points)

E-3. Ann wrote down all the perfect squares from one to one million (all in a single line). However, at night, an evil elf erased one of the numbers. So the next day, Ann saw an empty space between the numbers 760384 and 763876. What is the sum of the digits of the erased number?

(3 points)

E-4. We have a positive integer n, whose sum of digits is 100 . If the sum of digits of $44n$ is 800 then what is the sum of digits of $3n$?

(3 points)

E-5. The hexagon $ABCDEF$ has all angles equal . We know that four consecutive sides of the hexagon have lengths 7, 6, 3 and 5 in this order. What is the sum of the lengths of the two remaining sides?

(4 points)

E-6. We build a modified version of Pascal's triangle as follows: in the first row we write a 2 and a 3, and in the further rows, every number is the sum of the two numbers directly above it (and rows always begin with a 2 and end with a 3).

In the 13th row, what is the 5th number from the left?

$$\begin{array}{ccccccc} & & 2 & & 3 & & \\ & 2 & & 5 & & 3 & \\ 2 & & 7 & & 8 & & 3 \\ & & & \ldots & & & \end{array}$$

(4 points)

E-7. Santa Claus plays a guessing game with Marvin before giving him his present. He hides the present behind one of 100 doors, numbered from 1 to 100. Marvin can point at a door, and then Santa Claus will reply with one of the following words:

- "hot" if the present lies behind the guessed door,
- "warm" if the guess is not exact but the number of the guessed door differs from that of the present's door by at most 5,
- "cold" if the numbers of the two doors differ by more than 5.

At least how many such guesses does Marvin need, so that he can be certain about where his present is?

Marvin does not necessarily need to make a "hot" guess, just to know the correct door with 100% certainty.

(4 points)

E-8. The integers 1, 2, 3, 4, 5 and 6 are written on a board. You can perform the following kind of move: select two of the numbers, say a and b, such that $4a - 2b$ is nonnegative; erase a and b, then write down $4a - 2b$ on the board (hence replacing two of the numbers by just one). Continue performing such moves until only one number remains on the board. What is the smallest possible positive value of this last remaining number?

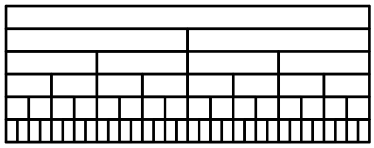
(4 points)

E-9. On a piece of paper, we write down all positive integers n such that all proper divisors of n are less than 18. We know that the sum of all numbers on the paper having exactly one proper divisor is 666. What is the sum of all numbers on the paper having exactly two proper divisors?

We say that k is a proper divisor of the positive integer n if $k \mid n$ and $1 < k < n$.

(5 points)

E-10. Soma has a tower of 63 bricks , consisting of 6 levels. On the k-th level from the top, there are 2^{k-1} bricks (where $k = 1, 2, 3, 4, 5, 6$), and every brick which is not on the lowest level lies on precisely 2 smaller bricks (which lie one level below) - see the figure. Soma takes away 7 bricks from the tower, one by one. He can only remove a brick if there is no brick lying on it. In how many ways can he do this, if the order of removals is considered as well?



(5 points)

E-11. The convex quadrilateral $ABCD$ has $|AB| = 8$, $|BC| = 29$, $|CD| = 24$ and $|DA| = 53$. What is the area of the quadrilateral if $\angle ABC + \angle BCD = 270°$?

(5 points)

E-12. We have a white table with 2 rows and 5 columns , and would like to colour all cells of the table according to the following rules:

- We must colour the cell in the bottom left corner first.
- After that, we can only colour a cell if some adjacent cell has already been coloured. (Two cells are adjacent if they share an edge.)

How many different orders are there for colouring all 10 squares (following these rules)?

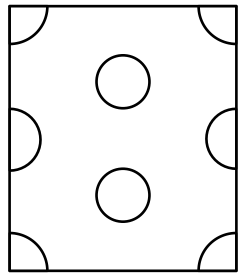
(5 points)

E-13. In triangle ABC we inscribe a square such that one of the sides of the square lies on the side AC, and the other two vertices lie on sides AB and BC. Furthermore we know that $AC = 5$, $BC = 4$ and $AB = 3$. This square cuts out three smaller triangles from $\triangle ABC$. Express the sum of reciprocals of the inradii of these three small triangles as a fraction $\frac{p}{q}$ in lowest terms (i.e. with p and q coprime). What is $p + q$?

(6 points)

E-14. How many ways are there to fill in the 8 spots in the picture with letters A, B, C and D, using two copies of each letter, such that the spots with identical letters can be connected with a continuous line that stays within the box, without these four lines crossing each other or going through other spots?
The lines do not have to be straight.

(6 points)



E-15. In a movie theatre there are 6 VIP chairs labelled from 1 to 6. We call a few consecutive vacant chairs a block. In the online VIP seat reservation process the reservation of a seat consists of two steps: in the first step we choose the block, in the second step we reserve the first, last or middle seat (in case of a block of size even this means the middle chair with the smaller number) of that block. (In the second step the online system offers the three possibilities even though they might mean the same seat.) Benedek reserved all seats at some screeining. In how many ways could he do it if we distinguish two reservation if there were a step when Benedek chose a different option?

For instance, if the seats 1 *and* 6 *are reserved, then there are two blocks, the first one consists of the seat* 1*, the second block consists of the seats* $3, 4$ *and* 5*. Two reservation orders are different if there is a chair that was reserved in a different step, or there is a chair that was reserved with different option (first, last or middle). So if there were* 2 *VIP chairs, then the answer would have been* 9*.*

(6 points)

E-16. Dora has 8 rods with lengths $1, 2, 3, 4, 5, 6, 7$ and 8 cm. Dora chooses 4 of the rods and uses them to assemble a trapezoid (the 4 chosen rods must be the 4 sides). How many different trapezoids can she obtain in this way?

Two trapezoids are considered different if they are not congruent.

(6 points)

1.4.4 Category E$^+$

E$^+$-1. We have a positive integer n, whose sum of digits is 100 . If the sum of digits of $44n$ is 800 then what is the sum of digits of $3n$?

(3 points)

E$^+$-2. The hexagon $ABCDEF$ has all angles equal . We know that four consecutive sides of the hexagon have lengths 7, 6, 3 and 5 in this order. What is the sum of the lengths of the two remaining sides?

(3 points)

E$^+$-3. We build a modified version of Pascal's triangle as follows: in the first row we write a 2 and a 3, and in the further rows, every number is the sum of the two numbers directly above it (and rows always begin with a 2 and end with a 3).

In the 13th row, what is the 5th number from the left?

$$\begin{array}{ccccccc} & & 2 & & 3 & & \\ & 2 & & 5 & & 3 & \\ 2 & & 7 & & 8 & & 3 \\ & & & \cdots & & & \end{array}$$

(3 points)

E$^+$-4. Santa Claus plays a guessing game with Marvin before giving him his present. He hides the present behind one of 100 doors, numbered from 1 to 100. Marvin can point at a door, and then Santa Claus will reply with one of the following words:

- "hot" if the present lies behind the guessed door,
- "warm" if the guess is not exact but the number of the guessed door differs from that of the present's door by at most 5,
- "cold" if the numbers of the two doors differ by more than 5.

At least how many such guesses does Marvin need, so that he can be certain about where his present is?

Marvin does not necessarily need to make a "hot" guess, just to know the correct door with 100% certainty.

(3 points)

E$^+$-5. On a piece of paper, we write down all positive integers n such that all proper divisors of n are less than 30. We know that the sum of all numbers on the paper having exactly one proper divisor is 2397. What is the sum of all numbers on the paper having exactly two proper divisors?

We say that k is a proper divisor of the positive integer n if $k \mid n$ and $1 < k < n$.

(4 points)

E$^+$-6. Positive integers a, b and c are all less than 2020. We know that a divides $b + c$, b divides $a + c$ and c divides $a + b$. How many such ordered triples (a, b, c) are there?

Note: In an ordered triple, the order of the numbers matters, so the ordered triple $(0, 1, 2)$ is not the same as the ordered triple $(2, 0, 1)$.

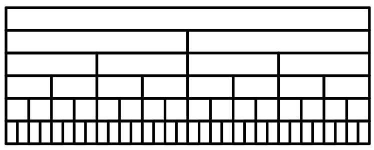
(4 points)

E$^+$-7. There are red and blue balls in an urn : 1024 in total. In one round, we do the following: we draw the balls from the urn two by two. After all balls have been drawn, we put a new ball back into the urn for each pair of drawn balls: the colour of the new ball depends on that of the drawn pair. For two red balls drawn, we put back a red ball. For two blue balls, we put back a blue ball. For a red and a blue ball, we put back a black ball. For a red and a black ball, we put back a red ball. For a blue and a black ball, we put back a blue ball. Finally, for two black balls we put back a black ball.

Then the next round begins. After 10 rounds, a single ball remains in the urn, which is red. What is the maximal number of blue balls that might have been in the urn at the very beginning?

(4 points)

E$^+$-8. Soma has a tower of 63 bricks , consisting of 6 levels. On the k-th level from the top, there are 2^{k-1} bricks (where $k = 1, 2, 3, 4, 5, 6$), and every brick which is not on the lowest level lies on precisely 2 smaller bricks (which lie one level below) - see the figure. Soma takes away 7 bricks from the tower, one by one. He can only remove a brick if there is no brick lying on it. In how many ways can he do this, if the order of removals is considered as well?



(4 points)

E$^+$-9. The convex quadrilateral $ABCD$ has $|AB| = 8$, $|BC| = 29$, $|CD| = 24$ and $|DA| = 53$. What is the area of the quadrilateral if $\angle ABC + \angle BCD = 270°$?

(5 points)

E$^+$-10. We have a white table with 2 rows and 5 columns , and would like to colour all cells of the table according to the following rules:

- We must colour the cell in the bottom left corner first.

- After that, we can only colour a cell if some adjacent cell has already been coloured. (Two cells are adjacent if they share an edge.)

How many different orders are there for colouring all 10 squares (following these rules)?

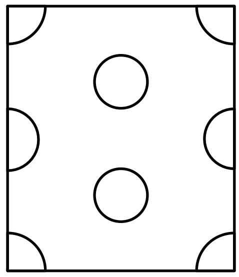
(5 points)

E$^+$-11. In triangle ABC we inscribe a square such that one of the sides of the square lies on the side AC, and the other two vertices lie on sides AB and BC. Furthermore we know that $AC = 5$, $BC = 4$ and $AB = 3$. This square cuts out three smaller triangles from $\triangle ABC$. Express the sum of reciprocals of the inradii of these three small triangles as a fraction $\frac{p}{q}$ in lowest terms (i.e. with p and q coprime). What is $p + q$?

(5 points)

E$^+$-12. How many ways are there to fill in the 8 spots in the picture with letters A, B, C and D, using two copies of each letter, such that the spots with identical letters can be connected with a continuous line that stays within the box, without these four lines crossing each other or going through other spots?
The lines do not have to be straight.

(5 points)



E$^+$-13. In the game of Yahtzee , players have to achieve various combinations of values with 5 dice. In a round, a player can roll the dice three times. At the second and third rolls, he can choose which dice to re-roll and which to keep. What is the probability that a player achieves at least four 6's in a round, given that he plays with the optimal strategy to maximise this probability?

Writing the answer as $\frac{p}{q}$ where p and q are coprime, you should submit the sum of all prime factors of p, counted with multiplicity. So for example if you obtained $\frac{p}{q} = \frac{3^4 \cdot 11}{2^5 \cdot 5}$ then the submitted answer should be $4 \cdot 3 + 11 = 23$.

(6 points)

E$^+$-14. In a movie theatre there are 6 VIP chairs labelled from 1 to 6. We call a few consecutive vacant chairs a block. In the online VIP seat reservation process the reservation of a seat consists of two steps: in the first step we choose the block, in the second step we reserve the first, last or middle seat (in case of a block of size even this means the middle chair with the smaller number) of that block. (In the second step the online system offers the three possibilities even though they might mean the same seat.) Benedek reserved all seats at some screeining. In how many ways could he do it if we distinguish two reservation if there were a step when Benedek chose a different option?

For instance, if the seats 1 *and* 6 *are reserved, then there are two blocks, the first one consists of the seat* 1*, the second block consists of the seats* $3, 4$ *and* 5*. Two reservation orders are different if there is a chair that was reserved in a different step, or there is a chair that was reserved with different option (first, last or middle). So if there were* 2 *VIP chairs, then the answer would have been* 9.

(6 points)

E$^+$-15. The function f is defined on positive integers : if n has prime factorization $p_1^{k_1} p_2^{k_2} \cdot \ldots \cdot p_t^{k_t}$ then $f(n) = (p_1 - 1)^{k_1+1}(p_2 - 1)^{k_2+1} \cdot \ldots \cdot (p_t - 1)^{k_t+1}$. If we keep using this function repeatedly, starting from any positive integer n, we will always get to 1 after some number of steps. What is the smallest integer n for which we need exactly 6 steps to get to 1?

(6 points)

E$^+$-16. Dora has 10 rods with lengths $1, 2, 3, 4, 5, 6, 7, 8, 9$ and 10 cm. Dora chooses 4 of the rods and uses them to assemble a trapezoid (the 4 chosen rods must be the 4 sides). How many different trapezoids can she obtain in this way?

Two trapezoids are considered different if they are not congruent.

(6 points)

2 Solutions

2.1 Online round

2.1.1 Category C

1. In each 2×2 part of the chessboard, the sum of the numbers is $2 + 3 + 3 + 2 = 10$. As the board can be subdivided into 16 such 2×2 squares, the total sum of the numbers on the board is $16 \cdot 10 = 160$.

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2. The six goats have a total of $6 \cdot 4 = 24$ legs, whereas the 14 chickens have $14 \cdot 2 = 28$. So the cats must have the remaining $100 - 24 - 28 = 48$ legs, which means there must be $48/4 = 12$ cats.

(Back to problems)

3. Let α be the fifth angle of the pentagon. We know that the sum of the internal angles of a pentagon is $3 \cdot 180° = 540°$. So we have $540° = 90° + 100° + 110° + 120° + \alpha = 420° + \alpha$, so $\alpha = 540° - 420° = 120°$.

(Back to problems)

4. Determining the positions of the 2 yellow cubes determines the whole tower, so it is enough for us to concentrate on the yellow cubes. The first one can be put in 6 possible places and the second can be put in one the remaining 5 places, and as we do not distinguish between cubes of the same colour, we have counted all cases twice, so the solution is $\frac{6 \cdot 5}{2} = 15$.

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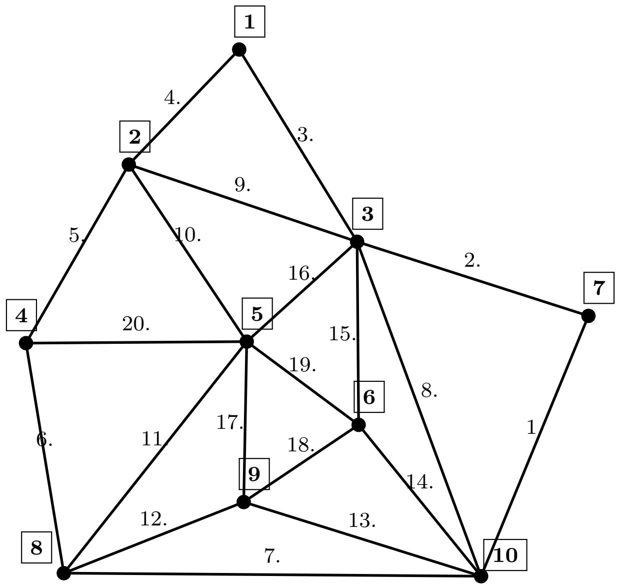
5. On the following figure we can see a possible walk that ends at clearing 4.



We will now show that this is the only possibility. Every time Johnny enters a clearing via a path, he then exits it via another path. So for every clearing, the paths emerging from it are used up in pairs. The only exceptions are the clearings in which the walk starts or ends, as these are the only two occasions when it is possible for Johnny to only use up one of the paths that belong to a clearing.

So for every clearing apart from the starting and finishing ones, the number of paths emerging from it must be even. This is true for all clearings except 4 and 10, so this means clearing 4 must be the place where Johnny's walk ends.

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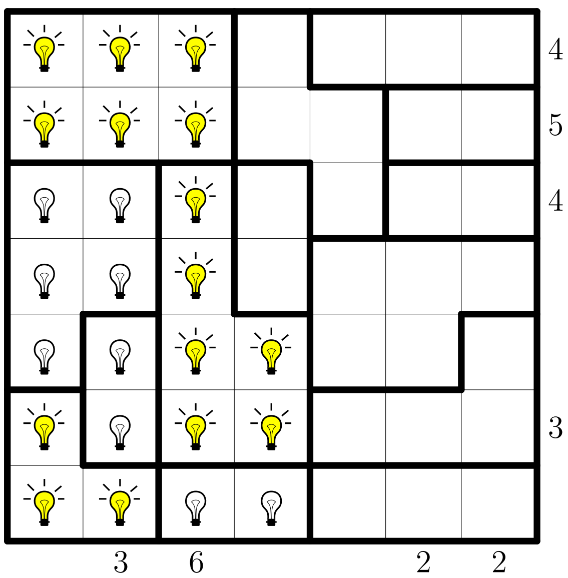
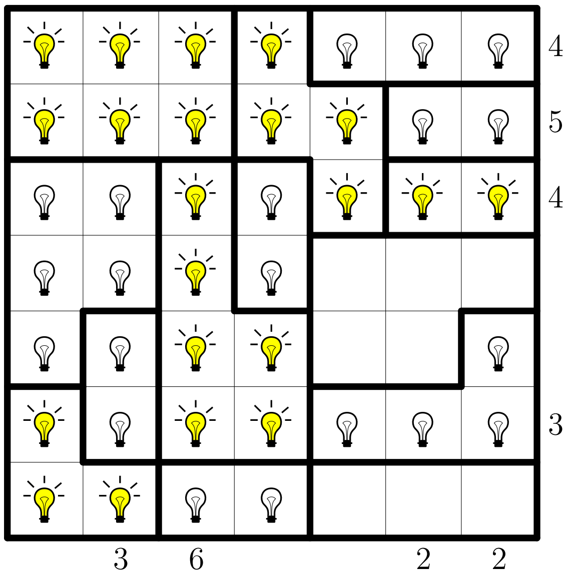
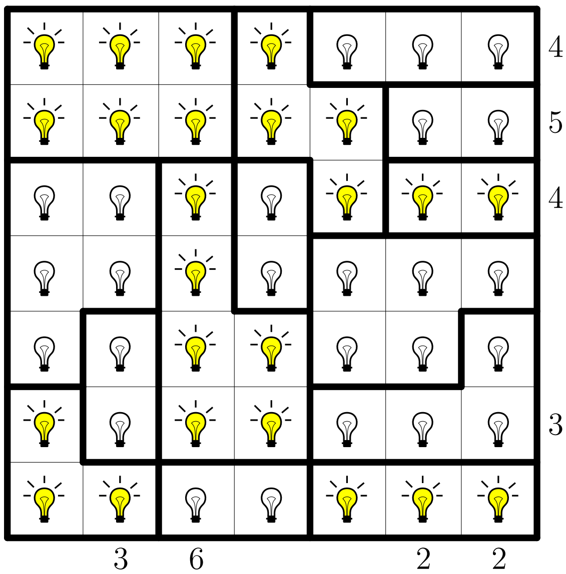
6. Based on the data from the second and third columns, we know the following:



Now use the information that we know about the rows:



The remaining squares can be filled in using the sixth column:



So we have finished solving this puzzle; in all rows and columns the correct number of lights is turned on. So in total, 24 of the lights are on.

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7. Let t be the number of Süsü's brothers. Then $\frac{1+3t+7+19}{t+3} = 6$, so $\frac{3t+27}{t+3} = 6$. From here, $3t + 27 = 6t + 18$, so $9 = 3t$, so Süsü has 3 brothers.

Note. This problem gives us "too much" information; the solution did not use all of the given conditions.

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8. Between any two consecutive question marks there are two periods, so there are a total of $5 \cdot 2 = 10$ periods in the 5 "gaps" between the 6 question marks. Together with the sentence ending in an exclamation mark, there are $6 + 10 + 1 = 17$ sentences in total, and each of them contains one punctuation mark (a comma) in the middle and one at the end, so the story contains a grand total of $2 \cdot 17 = 34$ punctuation marks.

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9. We will count the possible combinations by first determining the number of all possible orders of Ákos, and then subtracting those which consist only of meat-free dishes.

Ákos has 3 ways to choose a soup, 5 to choose a main course and 2 for a dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possible orders he can make. However 2 soups, 2 main courses and 2 desserts do not contain meat; these can be combined in $2 \cdot 2 \cdot 2 = 8$ ways to get a fully vegetarian order. Subtracting these, we get that there are $30 - 8 = 22$ possible orders which do contain meat.

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2.1.2 Category D

1. In each 2×2 part of the chessboard, the sum of the numbers is $4 + 6 + 6 + 4 = 20$. As the board can be subdivided into 16 such 2×2 squares, the total sum of the numbers on the board is $16 \cdot 20 = 320$.

(Back to problems)

2. If Pooh found a pots of size 4 dl and b of size 3 dl, then $4a + 3b = 13$.
If $a = 0$, then $3b = 13$, which is impossible since $3 \nmid 13$.
If $a = 1$, then $3b = 9$, so $b = 3$.
If $a = 2$, then $3b = 5$, which is impossible as $3 \nmid 5$.
If $a = 3$, then $3b = 1$, which is also absurd.
If $a > 3$, then the total volume of the 4 dl pots would be greater than 13 dl, so this is also impossible.
So the only possibility is that $a = 1$ and $b = 3$, so Pooh found a total of $a + b = 4$ pots.

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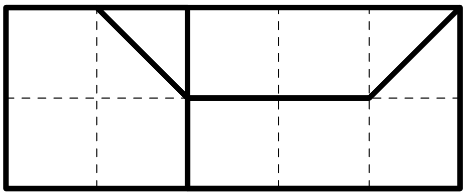
3. The pieces have areas 3.5 cm^2, 2.5 cm^2, 3.5 cm^2 and 0.5 cm^2 respectively, for a total area of 10 cm^2. It is easy to see that the rectangle's side lengths must be integers, so the area of 10 cm^2 can only arise as 1 cm $\times$ 10 cm or 2 cm $\times$ 5 cm. As it is impossible to fit the leftmost piece in the former one, our rectangle has to be 2 cm $\times$ 5 cm. Now this has a perimeter of $2 \cdot (2 \text{ cm} + 5 \text{ cm}) = 14$ cm.

It is indeed possible to make such a rectangle from the pieces, as shown below:



(Back to problems)

4. For the solution, see Category C Problem 5.

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5. Suppose that a hotdog costs A forints and a meatball costs B. Then we get a system of two equations:

$$5A + 8B = 1320$$

$$3A + 7B = 990$$

We are looking for the value of $7A+9B$. Observe that $7A+9B = 2\cdot(5A+8B)-(3A+7B)$, so $7A + 9B = 2 \cdot 1320 - 990 = 1650$.

Second solution: We can also solve this problem by calculating A and B explicitly: if we subtract 3 times the first equation from 5 times the second one, we get $11B = 5\cdot 990 - 3\cdot 1320 = 990$, so $B = 90$.

Substitute this into the first equation to get $A = 120$. So the sum we are looking for is $7A + 9B = 7 \cdot 120 + 9 \cdot 90 = 1650$ forints.

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6. In total, $3^3 = 27$ possible 3-scoop cones can be made out of the 3 flavours. But we have to subtract the cones which have chocolate on top of lemon: this can occur either for the top two or for the bottom two scoops. In both cases there are 3 possibilities for the third scoop so we get 6 "bad" cones in total, and these six cones are all distinct from each other. So Dani can order a "good" cone in $27 - 6 = 21$ possible ways.

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7. The top boundary of a vertical tile is a quarter-circular arc whose centre is the centre of the tile by the conditions given in the problem, so this arc has radius 6 cm, so the four corners of the tile are also 6 cm away from the centre. Connecting the centre with the four corners splits up the tile into four parts, which can be rearranged to from two squares of side length 6 cm. The total area of these two squares (and hence of our original tile) is $2 \cdot (6\text{ cm})^2 = 72\text{ cm}^2$.

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8. Choose two distinct rays coming from the left hand intersection point, and also two distinct rays from the top one. These four rays determine a convex quadrilateral, and each convex quadrilateral arises in this way, so it suffices to count the number of ways these four rays can be chosen.

There are $\binom{4}{2} = 6$ ways to choose two rays out of the four coming from the left, and $\binom{5}{2} = 10$ ways to choose two out of the five coming from the top. This gives a total of $6 \cdot 10 = 60$ convex quadrilaterals.

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9. As we are adding together two four-digit integers, the sum will be less than $2 \cdot 9999 < 20000$, so we have $D = 1$. The two summands and the sum all have R at the hundreds place, and this is only possible in two ways:

Case 1: $R = 0$, and there is no carry at the tens place.

We would like to express all letters in the summation in terms of E. We know that $S + \H{O} = 10$. We also know that $\H{O} + D + 1 = \H{O} + 1 + 1 = E$ (here we are using the fact that there is a carry in the ones place but not in the tens place). Since there was no carry in the hundreds place either, $\ddot{U} = 2 \cdot E - 10$. Using the relations above, we can express everything in terms of E:

$$\H{O} = E - 2$$

$$S = 12 - E$$

$$\ddot{U} = 2 \cdot E - 10$$

The third equation gives $E \geq 6$, but the second equation excludes the possibility that $E = 6$. If we had $E = 7$, then we would have $\H{O} = S = 5$ which is impossible. If $E = 8$, we would have $\H{O} = \text{Ü} = 6$, which is also impossible. The only case which does not lead to a contradiction is $E = 9$, and this gives a valid solution: $9073 + 9017 = 18090$.

Case 2: $R = 9$, and there is a carry at the tens place.

We know that $S + \H{O} = 9$, so there is no carry at the ones place. This means $\H{O} + D = \H{O} + 1 = 10 + E$ (here we are using the fact that there was no carry at the ones place but there was one at the tens place), so $\H{O} = E + 9$. But this means $E = 0$ and $\H{O} = 9$, which is impossible (since we already have $R = 9$). So this case does not give us any solutions.

So the only solution is $D = 1, E = 9, \H{O} = 7, R = 0, S = 3, \ddot{U} = 8$. So our answer is $\ddot{U}RES = 8093$.

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2.1.3 Category E

1. $11 + 22 + ... + 99 = 1 \cdot 11 + 2 \cdot 11 + ... + 9 \cdot 11 = (1 + 2 + ... + 9) \cdot 11 = 11 \cdot 45 = 495.$

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2. We know that the sum of internal angles in a decagon is $8 \cdot 180° = 1440°$. In a regular decagon, all angles are equal, so each angle has size $\frac{1440°}{10} = 144°$. Let α be the angle we want to calculate. Two of the angles of our new nonagon have size α whereas the other seven are $144°$. We also know that the sum of internal angles in a nonagon is $7 \cdot 180° = 1260°$. So we have $2\alpha + 7 \cdot 144° = 1260°$, so $\alpha = \frac{1260° - 7 \cdot 144°}{2} = 126°$.

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3. For the solution, see Category D Problem 5.

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4. For the solution, see Category C Problem 9.

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5. For the solution, see Category D Problem 8.

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6. If n dragons remain after kicking out Süsü with an average of 9 heads, then they have $9n$ heads in total. Before kicking him out, there were $n + 1$ family members, having a total of $8 \cdot (n + 1)$ heads. When Süsü was kicked out, the total number of heads decreased by 1, so $8(n + 1) - 1 = 9n$, meaning $n = 7$.

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7. For the solution, see Category D Problem 7.

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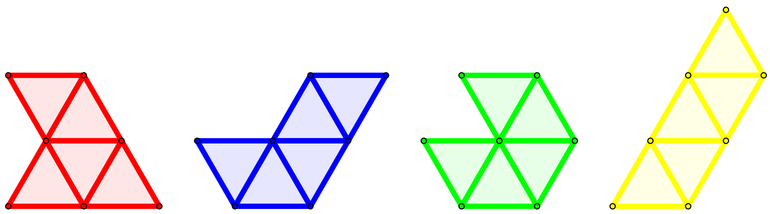
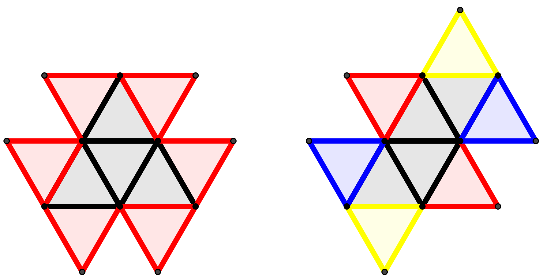
8. After a little searching, we can find the following four tiles:



Indeed, these are all the possible tiles: we can see that if we add a new triangle anywhere to a 4-triangle tile, we always get one of the four tiles pictured above. This is demonstrated in the figure below: the three possible 4-triangle tiles are shown in grey, and every possible location where we can add a new triangle is colour-coded according to which of the four 5-triangle tiles it yields.



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9. As the remainder of any integer is the same as the remainder of its sum of digits when dividing by 3 or 9, the numbers of Kartal, Bálint and Gábor all give the same remainder when dividing by 3. This means Timi's number is divisible by 3. But this also means that the numbers of the three boys are also divisible by 3, and they have the same remainder when divided by 9, so Timi's number must be divisible by 9.

At least one of the cards displayed must be a 1, or otherwise the three boys' numbers are all at least 2000, so Timi's number is at least 6000, a contradiction.

This leaves 7 possibilities for Timi's number (which must be a number between 4210 and 4567 divisible by 9 and containing a 1):

$$4212, 4221, 4311, 4401, 4410, 4419, 4518$$

We also know that all of the boys' numbers must start with either a 1 or a 2: if some boy's number is at least 3000 on his card, then as all three numbers are at least 1000, the sum must be at least 5000, again a contradiction.

The largest of the boys' three numbers must be at least a third of the sum. This lets us exclude the following possibilities for Timi's number: 4311, 4401 and 4410, as for neither of these three numbers is it possible to rearrange its digits to get a number which is at least a third of the original and starts with a 1 or a 2.

In the case of 4212 or 4221, none of the boys can have a number starting with a 2, as then that number is ≥ 2124, and the other two are ≥ 1224 so the sum of the three numbers would be $> 2100 + 1200 + 1200 = 4500$. So all of the boys' numbers begin with a 1, and the largest one (which is at least $4212/3 = 1404$) must be 1422. If all three boys have 1422 then the sum is 4266, which is not what we want. Otherwise the sum is $\leq 1422 + 1422 + 1242 = 4086$, which is too small. So these cases are impossible.

If Timi has 4419, then all of the boys have at least 1449, the next smallest possible number is 1494 and the next is 1944. If anybody has at least 1944, the sum is too big ($\geq 1944 + 2 \cdot 1449 = 4842$). So everybody has either 1449 or 1494. But using these two numbers it is not possible to make a sum of 4419, as $1449+1449+1494 = 4392 < 4419$ but $1449+1494+1494 = 4437 > 4419$. So this case is also impossible.

This leaves 4518 as Timi's number. And this is indeed possible, as $1485+1485+1548 = 4518$.

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2.2 Regional round

2.2.1 Category C

1. We know that Dávid scored at least 1 goal in the PE lesson, so Ákos scored at least $4 \cdot 1 + 1 = 5$ goals. Dávid knows the seats of more counties than this, so he knows the seats of at least 6 counties. Ákos knows the seats of more than three times as many counties, i.e. at least $3 \cdot 6 + 1 = 19$ counties. (Remark: We used the fact that the number of goals scored and the number of county seats known by a person are integers.)

Assume that Ákos knows the seats of more than 19, i.e. at least 20 counties. Then Dávid's first statement means that he had more than 20 points on last year's Dürer competition, so Ákos had more than $3 \cdot 20 = 60$ points, which is impossible.

So Ákos knows the seats of 19 counties.

Remark: There are 19 counties in Hungary.

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2. **a)** Let us consider what can be the largest digit of a 5-digit unlucky number. The sum of the four smallest digits is $0 + 1 + 2 + 3 = 6$, so the largest digit of the number is at most $13 - 6 = 7$.

If the number contains digit 7, its other four digits must be 0, 1, 2 and 3.

If the largest digit is 6, then the number cannot contain digit 5 (as $0+1+2+4+6 = 14 > 13$). So the remaining four digits must be all but one of 0, 1, 2, 3, 4. The sum of the five digits only works out if the digits are 0, 1, 2, 4, 6.

If the largest digit is 5, the five digits of the number must be 0, 1, 3, 4, 5.

The largest digit cannot be 4 (or less), as $0 + 1 + 2 + 3 + 4 = 10 < 13$.

In each case there are 4 possible digits to go to the ten thousands position (the number cannot start with digit 0). Then we can choose from 4 different digits for the thousands position, 3 digits for the hundreds, 2 for the tens and we have only one digit left for the ones position. So we get $4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96$ different numbers in each case. Hence there are $3 \cdot 96 = 288$ unlucky 5-digit numbers in total.

b) A number is even if its last digit is even.

If the last digit is 0, then we get $4 \cdot 3 \cdot 2 \cdot 1 = 24$ numbers in each case.

In the first case 2 is the only non-zero even digit, in the second case 2, 4, and 6 are such digits, in the third case only 4 is such a digit. We can count these numbers similarly to the above procedure (keeping in mind that the first digit cannot be 0). We get $3 \cdot 3 \cdot 2 \cdot 1 = 18$ different numbers in each case.

So there are $3 \cdot 24 + 5 \cdot 18 = 162$ unlucky even 5-digit numbers in total.

c) A number is divisible by 3 if the sum of its digits is divisible by 3. Note that 3 does not divide 13, so there is no number with the required property.

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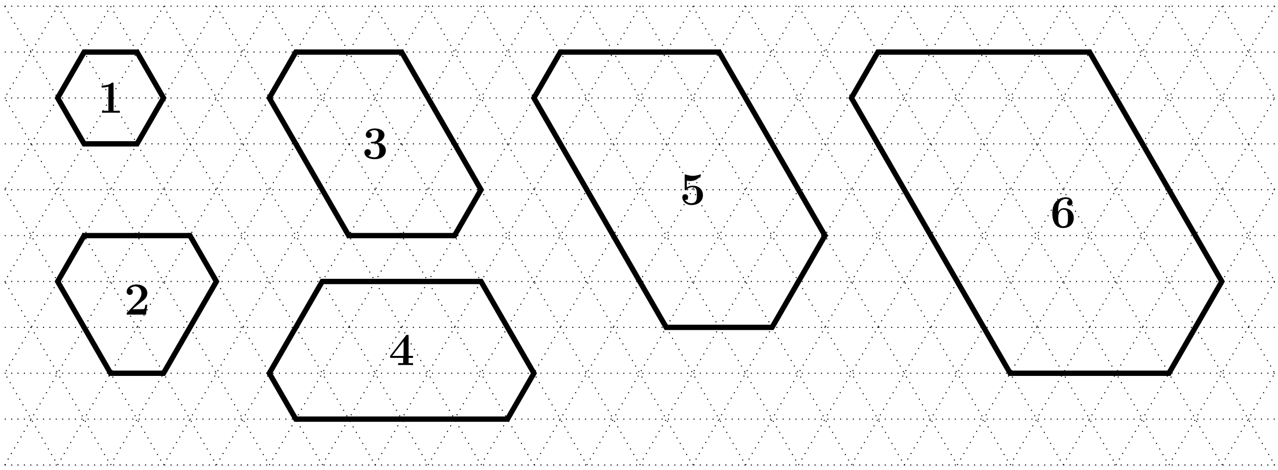
3. Any label from 1 to 6 is possible. We will give examples on a regular triangular grid. If we scale the grid so that the side length of each little triangle is 1 cm, then the following hexagons have integer side lengths.



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4. **a)** Yes, it is possible. The three students appear in the register book in order Lenger, Miller, Smith. To get a solution, let for example Lenger be the 7th, Miller the 13th and Smith the 20th student in the register. If Miller is on duty after Smith, then the next person on duty will be the student number $20 + 13 - 26 = 7$, i.e. Lenger.

b) It is not possible. Let us denote the numbers of Lenger, Smith and Miller by l, s, m respectively. Miller will be on duty after Lenger and Smith if $m = l + s$ or $m = l + s - 26$ holds. Is it possible to have $m = l + s$? We know that $1 \leq l < m < s \leq 26$, so $1 \leq l$ and $m < s$. Adding these two inequalities we get $m + 1 \leq l + s$, so $m < l + s$. So we cannot have $m = l + s$. How about the other case, $m = l + s - 26$? We know that $m > l$ and $26 \geq s$, so $m + 26 > s + l$, i.e. $m > s + l - 26$. So this case is not possible either.
This means that the three students cannot be on duty in this order.

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5. **a)** First take away one grain from every cell in the first row, and then double the grains in the first column. Now we have 4 grains on both cells of the first row, so we can take them all away. In the second row we have 10 and 3 grains on the cells. Double the grains in every cell of the second column, and take away 2-2 grains from the cells in the second row. Doubling the grains in the second column leaves us with 4 grains on the cells in the second row, so we can take them all away, emptying the table.

b) Suppose the table looks like this:

a	b
c	d

First we want to empty the first row. If $a = b$, take away a grains from every cell of the first row. If $a < b$, then keep doubling the first column until we reach $b \geq a' \geq b - a'$ (Here a' is the number of grains on the top-left cell). Then we take away $2a' - b$ grains from each cell of the first row, leaving us with $b - a'$ grains on the top left cell, and $2b - 2a'$ on the top right cell. Double the left column and take away $2b - 2a'$ grains from the first row's cells. This empties the top two cells. If $b < a$, we can do a similar algorithm.

These steps never decrease the number of grains in the second row, so after these steps we will have a positive number of grains on the other 2 cells of your table. We can use the same algorithm to empty these, and it never puts new grains on an empty cell, so thi way we can empty the whole table.

c) We will show that we can empty an arbitrary row from any starting table.

Take an arbitrary row, and look at the 2 cells in the row with the least amount of grains on them. Using the algorithm above we can make these equal (in example, we execute every step of the algorithm until we reach $2b - 2a'$ grains on both of these cells). Observe that none of the other cells in the table will become empty throughout this process. From now on, we consider these two columns the same, so that we will do the same steps on both of them. Then we have 7 columns that we consider differently, so we can use the same algorithm as above, leaving us with 6 differently considered columns. Repeat until all of the columns are considered the same. This means that every cell on the row that we took at the beginning has an equal number of grains. Then we take all of them away.

This way we can empty an arbitrary row, keeping the other cells in the table nonempty. So doing this for every row empties every cell, so we proved what we wanted.

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2.2.2 Category D

1. Absolutely unlucky numbers can only contain digits 1 and 3 and the sum of the digits is given, therefore giving the number of 3's will determine the number of 1's too. Let us group the absolutely unlucky numbers based on how many 3's they contain. They can contain $0, 1, 2, 3$ or 4 3's. (Cannot contain more than 4, as the sum of digits is 13.)

If there are k 3's in an absolutely unlucky number, it must contain $n = 13 - 3k$ 1's. There are $\binom{n+k}{k}$ numbers consisting of k 3's and n 1's (there are $n + k$ digits in total and we have to choose the position of the k 3's, which can be done $\binom{n+k}{k}$ different ways).

- If there are 0 3's, there will be $13 - 0 \cdot 3 = 13$ 1's. There is $\binom{13}{0} = 1$ such number.
- If there is 1 digit 3, there will be $13 - 1 \cdot 3 = 10$ 1's. There are $\binom{11}{1} = 11$ such numbers.
- If there are 2 3's, there will be $13 - 2 \cdot 3 = 7$ 1's. There are $\binom{9}{2} = 36$ such numbers.
- If there are 3 3's, there will be $13 - 3 \cdot 3 = 4$ 1's. There are $\binom{7}{3} = 35$ such numbers.
- If there are 4 3's, there will be $13 - 4 \cdot 3 = 1$ digit 1. There are $\binom{5}{1} = 5$ such numbers.

We get different absolutely unlucky numbers in the different cases (the number of 3's is different) and we considered each absolutely unlucky number. So we get the number of absolutely unlucky numbers by adding up how many numbers we found in each different case.
So there are $1 + 11 + 36 + 35 + 5 = 88$ absolutely unlucky numbers in total.

(Back to problems)

2. Let a_n be the number of rabbits and b_n be the number of foxes on the island n years from now. We know that $a_0 = 24$, $b_0 = 2$ and $b_n = 2 \cdot b_{n-1}$, $a_n = 2 \cdot a_{n-1} - b_{n-1}$ (assuming there are enough rabbits on the island). We immediately get that $b_n = 2 \cdot 2^n$, and we will also prove that $a_n = 2^n(24 - n)$.

We prove this by induction: In case $n = 0$: $a_0 = 24$, so the statement holds. In the spring of the first year there will be $24 \cdot 2 = 48$ rabbits and in autumn two of them get eaten by foxes, so there will be 46 rabbits and $2 \cdot 2 = 4$ foxes on the island. Indeed, $a_1 = 2^1(24 - 1) = 46$. Assume that $a_n = 2^n(24 - n)$ holds for some n. Then

$$a_{n+1} = 2 \cdot a_n - b_n = 2 \cdot a_n - 2 \cdot 2^n = 2 \cdot 2^n(24 - n) - 2 \cdot 2^n$$

by induction hypothesis, therefore

$$a_{n+1} = 2 \cdot 2^n(24 - n) - 2 \cdot 2^n = 2^{n+1}(24 - n - 1) = 2^{n+1}(24 - (n + 1)),$$

as required.

a) , **b)** Two years from now there will be $a_2 = 2^2(24 - 2) = 88$, ten years from now there will be $a_{10} = 2^{10}(24 - 10) = 14 \cdot 1024 = 14336$ rabbits on the island.

c) As long as there are some rabbits on the island, some foxes can survive. But if all rabbits die, so will all foxes in the following year. The rabbits go extinct in the first year when $a_n \leq 0$ ($a_n < 0$ meaning that there were more foxes than rabbits, so all the rabbits got eaten up, and some foxes died too). $a_n = 2^n(24 - n) \leq 0$ holds if and only if $24 - n \leq 0$, i.e. $24 \leq n$. So the rabbits go extinct in the 24th year and the foxes follow them in the 25th year.

(Back to problems)

3. Let us look at the parity of p and q.

If p, q are both odd, then p^q and q^p are both odd, so their sum r is even. Since r is a prime, it means $r = 2$. But p, q are also primes, so the left-hand side is greater than 2, which is a contradiction.

If p, q are both even, then both are 2, therefore $r = 4 + 4 = 8$, which is not a prime, contradiction.

If one of p, q is even and the other one is odd, let us assume that p is even, i.e. $p = 2$. Then $2^q + q^2 = r$. Let us look at the remainder modulo 3 of both sides. q is odd, so 2^q will give remainder 2 ($2^{odd} = 2^{2k+1} = 2^1 \cdot 2^{2k} = 2 \cdot 4^k$ and 4 gives remainder 1 mod 3, so $4 \cdot 4$, $4 \cdot 4 \cdot 4$ ... also give remainder 1).

If $q > 3$, then it has remainder 1 or 2 mod 3, so its square q^2 will have remainder 1 mod 3. It means that the left-hand side is divisible by 3, so r is divisible by 3, i.e. $r = 3$. The left-hand side is surely greater than 3, so this is not possible.

The only remaining case is when $q = 3$. Then $q^2 + 2^q = 17$, which is a prime.

So the two solutions are $p = 2,\ q = 3,\ r = 17$, and $p = 3,\ q = 2,\ r = 17$.

(Back to problems)

4. Let us define the distance between two vertices of the octagon as the number of sides we have to walk along, if we want to go from one vertex to the other on the perimeter. The distance between two distinct vertices can range from 1 to 4. Two vertices at distance 1 will also be called adjacent or consecutive, and two vertices at distance 4 will also be called opposite.

Now observe the following: if we take any pair of consecutive vertices, the segment connecting them is parallel to the one connecting the immediately adjacent vertices, and also to the one connecting the vertices adjacent to the last ones, and finally it is also parallel to the segment connecting the remaining two points (this is shown on the first diagram below). This is obvious for symmetry reasons.

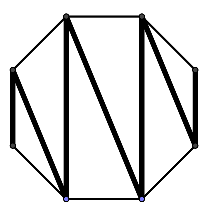
As there are four pairs of parallel sides, this means that we managed to place $4 \cdot 4 = 16$ of the segments into four classes of parallels. These are precisely the segments that connect vertices at distance 1 or 3.

The segments with distance 2 and 4 can be similarly placed into 4 classes based on their direction (so that each class contains segments which are parallel to each other). Every such class contains one segment of distance 4 and two of distance 2.

This means that we have managed to divide all $\binom{8}{2} = 28$ diagonals and sides into 8 classes based on their direction. We will get very close to the solution if we can calculate all possible angles between these 8 directions.

It is easy to find representatives of four of the directions: these are the main diagonals, which all pass through the centre of the octagon. We also want to find representatives of the other 4 directions passing through the centre: these will be the perpendicular bisectors of the sides of the octagon. These 8 lines divide 360° into 16 equal angles, since for any line, its two neighbours are symmetric with respect to it, so any two neighbouring angles are equal.

So all the angles arising between the directions are multiples of $\frac{360^\circ}{16} = 22.5^\circ$. So it just suffices to find such angles on the diagram, and we will be done: angles of measure $22.5^\circ, 45^\circ, 67.5^\circ, 90^\circ, 112.5^\circ$ and 135° can be found at a vertex. (These values can all be obtained by taking a suitable multiple of 22.5°.) An angle of measure 157.5° cannot be found on the figure, as this would require two diagonals of neighbouring directions (i.e. directions that are 22.5° apart) to intersect each other in the interior of the octagon. But this is impossible, as segments of neighbouring directions form the following shape:



These segments never intersect each other in the interior. So the only angles appearing in the figure are the 6 angles listed above.
Note: It is easy to find an angle of $180°$ *at an interior intersection point, however this does not arise from two distinct intersecting segments. No points were deducted from teams missing this case.*

(Back to problems)

5. **a)** For example $\{0, 3, 4, 6, 8, 13\}$ is a 6-tuple with the required property. Their pairwise sums in increasing order are $3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 19, 21$. We see that they are all different and the difference of any two consecutive numbers is at most 2.

We can easily form new 6-tuples from the above one with the required property by "translating" it (adding the same thing to each number) or "reflecting" it (multiplying each number by (-1)*), as these operations do not change the sizes of the differences between the pairwise sums.*

b) We will show that it is not possible to give 7 or more numbers with the required property.

We will prove the statement by contradiction. Suppose for contradiction that there are $n > 6$ numbers with the given property, let us call them $a_1, ..., a_n$. They must be all different, because $a_k = a_l$ with $k \neq l$ would mean that for any $m \neq k$, $m \neq l$ (such an m exists as $n > 6$) we get $a_m + a_l = a_m + a_k$.

So we can assume that $a_1 < a_2 \ldots < a_n$ (relabel if necessary). Then the two smallest sums will be $a_1 + a_2$ and $a_1 + a_3$.

Similarly the two largest sums will be $a_n + a_{n-1}$ and $a_n + a_{n-2}$.

We know that these sums are consecutive sums, so their difference is at most 2, and since they are not equal, their difference is at least 1. That is, $1 \leq (a_n + a_{n-1}) - (a_n + a_{n-2}) \leq 2$, thus $1 \leq a_{n-1} - a_{n-2} \leq 2$. Similary, $1 \leq a_3 - a_2 \leq 2$. These two differences cannot be the same since then $a_{n-1} + a_2$ would be equal to $a_{n-2} + a_3$ (Since $n > 6$ these four numbers are different.).

Simiilarly, if there are 2 pairs of numbers consisting of 4 different numbers such that $a_i - a_j = a_k - a_l$, then we would have $a_i + a_l = a_k + a_j$.

We can assume that $a_3 - a_2 = 1$ and $a_{n-1} - a_{n-2} = 2$ since if $a_3 - a_2 = 2$ and $a_{n-1} - a_{n-2} = 1$ we can use the trick that we have seen at part **a)** about translation and rotation (multiplication of -1), and we could reduce the problem to $a_3 - a_2 = 1$ and $a_{n-1} - a_{n-2} = 2$.

We show that this implies that $a_4 - a_3 \geq 3$. If $a_4 - a_3 = 1$, then $a_4 - a_2 = 2$ and $a_{n-1} - a_{n-2} = 2$ would imply equal cross sums. If $a_4 - a_3 = 2$ and $a_{n-1} - a_{n-2} = 2$ would imply equal cross sums. Note that $n \geq 7$, and so $(n-2) \geq 4$, that is, the four numbers are indeed pairwise different. This shows that $a_4 - a_3 \geq 3$.

Then in the list of pairwise sums the numbers $a_1 + a_3$ and $a_1 + a_4$ cannot be consecutive elements since their difference is at least 3. So after $a_1 + a_3$ we can only have the sum $a_2 + a_3$ in the list. Indeed, if $i \geq 2$ and $j > i$, then $a_1 + a_3 < a_2 + a_3 < a_i + a_j$, and if $i = 1$ and $j \geq 4$, then $a_1 + a_3 < a_1 + a_4 < a_1 + a_j$. So we had only two possibilities for the next number after $a_1 + a_3$, namely $a_1 + a_4$ and $a_2 + a_3$, but we excluded $a_1 + a_4$.

Then we know that $(a_2 + a_3) - (a_1 + a_3) \leq 2$, that is, $a_2 - a_1 \leq 2$. In other words, $a_2 - a_1$ is 1 or 2. But none of these can occur. If $a_2 - a_1 = 1$, then $a_3 - a_1 = 2$ (since we know that $a_3 - a_2 = 1$) and $a_{n-1} - a_{n-2} = 2$ yields equal cross sums. If $a_2 - a_1 = 2$, then $a_2 - a_1 = 2$ and $a_{n-1} - a_{n-2} = 2$ yields equal cross sums.

Hence there is no such set for $n \geq 7$.

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2.2.3 Category E

1. **a)** A good substitution is $a = 3$, $b = 1$, since $2^2 = 9 + 4 - 9$.

b) If Albrecht has got a correct result, it means that $(a + 2b - 3)^2 = a^2 + 4b^2 - 9$. This expands to:

$$a^2 + 4b^2 + 9 + 4ab - 6a - 12b = a^2 + 4b^2 - 9$$

After simplification:

$$2ab - 3a - 6b + 9 = 0$$

Factorize the left side as:

$$(2b - 3)(a - 3) = 0$$

Now, since the product is zero, there are two cases.

First case: $2b - 3 = 0$, from which $b = \frac{3}{2}$, this is not possible since b has to be an integer.

Second case: $a - 3 = 0$, from which $a = 3$.

So Albrecht gets the correct result if and only if $a = 3$ and in this case he can substitute any positive integer for b.

(Back to problems)

2. For the solution, see Category C Problem 5.

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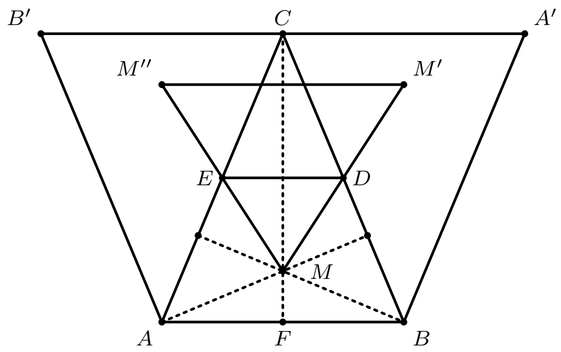
3. **a)** Yes, for example the sum of divisors is 12 for 6 and 11 as well.

b) First of all we prove that the product of the positive divisors of n is $n^{\frac{d(n)}{2}}$, where $d(n)$ denotes the number of divisors of n. The divisors can be arranged in pairs, so if $k \mid n$ then $\frac{n}{k} \mid n$ and $k \cdot \frac{n}{k} = n$. From this we get the formula easily.

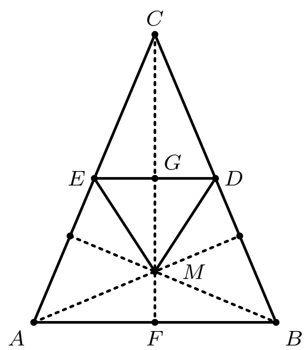
After this we need to prove that if $n^{\frac{d(n)}{2}} = m^{\frac{d(m)}{2}}$ then $n = m$. So if any prime number divides n, then it divides m as well. Let us consider a prime p that divides n. Let k be the positive integer for which $p^k \mid n$ but $p^{k+1} \nmid n$, and similarly let l be the number for which $p^l \mid m$ but $p^{l+1} \nmid m$. In this case since $n^{\frac{d(n)}{2}} = m^{\frac{d(m)}{2}}$, therefore the exponent of p is equal on both sides, so $\frac{k \cdot d(n)}{2} = \frac{l \cdot d(m)}{2}$, this means that $\frac{k}{l} = \frac{d(m)}{d(n)}$. This is true for all prime numbers, so if $d(n) < d(m)$, then the exponent of prime p, where p divides n, is greater in n than in m, so $n > m$ and every divisor of m divides n as well, therefore the products of divisors cannot be equal. Similarly $d(n) > d(m)$ is not possible either, so for $d(n) = d(m)$ to hold, the exponent of every prime should be equal in n and in m, which is equivalent to $n = m$.

(Back to problems)

4. Let us use the notations from the figure. Let the midpoints of the sides of triangle ABC be D, E, F respectively, the orthocenters of ABC, $A'BC$ and $B'AC$ be M, M' and M'' respectively.



Since triangles ABC and $A'BC$ are mirror images of each other with respect to point D, this also holds for their orthocenters, so the midpoint of section MM' has to be D. Similarly the midpoint of section MM'' is point E. This means that segment DE is a midline in triangle $MM'M''$, so triangle DEM is also equilateral. From this follows that M point is on the perpendicular bisector of segment DE, which is also perpendicular to side AB, since DE is the midline corresponding to side AB in trangle ABC. So vertex C is also on the perpendicular bisector of DE, this means that triangle ABC is isosceles.



Now we know that triangle ABC is isosceles and triangle DEM is equilateral. Assume that point M lies between lines DE and AB. Let the midpoint of segment DE be G. Let us denote the length of the altitude of ABC through C (segment CF) be m. Since segment DE is midline, the length of segment CG is $\frac{m}{2}$. Triangle MED is equilateral, so the altitude over side DE (segment GM) is of length $\frac{\sqrt{3}}{2}DE = \frac{\sqrt{3}}{4}$. The angles of triangles AFC and MFA are pairwise the same, so they are similar, so

$$\frac{AF}{FC} = \frac{MF}{FA},$$

$$MF = \frac{AF}{FC} \cdot FA = \frac{\frac{1}{2}}{m} \cdot \frac{1}{2} = \frac{1}{4m}.$$

Since $CF = CG + GM + MF$ we get the following equation:

$$m = \frac{m}{2} + \frac{\sqrt{3}}{4} + \frac{1}{4m},$$

$$\frac{m}{2} - \frac{\sqrt{3}}{4} - \frac{1}{4m} = 0,$$

$$2m^2 - \sqrt{3}m - 1 = 0,$$
$$m_{1,2} = \frac{\sqrt{3} \pm \sqrt{11}}{4}.$$

Since the length of the altitude is positive, the length of the altitude of ABC through C is $m = \frac{\sqrt{3}+\sqrt{11}}{4}$.

(Back to problems)

5. We will solve the problem generally for any n.

Observe that it is enough to determine the parity of the elements of the table, as we can reconstruct any element if we know the parity of every element (we just count the number of even elements in its row and column). From now on we will denote odd elements as 1 and even ones as 0.

In this case we will prove that two adjacent columns are either the same in every entry (by row) or different in each one. From this it would follow that the same holds for any two columns and by symmetry for any two rows.

It is enough to prove the statement for subtables with 2 rows and columns (we define subtable as the intersection of some adjacent rows and columns). Let our table look like this:

A	C	E
B	D	F
G	H	

Where the letters denote the number of even numbers in their respective regions. Evidently the parity of an element in the subtable in top-left corner is the opposite of the parity of the letter corresponding to it, but it is still enough to prove that A, B and C, D are either the same or are opposite.

Now we can write the following congruences based on our 2×2 subtable:

$$A \equiv C + E + B + G \pmod 2$$
$$B \equiv D + F + A + G \pmod 2$$
$$C \equiv A + E + D + H \pmod 2$$
$$D \equiv B + F + C + H \pmod 2$$

The sum of these equations is:

$$A + B + C + D \equiv 2A + 2B + 2C + 2D + 2E + 2F + 2G + 2H \equiv 0 \pmod 2$$
$$A + C \equiv B + D \pmod 2$$

Which means that if $A \equiv C \pmod 2$, then $B \equiv D \pmod 2$, and if $A \not\equiv C \pmod 2$, then $B \not\equiv D \pmod 2$, which is what we set out to prove.

We will introduce some notations: We call a table and its elements self-descriptive if they satisfy the properties required in the statement of this problem. We will denote the intersection of the i-th row and j-th column as (i, j), and the value of (i, j) as $f(i, j)$. We call an element good if the parity of the number of even values in its row and column (without itself) is the same as its parity. Let's call a table and its elements nice if every column is either the same as the first one or the opposite. Observe that we proved above that every self descriptive table is nice.

Now we will show that if n is even then there is only one self-descriptive table, which is the one where every entry is even (therefore $n - 2$).

Let us suppose that there is an odd element in a self-descriptive table. As our statement is invariant for the reordering of rows and columns we can suppose that $f(1, 1) = 1$. According to the description of the problem this means that either the first row or column has an odd number of even values, thus there is at least one even number in the same row or column. Therefore we can suppose that $f(1, 2) = 0$. Let x be the number of even values in the first column. As the table is nice and in the first row the first two elements are not the same, the first two columns must be the inverse of each other. Therefore the number of even numbers in the second column must be $n - x$. Let k be the number of even numbers in the first row excluding the first element.

1	0	k
x	$n - 1 -$ $-x$	

Let us write down the statement of the problem for the element $(1, 1)$:

$$1 \equiv k + 1 + x \pmod 2$$

And for element $(1, 2)$:

$$0 \equiv k + n - 1 - x \equiv k + x + 1 \pmod 2$$

Contradiction.

In the case where n is odd we will prove that there are 2^{2n-2} different self-descriptive tables.

Let us choose the elements of first row except for the last element and the entire first column. These are 2^{2n-2} cases. We will prove that we can complete the table exactly one way for every choice. As $(1,1)$ must be a good element, it determines the value of $(1,n)$. As the table must be nice and we know the first column and the first element in every column we can determine the value of any element. Now we only need to prove that every such table is self-descriptive. This requires that every element is good.

We know that $(1,1)$ is good. Let us prove that the first row and column is good, or equivalently that $(1,2)$ is good. If $f(1,1)=f(1,2)$ then the entire second column is the same as the first one thus the number of even numbers is the same in their columns, and they are in the same row, and their value is the same, thus the sum corresponding to them must be the same as well. If $f(1,1)\neq f(1,2)$ then let x-be the number of even numbers in the first column except for $(1,1)$, as $f(1,1)\neq f(1,2)$ the second row without $(1,2)$ must have $n-1-x$ even entries, which as $n-1$ is even means that the sum in the column of $(1,1)$ and $(1,2)$ is the same (without the first row), but the sum in their row without themselves is different, as they are in the same row, but have different values. Therefore the sum will differ by one between $(1,1)$ and $(1,2)$ which is what we wanted to prove. (And from that we have also proved that the entire first row is good).

We can use the same method to prove that any (i,j) is good from the fact that $(1,j)$ is good. Therefore the table is self-descriptive.

In conlusion: if n is even the answer is 1, if it is odd there are 2^{2n-2} different tables. So the answers: **a)** $2^4=16$, **b)** 1, **c)** $2^8=256$.

(Back to problems)

2.2.4 Category E$^+$

1. a) Yes, for example the sum of divisors is 12 for 6 and 11 as well.

b) No, it is not possible that the product of the divisor of two different numbers are equal. First of all we prove that the product of the positive divisors of n is $n^{\frac{d(n)}{2}}$, where $d(n)$ denotes the number of divisors of n. The divisors can be arranged in pairs, so if $k\mid n$ then $\frac{n}{k}\mid n$ and $k\cdot\frac{n}{k}=n$. From this we get the formula easily.

After this we need to prove that if $n^{\frac{d(n)}{2}}=m^{\frac{d(m)}{2}}$ then $n=m$. So if any prime number divides n, then it divides m as well. Let us consider a prime p that divides n. Let k be the positive integer for which $p^k\mid n$ but $p^{k+1}\nmid n$, and similarly let l be the number for which $p^l\mid m$ but $p^{l+1}\nmid m$. In this case since $n^{\frac{d(n)}{2}}=m^{\frac{d(m)}{2}}$, therefore the exponent of p is equal on both sides, so $\frac{k\cdot d(n)}{2}=\frac{l\cdot d(m)}{2}$, this means that $\frac{k}{l}=\frac{d(m)}{d(n)}$. This is true for all prime numbers, so if $d(n)<d(m)$, then the exponent of prime p, where p divides n, is greater in n than in m, so $n>m$ and every divisor of m divides n as well, therefore the products of divisors cannot be equal. Similarly $d(n)>d(m)$ is not possible either, so for $d(n)=d(m)$ to hold, the exponent of every prime should be equal in n and in m, which is equivalent to $n=m$.

(Back to problems)

2. Observe that it is enough to determine the parity of the elements of the table, as we can reconstruct any element if we know the parity of every element (we just count the number of even elements in its row and column). From now on we will denote odd elements as 1 and even ones as 0.

In this case we will prove that two adjacent columns are either the same in every entry (by row) or different in each one. From this it would follow that the same holds for any two columns and by symmetry for any two rows.

It is enough to prove the statement for subtables with 2 rows and columns (we define subtable as the intersection of some adjacent rows and columns). Let our table look like this:

A	C	E
B	D	F
G	H	

Where the letters denote the number of even numbers in their respective regions. Evidently the parity of an element in the subtable in top-left corner is the opposite of the parity of the letter corresponding to it, but it is still enough to prove that A, B and C, D are either the same or are opposite.

Now we can write the following congruences based on our 2×2 subtable:

$$A \equiv C + E + B + G \pmod 2$$

$$B \equiv D + F + A + G \pmod 2$$
$$C \equiv A + E + D + H \pmod 2$$
$$D \equiv B + F + C + H \pmod 2$$

The sum of these equations is:

$$A + B + C + D \equiv 2A + 2B + 2C + 2D + 2E + 2F + 2G + 2H \equiv 0 \pmod 2$$

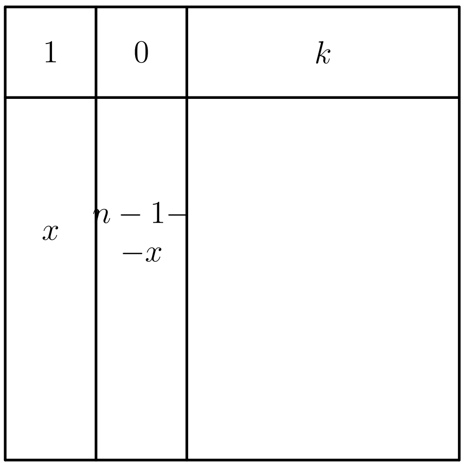
$$A + C \equiv B + D \pmod 2$$

Which means that if $A \equiv C \pmod 2$, then $B \equiv D \pmod 2$, and if $A \not\equiv C \pmod 2$, then $B \not\equiv D \pmod 2$, which is what we set out to prove.

We will introduce some notations: We call a table and its elements self-descriptive if they satisfy the properties required in the statement of this problem. We will denote the intersection of the i-th row and j-th column as (i, j), and the value of (i, j) as $f(i, j)$. We call an element good if the parity of the number of even values in its row and column (without itself) is the same as its parity. Let's call a table and its elements nice if every column is either the same as the first one or the opposite. Observe that we proved above that every self descriptive table is nice.

Now we will show that if n is even then there is only one self-descriptive table, which is the one where every entry is even (therefore $n - 2$).

Let us suppose that there is an odd element in a self-descriptive table. As our statement is invariant for the reordering of rows and columns we can suppose that $f(1,1) = 1$. According to the description of the problem this means that either the first row or column has an odd number of even values, thus there is at least one even number in the same row or column. Therefore we can suppose that $f(1,2) = 0$. Let x be the number of even values in the first column. As the table is nice and in the first row the first two elements are not the same, the first two columns must be the inverse of each other. Therefore the number of even numbers in the second column must be $n - x$. Let k be the number of even numbers in the first row excluding the first element.



Let us write down the statement of the problem for the element $(1,1)$:

$$1 \equiv k + 1 + x \pmod 2$$

And for element $(1,2)$:

$$0 \equiv k + n - 1 - x \equiv k + x + 1 \pmod 2$$

Contradiction.

In the case where n is odd we will prove that there are 2^{2n-2} different self-descriptive tables.

Let us choose the elements of first row except for the last element and the entire first column. These are 2^{2n-2} cases. We will prove that we can complete the table exactly one way for every choice. As $(1,1)$ must be a good element, it determines the value of $(1,n)$. As the table must be nice and we know the first column and the first element in every column we can determine the value of any element. Now we only need to prove that every such table is self-descriptive. This requires that every element is good.

We know that $(1,1)$ is good. Let us prove that the first row and column is good, or equivalently that $(1,2)$ is good. If $f(1,1) = f(1,2)$ then the entire second column is the same as the first one thus the number of even numbers is the same in their columns, and they are in the same row, and their value is the same, thus the sum corresponding to them must be the same as well. If $f(1,1) \neq f(1,2)$ then let x-be the number of even numbers in the first column except for $(1,1)$, as $f(1,1) \neq f(1,2)$ the second row without $(1,2)$ must have $n-1-x$ even entries, which as $n-1$ is even means that the sum in the column of $(1,1)$ and $(1,2)$ is the same (without the first row), but the sum in their row without themselves is different, as they are in the same row, but have different values. Therefore the sum will differ by one between $(1,1)$ and $(1,2)$ which is what we wanted to prove. (And from that we have also proved that the entire first row is good).

We can use the same method to prove that any (i,j) is good from the fact that $(1,j)$ is good. Therefore the table is self-descriptive.

In conlusion: if n is even the answer is 1, if it is odd there are 2^{2n-2} different tables.

Second solution: As in the previous solution we only need to determine the parity of each element, therefore we will count modulo 2 in the proof. Let $t_{i,j}$ be the value corresponding to the intersection of row i and column j modulo 2. Let s_i and o_i respectively be the sum of elements in a row or column modulo 2. Then

$$t_{i,j} \equiv s_i + o_j \tag{1}$$

.
So if we know all s_i and o_i for every i, then we can detemine every value in the table. We can express s_i from $t_{i,j}$ modulo 2, as we counted $t_{i,j}$ if $t_{i,j} \equiv 0$. Formally:

$$s_i \equiv \sum_{j=1}^{n}(1 + t_{i,j})$$

We can write this using (1) as:

$$s_i \equiv \sum_{j=1}^{n}(1 + t_{i,j}) \equiv \sum_{j=1}^{n}(1 + s_i + o_j) \equiv n + ns_i + \sum_{j=1}^{n} o_j \tag{2}$$

Similarly for o_i:

$$o_i \equiv \sum_{j=1}^{n}(1 + t_{j,i}) \equiv \sum_{j=1}^{n}(1 + s_j + o_i) \equiv n + no_i + \sum_{j=1}^{n} s_j$$

If n is even then the last two equations:

$$s_i \equiv \sum_{j=1}^{n} o_j \equiv 0$$

$$o_i \equiv \sum_{j=1}^{n} s_j \equiv 0$$

As the sums on the right sides are independent from i, therefore all of s_i and o_j are equal. But in this case we added the same number to itself even times thus the value is 0. Therefore $t_{i,j} = 0$ for every i,j. If n is even then this is the only solution.

If n is odd then from equation (2):

$$1 \equiv -n + (1-n)s_i \equiv \sum_{j=1}^{n} o_j \tag{3}$$

And alike:

$$1 \equiv \sum_{j=1}^{n} s_j \tag{3}$$

Let us choose s_i and o_i for every $i < n$. We will prove that this determines one unique solution. s_n and o_n are determined by the congruences (3), and from that (1) determines every $t_{i,j}$. For every choice of o_i and s_j we get a different table, so we only need to prove that every such solution is self-descriptive. This is true as:

$$t_{k,l} \equiv s_k + o_l \equiv 1 - \sum_{\substack{i=1\\ i\neq k}}^{n} s_i + 1 - \sum_{\substack{j=1\\ j\neq l}}^{n} s_i + (n-1)(s_k + o_l) \equiv \sum_{\substack{i=1\\ i\neq k}}^{n} (s_i + o_l) + \sum_{\substack{j=1\\ j\neq l}}^{n} (s_k + o_j) \equiv \sum_{\substack{i=1\\ i\neq k}}^{n} t_{i,l} + \sum_{\substack{j=1\\ j\neq l}}^{n} t_{k,j}$$

Which is what we wanted to prove.

In conlusion: if n is even the answer is 1, if it is odd there are 2^{2n-2} different tables.

(Back to problems)

3. Instead of solving the problem for 2019, we generalise it and solve for n, where $n > 3$ odd integer. For even n and $n = 3$ we can get a solution using the same ideas.

a) Answer: $n + 3$.

It is clear that we need at least $n + 1$ numbers. First we prove that $n + 1$ and $n + 2$ is not enough, then we show that there is a construction for $n + 3$.

Suppose that we have $n + 1$ numbers that satisfy the condition and let them be $a_1 \leq a_2 \leq ... \leq a_{n+1}$. Then $a_1 = a_2 + a_3 + ... + a_{n+1}$ and $a_{n+1} = a_1 + a_2 + ... + a_n$. Subtracting the second equality from the first one: $a_1 - a_{n+1} = a_{n+1} - a_1$, so $a_1 = a_{n+1}$, which means that $a_i = a_j$ for all i, j. Substituting back to the first equation, $a_1 = a_2 + a_3 + ... + a_{n+1} = n \cdot a_1$, which is not possible as $n > 1$ and $a_1 \neq 0$.

Suppose that we have $n + 2$ numbers that satisfy the condition and let them be $a_1 \leq a_2 \leq ... \leq a_{n+2}$. Then a_1 is equal to a sum of n other a_i, so $a_1 \geq a_2 + a_3 + ... + a_{n+1}$. Similarly, $a_{n+2} \leq a_2 + a_3 + ... + a_{n+1}$, which means that $a_1 \geq a_{n+2}$, so $a_1 = a_{n+2}$. As seen above, it is not possible, so $n + 2$ numbers are not enough.

It is enough to write down $n+3$ numbers: Write down $\frac{n+3}{2}$ (-1)-s and $\frac{n+3}{2}$ 1-s. Then every (-1) can be written as a sum of $\frac{n+1}{2}$ (-1)-s and $\frac{n-1}{2}$ 1-s, and similarly every 1 is the sum of $\frac{n+1}{2}$ 1-s and $\frac{n-1}{2}$ (-1)-s. So this is a good construction for $n + 3$ numbers.

b) Answer: $n + 4$.

Because of the above, we need at least $n + 3$ numbers.

First we prove that $n + 3$ numbers are not enough by contradiction: Suppose that we have $n + 3$ numbers satisfying the condition, and let the numbers be $a_1 < a_2 < ... < a_{n+3}$. Then a_1 can be written as a sum of n other numbers, so $a_1 \geq a_2 + a_3 + ... + a_{n+1}$. Similarly $a_{n+3} \leq a_3 + a_4 + ... + a_{n+2}$. Subtracting the first inequality from the second one $a_{n+3} - a_1 \leq a_{n+2} - a_2$, but this is not possible, since $a_1 < a_2$ and $a_{n+2} < a_{n+3}$.

Now we show that there is a solution with $n + 4$ numbers. Write n as $2k + 1$, where $k > 1$ is an integer. Then our construction for $n + 4 = 2k + 5$ numbers is: $-(k + 3),\ -(k + 2),\ \dots,\ -2,\ 1,\ 2,\ \dots,\ (k + 3)$.

The sum of these numbers is 1. When we want to write up l as a sum of $n = 2k + 1$ other numbers, then the 3 numbers that are not in the sum and are not l, sum to $1 - 2l$. The converse is also true, because if we find 3 numbers not equal to l, which sum to $1 - 2l$, then the remaining

$2k+1$ numbers sum to l. Because of this, it is enough to show that the following is true for our numbers: for any l from the numbers there are 3 other numbers, which sum to $1-2l$.

For $l=-k-3$: $2,\ (k+2),\ (k+3)$ sum to $1-2l$.
$3 \leq -l \leq (k+2)$: $1,\ (-l-1),\ (-l+1)$ sum to $1-2l$.
$l=-2$: $-3, 3, 5$ sum to $1-2l$ (only if $5 \leq (k+3)$, so $k \geq 2$).
$l=1$: $4, -2, -3$ sum to $1-2l$.
$l=2$: $3, -2, -4$ sum to $1-2l$.
$3 \leq l \leq (k+2)$: $1,\ (-l-1),\ (-l+1)$ sum to $1-2l$.
$l=k+3$:$-2,\ -k-1,\ -k-2$ sum to $1-2l$ (only if $k \geq 2$, so $-k-1 \leq -2$).
So our construction satisfies the condition.

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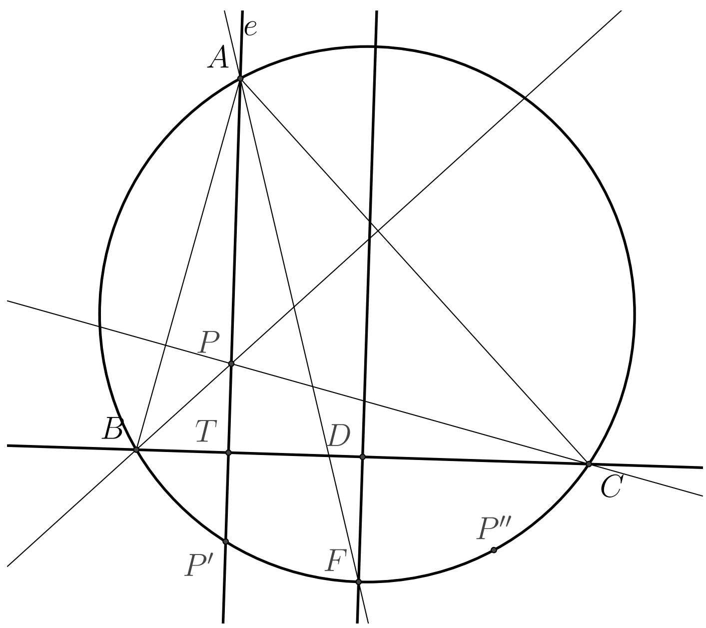
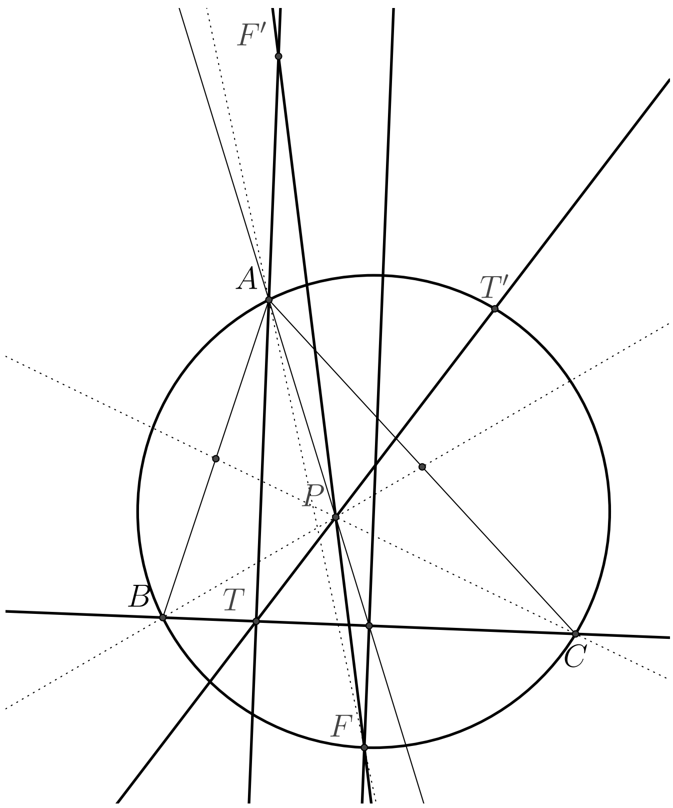
4. Throughout the solution let T denote the given foot of the altitude and let F denote the given midpoint of the arc.

a) First construct line PT, let this be e. Let the intersection of the line through F parallel with e and the line through T perpendicular to e be D. Then point D is the midpoint of side BC. Reflect P by T and D, let these points be P' and P'' respectively. It is known that P' and P'' lie on the circumscribed circle of ABC, this can be verified by some angle chasing. P', F and P'' are pairwise different since ABC is scalene, so by constructing the circle $P'FP''$ we get the circumscribed circle of ABC, intersecting it with lines PT and TD we get points A, P', B and C.

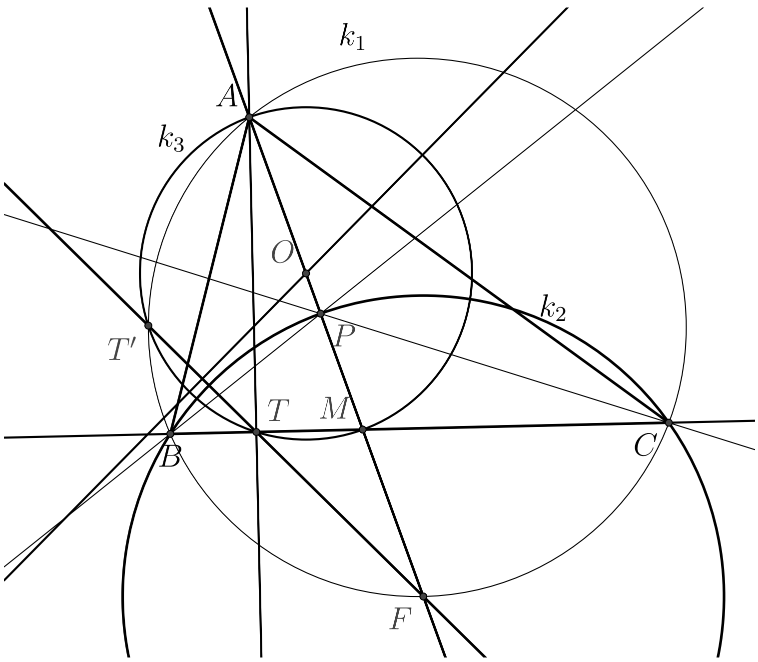


b) We will use the following fact multiple times: the centroid divides the medians in the ratio of 2 : 1. It follows from this fact that point T', which we get by enlarging point T from P with scale factor -2, lies on the circumcircle of ABC and coincides with the the reflection of point A by the perpendicular bisector of side BC. It also follows that F', which we get by enlarging point F from point P with scale factor -2, lies on line AT. Now let us start the construction. We can construct points F' and T'. Consider the line that is the enlargement of line TF' from P with scale factor $-1/2$. This will be the perpendicular bisector of side BC; by reflecting T' by this line, we get A. Since triangle ABC is scalene, this means that $A \neq T'$ so we can construct the circumscribed circle of ABC, knowing its three different points. By intersecting this circle with the line perpendicular to TF' through T, we get B and C.



c) A few remarks. Let the circumcircle of triangle ABC be k_1. It is known that points B, P and C lie on a circle with center F let this be k_2 (this can be verified by angle chasing). Let the intersection of lines AF and BC be M. Let the circle with diameter AM be k_3, denote its center with O. Since T is the foot of the altitude through A, triangle ATM is right, this means that T lies on k_3. Let T' denote the second intersection of line FT with k_3. Since the inversion with respect to k_2 maps k_1 to line BC, so the image of pont M is point A, so the image of k_3 is itself, it means that the image of T is T'.

The construction: (we are using the same notations as in the remarks). First construct k_2 (knowing its center and one point on the perimeter). Construct T', the image of T in the inversion with respect to k_2. Now the intersection of line FP and the perpendicular bisector of TT' is O, so we can construct k_3. The intersection of k_3 and line FP farther from F will be A. The intersections of k_2 and the line through T perpendicular to AT will be points B and C.



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5. Throughout the proof we will compute modulo p, every congruence is modulo p.
Let $Q(x)$ be the polynomial. Assume that $k \mid p-1$, $k>1$ and Q attains every possible value modulo p. Since $Q(x) \equiv Q(x+p)$, the polynomial attains different values at $x = 0, 1, 2, \dots, p-1$.
Lemma:

$$\sum_{j=0}^{p-1} j^t \equiv \begin{cases} -1 & \text{if } t = p-1 \\ 0 & \text{if } t = 0, 1 \ldots p-2 \end{cases}$$

Proof: It is known that there exists a primitive root g modulo p. If $t \neq p-1$, by the sum of the geometric series:

$$\sum_{j=0}^{p-1} j^t \equiv \sum_{j=0}^{p-2} g^{jt} \equiv \frac{g^{(p-1)t}-1}{g^t-1} \equiv 0$$

If $t = p - 1$ then $x^t \equiv 1$ when $x \not\equiv 0$, by Fermat's little theorem. So

$$\sum_{j=0}^{p-1} x^{p-1} \equiv p - 1 \equiv -1.$$

With this we have proven the lemma.

Now suppose indirectly that the leading coefficient of Q is not divisible by p. Let $c = \frac{p-1}{k}$ and let $R(x) = Q^c(x) = \sum_{i=0}^{p-1} b_i x^i$. Firstly we can see that in R the coefficient b_{p-1} is not 0 modulo p since the leading coefficient of Q is nonzero and $b_{p-1} = a_k^c$.

Let us consider the following sum:

$$\sum_{j=0}^{p-1} R(j)$$

Since $k > 1$ and k is a divisor of $p-1$, c is a positive integer smaller than $p-1$. By our hypothesis $Q(x)$ attains every value modulo p, in other words $Q(0), Q(1), ...Q(p-1)$ is a permutation of $0, 1...p-1$, so the sum can we written the following way:

$$\sum_{j=0}^{p-1} R(j) = \sum_{j=0}^{p-1} Q^c(j) \equiv \sum_{j=0}^{p-1} j^c \equiv 0$$

The last congruence follows from our lemma. And writing the sum in a different way:

$$\sum_{j=0}^{p-1} R(j) = \sum_{j=0}^{p-1}\sum_{i=0}^{p-1} b_i j^i = \sum_{i=0}^{p-1} b_i \sum_{j=0}^{p-1} j^i \equiv -b_{p-1}$$

Now we used the lemma again. So $0 \equiv -b_{p-1}$, which is a contradiction since we have shown already that b_{p-1} is nonzero.

So the leading coefficient of Q must be divisible by p.

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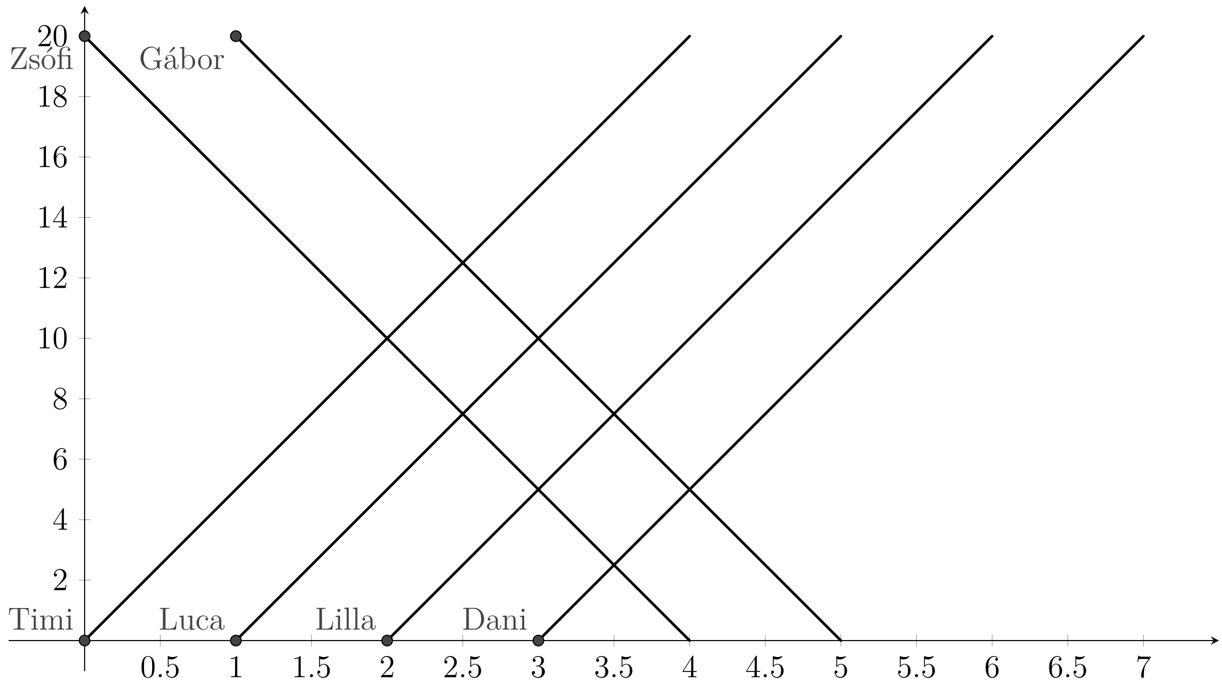
2.3 Final round – day 1

2.3.1 Category C

1. People starting at Miskolc meet with Zsófi first and then with Gábor, while those starting at Szikszó will meet Timi, Luca, Lilla and Dani in this order. The figure below represents how far each person is (in km) from Miskolc depending on the time passed since 8:00 (in hours).



If Timi has Marvin to begin with, then she meets with Zsófi first at 10:00, hence Marvin will be passed to Zsófi. The next person Zsófi meets is Luca (at 10:30), when she passes Marvin to her. Luca, on her way to Szikszó, only meets Gábor after Zsófi (at 11:00), so Marvin will be with him, until he passes Marvin to Lilla (at 11:30). Lilla will not meet anyone after Gábor, so she ends up carrying Marvin to Szikszó. Therefore, if Marvin starts off with Timi, then it ends up in Szikszó.

If Marvin starts off with Luca, then she passes it to Zsófi at 10:30. Then Zsófi meets Lilla at 11:00 passing Marvin to her. Lilla then meets Gábor on her way to Szikszó at 11:30, after which Gábor meets Dani at 12:00. Therefore, if Marvin starts off with Luca, then it will end up with Dani going to Szikszó.

If Marvin is with Lilla to begin with, then she will pass it to Zsófi at 11:00. The next person Zsófi meets is Dani (at 11:30), passing Marvin to him. Dani then only meets Gábor on his way to Szikszó (at 12:00). Since Gábor does not meet anyone after Dani, if Marvin starts off with Lilla, then it ends up in Miskolc.

If Dani has Marvin to begin with, then he passes it to the first person he meets, Zsófi at 11:30. Zsófi does not meet anyone after meeting with Dani, hence in this case, she ends up carrying Marvin to Miskolc.

To conclude, Marvin can only start off with Timi or Luca to end up in Szikszó.

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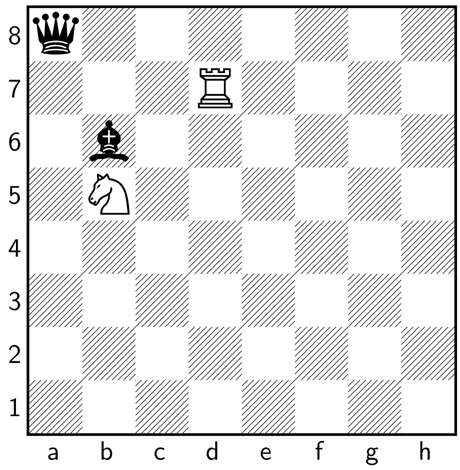
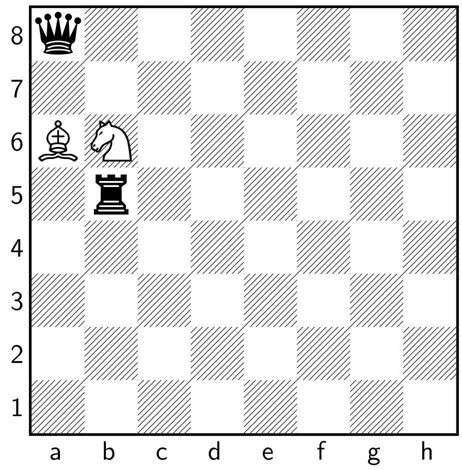
2. The answer for parts **a)** and **b)** are yes. The following are possible solutions.





Now we show that there is no solution for part **c)** . Suppose for contradiction that we can arrange the pieces according to the requirements of the problem. Observe that since every piece can capture exactly two others, there must be exactly 8 possible captures. Since all pieces can capture the same number of other pieces, this must be $8/4 = 2$. So even the knight can capture two other pieces. None of these can capture the knight since neither of the queen, bishop and rook can move in this fashion. Thus at most one piece can capture the knight, a contradiction.

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3. We show that it is possible to fill the table in the cases **a)**, **b)** and **d)**
The following table is good for parts **a)** and **b)**

4	9	2
3	5	7
8	1	6

In this table the sum of the elements in each row, column and diagonal is 15, so every sum is divisble by 3 and 5, respectively.

Naturally, in part **b)** it is not necessary to find an example, where each sum is divisible by 3, just as in part **a)** it is not necessary to find an example, where each sum is divisible by 5, this is just an addition. The sums do not even have to be equal.

The 3×3 magic square perfectly fits to this competition, the little brother of the Dürer 4×4 magic square!

The following example is good for part **d)**, and consequently for part **a)**:

1	8	9
2	3	4
6	7	5

Here every sum is divisibly by 9 since they are 9 or 18.

There is no table for part **c)** . We prove this statement indirectly. Suppose for contradiction that there exists a table, where each sum is divisible by 7. Let S denote the sum of all numbers in the table. Furthermore, let S_1, S_2, S_3 denote the sum of the numbers in the first, second an third row, respectively. Then $S = 1 + 2 + 3 + \ldots + 9 = \frac{9 \cdot 10}{2} = 45$ for all tables.
On the other hand, 7 divides S_1, S_2, S_3, and so their sum $S_1+S_2+S_3$. Note that $S = S_1+S_2+S_3$ since the sum of all numbers in the table is just the sum of row sums. Hence 7 divides $S = 45$, but this is, of course, a contradiction. Note that we did not even use that 7 divides the column and diagonal sums.

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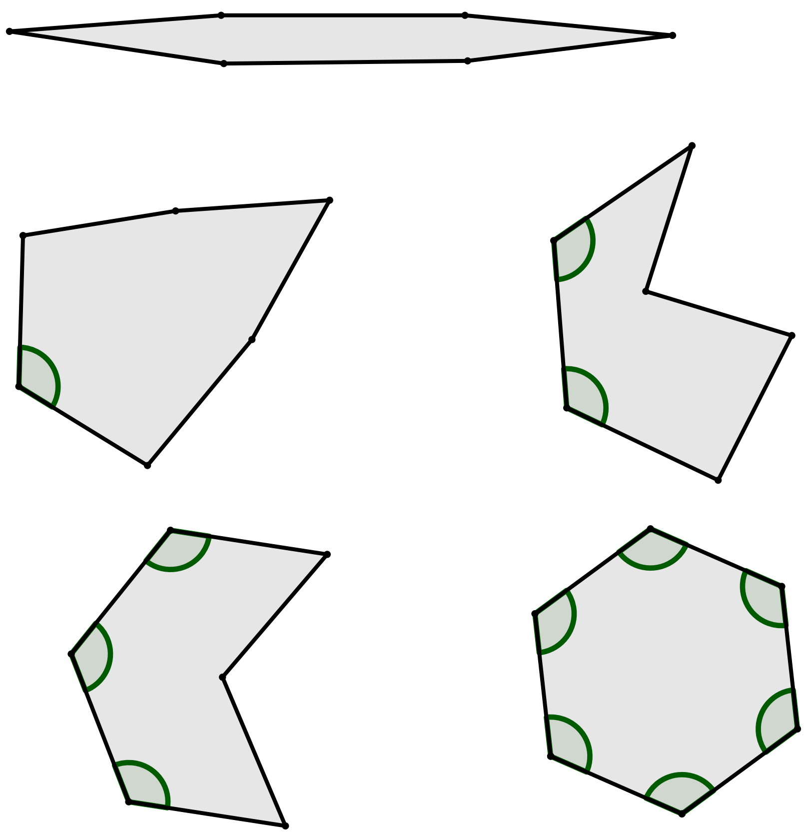
4. Since a hexagon has 6 angles, the possible values are between 0 and 6.

We show that the possible values for the number of nice angles are 0, 1, 2, 3 and 6. The figures at the end of the solution show that these values are indeed possible.

Let us show that other values are not possible.

Suppose that a hexagon has at least 4 nice angles. We show that in fact in this case there are 6 nice angles. Suppose that the two potentially non-nice angles are at vertices A and B (that are not necessarily consecutive). As all the sides have equal length, the part between A and B is uniquely determined on both sides. Hence the hexagon itself is also uniquely determined. Since the regular hexagon is a solution, it is the only solution. However, there the number of nice angles is 6, hence if there are at least 4 nice angles, there are in fact 6 of them.



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5. We show that one needs at least 253 shots, and this is also enough to destroy the tank.

First we describe how to destroy the tank. We first colour the fields of the table with black and white in a chessboard-like fashion such that the corners are black. First let us shoot at all the white fields. After these shots the tank must be on a black field since if originally it was on a white field, then it got a hit and moved to a black field. If the tank was on a black field, then it did not get shot and so stayed on its field. Then let us shoot at all the black fields. Now the tank got hit exactly once, and either it got destroyed (if it started on a white field), or moved to a white field. Then let us shoot again at all the white fields. If the tank was not destroyed before, then the tank now is surely destroyed. This way we used $84+85+84=253$ shots.

Next we show that fewer shots cannot be enough. If there were two neighbouring fields at which alltogether we shot at most two times, then it may occur that the tank was on the field that got the second –or just last if there was at most one shot– shot and moved to the other one. In this case the tank will be not destroyed. If we cover the 13×13 table except one field with 1×2 dominos, then we need at least $\frac{13\cdot 13-1}{2}\cdot 3+1=253$ shots since the exceptional field needs at least one shot too (otherwise the tank may stay there safely). So we need at least 253 shots.

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6. We show that the second player has a winning strategy. Let us number the fields from 1 to 9:

1	2	3
4	5	6
7	8	9

If the first player puts a disk to field 5, then the second should put one on the field 1. If then the first puts a disk to the field 9, then the second puts one on the field 3. From now on the second player can achieve a draw in the first part of the game. If the first player does not put her disk to the field 9 in the second move, then the second player can still achieve the draw in the first part. Then in the 10-th round the second player colors the field 5 to purple. Then no matter which disk the first player colours to purple, we can colour the opposite one to purple thereby winning the game.

If the first player does not put a disk on field 5, then the second player puts one on the field 5. From now on the second player can achieve a draw in the first part of the game using logical moves. The board can look two different ways as follows (not counting ones that can be obtained by rotations and reflections). Here x denotes the fields of the second player fields.

x		x
	x	
	x	

x		
	x	x
	x	

In the first case the second player colours the field 4 to purple. If the first player then colours field 1 or 5, then the second player can win in the next move. If the first player instead colours the field 3 or 8, the the second player colours the field 9, and can win in the next round.

In the second case the second player colours the field 3 to purple. Then the first player should colour the field 8 in order not to lose in the next round. In this case the first player can colour field 4. Then whatever the first player colours the second player can win in the next round.

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2.3.2 Category D

1. For the solution, see Category C Problem 3.

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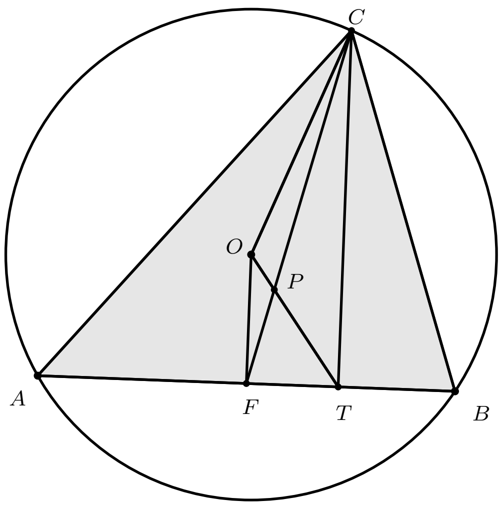
2. Let F be the midpoint of the segment AB. Furthermore, let P be the intersection of the segments FC and OT. It is sufficient to show that the area of the quadrilateral $BTOC$ is half of the triangle, that is, it has the same area as the triangle FBC since the median FC divides the original triangle to two triangles of equal area. Since O is the center of the circumscribed circle, it is on the perpendicular bisector of the segment AB, that is, $AB \perp FO$. The areas of the triangles TCO and TCF coincide as they have a common side and the length of the corresponding altitudes. Hence $T(BTOC) = T(BTC_\triangle) + T(TCO_\triangle) = T(BTC_\triangle) + T(TFC_\triangle) = T(BFC_\triangle)$.

Note: We only used the fact about the point O that it is inside the triangle ATC and incident to the perpendicular bisector of the segment AB. So every point with these properties the statement remain true.



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3. **a)** We first show that if a 1×9 or 9×1 rectangle has disks on its first 3 fields, then these disks can be transported to the last 3 fields.

Let us number the fields from 1 to 9. Initially, the disks are on fields 1,2 and 3. First reflect the disk on field 1 to the disk on field 3. Then we have disks on 2,3 and 5. Reflecting 2 and 3 to 5, we get to 5, 7 and 8. Now reflecting 5 to 7 we get to 7,8,9.

Using this, we show that the desired configuration can be reached.

First do the above described steps for the 3 lowest rows of the table. Then the 9 disks are in the lower-right 3×3 table, one on each field.

Now do the steps for the 3 rightmost columns. Then the 9 disks are in the upper-right 3×3 table, one on each field, hence we are ready.

b) Denote the field of the table by coordinates (x, y) for $(1 \leq x, y \leq 8)$. Initially, the disks are on the fields (x, y), $(1 \leq x, y \leq 3)$.

Let us look at how the parity of the coordinates change upon a reflection of (x, y) to (x_0, y_0). To be able to reflect, either $x = x_0$ or $y = y_0$. Suppose that $x = x_0$. Then after reflection, the new second coordinate is $y + 2 \cdot (y_0 - y)$. The parity of this number is the same as the parity of y as we added an even number to it. Hence upon a reflection, the parity of both coordinates remains the same.

Initially, there are 4 disks with (odd,odd) coordinates. However, in the desired final state, there is only one disk with (odd, odd) coordinates.

Hence the desired state cannot be reached by the reflections.

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4. For the solution, see Category C Problem 4.

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5. We prove the statement by induction on n. For $n = 1$ we have $2^{2^1} + 2^{2-1} + 1 = 7$ that has at least 1 prime divisor. For $n = 2$ we have $2^{2^2} + 2^{2^{2-1}} + 1 = 21 = 3 \cdot 7$ that has at least two prime divisors.

Now let $n > 1$ and assume that we already proved the statemnt till n.

Observe that if $x = 2^{2^{n-1}}$, then the expression in the problem is $2^{2^n} + 2^{2^{n-1}} + 1 = x^2 + x + 1$, while for $n+1$ we have $2^{2^{n+1}} + 2^{2^{(n+1)-1}} + 1 = x^4 + x^2 + 1$. The right hand side can be factorised as follows:

$$x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1).$$

That is, $2^{2^{n+1}} + 2^{2^{(n+1)-1}} + 1 = (2^{2^n} - 2^{2^{n-1}} + 1)(2^{2^n} + 2^{2^{n-1}} + 1)$. The first term is exactly the previous sum, so all the previous prime divisors will divide the new number too.

We show that the numbers $2^{2^n} - 2^{2^{n-1}} + 1$ and $2^{2^n} + 2^{2^{n-1}} + 1$ are realtively primes. Indeed, if they had a common divisor d, then it would divide their difference which is a 2-power. Since both numbers are odd, there greatest common divisor must be 1.

The first term in the factorisation is greater that 1, so it has at least one prime divisor. From this and the inductive hypothesis we get that the product has at least $n + 1$ different prime divisors.

(Back to problems)

6. Let us call a number n x-winning for $1 \leq x < 13$ if the person stepping from $n-x$ to n have a strategy to win the game. Let us call n a winning number if n is x-winning for each $1 \leq x < 13$. Let us call n a losing number if it is not x-winning for any $1 \leq x < 13$.

m and the numbers larger than m are losing numbers, since we have lost if we step at them.

$m-1$ is a winning number, since the opponent can only step on losing numbers from there.

$m-2$ is only x-winning for $x=12$, since if we do not step on it from $m-14$, then the opponent can step onto $m-1$, but if we step on it from $m-14$, then the opponent can only step on losing states.

$m-3$ is only x-winning for $x=11$, since if we do not step on it from $m-14$, then the opponent can step onto $m-1$, but if we step on it from $m-14$, then the opponent can only step on losing states, or step on $m-2$ from $m-3$, but $m-2$ is not 1-winning.

Similarly, for $2 \leq k \leq 13$, $m-k$ is only x-winning for $x = 14-k$.

$m-14$ is a losing number since no matter which number we add, the opponent can step on a losing position.

From now on, everything that happened between $m-1$ and $m-14$ is repeated periodically. Indeed, from a number smaller than $m-14$, we cannot step to a winning position that is at least $m-14$. (That is, $m-14$ behaves the same way as m.)

Hence winning and losing positions are the following ($k \geq 0$):

x-winning	1	2	3	4	5
number	$m-14k-13$	$m-14k-12$	$m-14k-11$	$m-14k-10$	$m-14k-9$

x-winning	6	7	8	9	10
number	$m-14k-6$	$m-14k-5$	$m-14k-4$	$m-14k-8$	$m-14k-7$

x-winning	11	12	winning number	losing number
number	$m-14k-3$	$m-14k-2$	$m-14k-1$	$m-14k$

Hence if m gives remainder 1 when divided by 14, then the second player has a winning strategy. In all other cases, the first player has a winning strategy.

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2.3.3 Category E

1. For the solution, see Category D Problem 2.

(Back to problems)

2. For the solution, see Category C Problem 5.

(Back to problems)

3. We get the least common multiple as follows: we take the prime divisors of the numbers and we raise it to the highest power that appears among the exponents. Since the least common multiple is a square, all such exponents must be even. We know that primes greater than 3 may divide at most one number, so if a prime divides any of the numbers, then it is on an even exponent in the corresponding number.

Suppose for contradiction that the statement is true. Then there will be a number among them which is not divisible by 2 and 3. So only primes bigger than 3 may divide it, and consequently on even powers, so it must be a perfect square. If the smallest number is 1, then we get no solution. Hence there are at most one perfect squares among the numbers. So the other numbers are not perfect squares.

Since there must be a number divisible by 2, there must be a number where it appears on an even power. This number cannot be a perfect square since the only perfect square is not divisible by 2. So there must be a prime number which appears on an odd power in this number. This prime can only be 3. Similarly, there is a number in which the 3 appears on an positive even power, and the 2 appears on an odd power. But this shows that there are two numbers that are divisible by 6. But there must be at most one number divisible by 6 among 5 consecutive integers, a contradiction.

(Back to problems)

4. a) We show that such n-tuples exist for each n. For a given n, take the tuple $\{-1, -1, -1, \dots, -1, n-1\}$ and the tuple $\{1, 1, 1, \dots, 1, -n+1\}$. We show that for these two tuples, the numbers on the blackboard will be the same.

For a subset in one tuple, take the negatives of the unchosen elements from the other tuple. As the elements in one tuple sum to 0, this way we get two subsets that have the same sum. This way we paired up the subsets such that pairs have the same sum, hence we indeed get the same sums for the two tuples.

If fact, this proof did not even use the actual values of the numbers, and with the same proof, we can prove the following statement:

Lemma. If $\sum_{i=1}^{n} a_i = 0$, then for the tuples $\{a_1, a_2, \dots, a_n\}$ and $\{-a_1, -a_2, \dots, -a_n\}$ the numbers on the blackboard are the same.

All of these examples use negative numbers. Part b says that this is necessary.

b) The emptyset will have sum zero. For every other set, the sum will be positive. Erase zero from the blackboard. Let the remaining smallest number be a_1. a_1 has to be the sum of a one-element set, otherwise it could be made smaller by leaving out an element from the sum. Hence a_1 is on the paper, moreover, there is no smaller number on the paper. Erase a_1 from the blackboard.

Let a_2 be the remaining smallest element on the blackboard. a_2 also has to be the sum of a one-element set, since otherwise there would be at least one element in the sum which is not a_1. Leaving out all but one element so that the remaining one is not a_1, we would get a smaller sum that is still on the blackboard. Erase a_2 and $a_1 + a_2$ from the table.

Let us continue like this. Suppose that we found the $k-1$ first numbers $a_1, a_2, \dots a_{k-1}$ on the paper in increasing order, and we erased them and all the sums of their subsets from the blackboard. Let a_k be the smallest remaining number on the blackboard. We show that a_k is the kth least number on the paper. Since the sum of any subset of $a_1, \dots a_{k-1}$ is already erased from the blackboard, a_k corresponds to a sum containing at least one element other then $a_1, \dots, a_{k-1}$. If a_k would have other summands as well, then we could make it smaller by keeping only the smallest summand different from $a_1, \dots, a_{k-1}$, and that sum would still be on the board. Hence a_k is the smallest number on the paper after a_{k-1}.

Using this method, we can recover each number on the paper.

(Back to problems)

5. Throughout the solution $f^n(x)$ denotes the value obtained by applying the function n times to the number x.

a) We show that there is no such function. Suppose for contradiction that f satisfies the equation. Using the above notation the equation is of the form $x = f(x) - f^2(x)$. Substituting $x := f(x)$ into the equation we get that $f(x) = f^2(x) - f^3(x)$. Adding this equation to the original one we get that $x + f(x) = f(x) - f^2(x) + f^2(x) - f^3(x)$, that is, $x = -f^3(x)$. By substituting $x := 2020$ into this equation we get that $f^3(2020) = -2020$. On the other hand, this contradicts to the fact that $f^3(2020) \in H$ but $-2020 \notin H$. Hence there is no such function.

b) The equation has the form $x = f(x) + f^2(x) - f^3(x)$. Let us substitute $x := f(x)$ into the equation. Then we get that

$$f(x) = f^2(x) + f^3(x) - f^4(x).$$

Adding this equation to the original one we get that

$$x + f(x) = f(x) + 2f^2(x) - f^4(x).$$

After rearrangement we get that

$$x - f^2(x) = f^2(x) - f^4(x).$$

We show by induction on n that $x - f^2(x) = f^{2n}(x) - f^{2n+2}(x)$ is satisfied for all positive integer n. For $n = 1$ we have already seen that this statement is satisfied. If this equation is satisfied for some positive integer n, then by substituting $x := f^2(x)$ we get that

$$f^{2n+2}(x) - f^{2n+4}(x) = f^2(x) - f^4(x) = x - f^2(x),$$

implying that the statement for $n + 1$. Hence

$$x - f^2(x) = f^2(x) - f^4(x) = f^4(x) - f^6(x) = f^6(x) - f^8(x) = \dots,$$

This shows that the sequence $x, f^2(x), f^4(x), \dots$ is an arithmetic progression. On the other hand, each of its element is bounded as it belongs to the set H. Hence this sequence is constant, that is, $f^2(x) = x$.
The functions satisfying $x = f^2(x)$ are all soultion since in this case we get that $f(x) = f^3(x)$ after substituting $x := f(x)$. Then $x = f(x) + f^2(x) - f^3(x)$.

(Back to problems)

6. For the solution, see Category D Problem 6.

(Back to problems)

2.3.4 Category $\mathbf{E}^+$

1. We prove a more general statement. Let $m > 1$ be an arbitrary positive integer, and let $b_n(m)$ be the remainder of a_n when divided by m. We prove that for each m, the sequence $b_1(m), b_2(m), \ldots$ is eventually constant. The original statement is the case $m = 2020$.

We prove the statement by induction on m. It is clear that for $m = 2$ the sequence $b_n(2)$ is constant 0.

Suppose that we know the statement for each positive integer smaller than m. Now let us prove it for m: Let $m = 2^c \cdot t$ where t is odd. If $t = 1$, thus m is a power of two, then $b_n(m)$ is eventually 0, since the sequence a_n is an increasing sequence of powers of two.

Now suppose that $t > 1$. The remainder of a number when divided by m can be determined if we know the remainder when divided by 2^c and the remainder when divided by t. Hence if $c > 0$, then for some index n_0 we have $b_n(2^c) = 0$ for each $n > n_0$. By induction, since $t < m$, there exist also an index n_1 such that $b_n(t)$ is constant for $n \geq n_1$. Hence for $n \geq \max\{n_0, n_1\}$, $b_n(m)$ is also constant.

The only remaining case is the case $c = 0$, thus, m is odd. In this case $gcd(m, a_n) = 1$ for each n, since a_n is a power of two. Let φ denote Euler's function. Since $\varphi(m) < m$, we already know the statement for $\varphi(m)$, that is, for some index n_0, $b_n(\varphi(m))$ is constant for $n \geq n_0$. This is equivalent to $\varphi(m) \mid a_n - a_{n-1}$ for $n \geq n_0 + 1$. Then by the Euler–Fermat theorem $2^{a_n - a_{n-1}} \equiv 1 \pmod{m}$. To show that $b_n(m)$ is constant from some point, we need to show that from some point $m \mid a_{n+1} - a_n$. This will be true for $n \geq n_0 + 2$, since

$$a_{n+1} - a_n = 2^{a_n} - 2^{a_{n-1}} = 2^{a_{n-1}}(2^{a_n - a_{n-1}} - 1) \equiv 0 \pmod{m}.$$

This finishes the proof of the more general statement, thereby proving the $m = 2020$ case as well.

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2. Throughout the solution $f^n(x)$ denotes the value obtained by applying the function n times to the number x.

a) The equation has the form $x = f(x) + f^2(x) - f^3(x)$. Let us substitute $x := f(x)$ into the equation. Then we get that

$$f(x) = f^2(x) + f^3(x) - f^4(x).$$

Adding this equation to the original one we get that

$$x + f(x) = f(x) + 2f^2(x) - f^4(x).$$

After rearrangement we get that

$$x - f^2(x) = f^2(x) - f^4(x).$$

We show by induction on n that $x - f^2(x) = f^{2n}(x) - f^{2n+2}(x)$ is satisfied for all positive integer n. For $n = 1$ we have already seen that this statement is satisfied. If this equation is satisfied for some positive integer n, then by substituting $x := f^2(x)$ we get that

$$f^{2n+2}(x) - f^{2n+4}(x) = f^2(x) - f^4(x) = x - f^2(x),$$

implying that the statement for $n+1$. Hence

$$x - f^2(x) = f^2(x) - f^4(x) = f^4(x) - f^6(x) = f^6(x) - f^8(x) = \dots,$$

This shows that the sequence $x, f^2(x), f^4(x), \dots$ is an arithmetic progression. On the other hand, each of its element is bounded as it belongs to the set H. Hence this sequence is constant, that is, $f^2(x) = x$.
The functions satisfying $x = f^2(x)$ are all soultion since in this case we get that $f(x) = f^3(x)$ after substituting $x := f(x)$. Then $x = f(x) + f^2(x) - f^3(x)$.

b) We show that there is no such function. We prove it indirectly. Suppose for contradiction that f satisfies the equation.

The function f is injective since in case of $f(i) = f(j)$ we get that

$$i = f(i) + 2f^2(i) - 3f^3(i) = f(j) + 2f^2(j) - 3f^3(j) = j.$$

Since H is a finite set, and f maps H to H we get that f is bijective, that is, the numbers $f(-2019), f(-2018), \dots, f(2020)$ are exactly the elements of H in some order. Applying f one more times we get that $f^2(-2019), f^2(-2018), \dots, f^2(2020)$ are again the elements of H in some order, and in general, for every positive integer n the numbers $f^n(-2019), f^n(-2018), \dots, f^n(2020)$ are exactly the elements of H in some order. Thus for all n we have

$$\sum_{i \in H} f^n(i) = \sum_{j=-2019}^{2020} j = 2020.$$

Let us substitute each element of H into the original equation and add up all these equations. Then

$$2020 = \sum_{i \in H} i = \sum_{i \in H} f(i) + 2f^2(i) - 3f^3(i) = \sum_{i \in H} f(i) + 2\sum_{i \in H} f^2(i) - 3\sum_{i \in H} f^3(i) = 2020 + 4040 - 6060 = 0,$$

a contradiction. Hence there is no such function.

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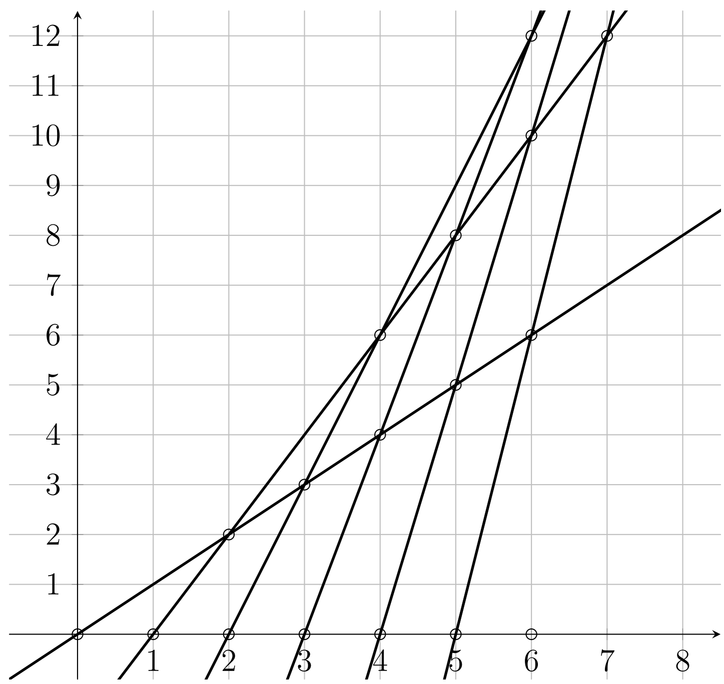
3. First we show that for all n one can give n such lines. Let us consider n lines of equation $ax + by = c$, where a, b, c are rational, and they are in general position. It is clear that this can be done. To obtain the intersection of two lines we need to express x and y from their equations. To do this, we only need to divide, multiply, add and subtract numbers, so we do the computations only with rational numbers. So the coordinates of the intersection are rational. Since we have only a finite number of intersections we can consider the least common multiple of all denominators of the coordinates of the intersections. By enlarging the whole picture by A from the origin we get that the coordinates of all intersections are integers. This way we obtained a solution for each n.

Perhaps surprisingly we can even give infinitely many such lines. Let us consider the following lines: the n-th line has slope n and goes through the point $(n-1, 0)$. With formula this is $\{y = nx - n(n-1) \mid n \in \mathbb{Z}\}$.



We show that these lines satisfy the condition of the problem. Since their slopes are different, there is no two parallel among them. We still need to show that any two intersects in a lattice point, and there is no three lines passing through the same point. Let us consider two such lines. Let the lines be given by the equations $y = nx - n(n-1)$ and $y = mx - m(m-1)$. Then

$$nx - n(n-1) = mx - m(m-1),$$

$$x = \frac{n(n-1) - m(m-1)}{n-m} = \frac{n^2 - m^2 + m - n}{n-m} = n + m - 1,$$

$$y = nx - n(n-1) = n(n+m-1) - n(n-1) = nm.$$

Thus their intersection has coordinates $(n+m-1, nm)$, that is, it is indeed a lattice point. We still need to show that there is no third line passing through this point. The line with equation $y = kx - k(k-1)$ intersects the line with equation $y = nx - n(n-1)$ in the point $(n+k-1, nk)$. This is different from the point $(n+m-1, nm)$ for $m \neq k$. Hence we have given infinitely many lines satisfying the conditions of the problem.

Note: An interesting fact for those who heard about cardinalities of infinite sets: we have given countalbly infinite lines such that any two of them intersects in a lattice point. On the other hand, it is not possible to give uncountably infinite lines with the same property since uncountably infinite lines have uncountably infinite intersections, but there are only countably infinite lattice points.

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4. Let k be the circumscribed circle of the triangle ABC. Furthermore, let D be the second intersection of the bisector of the angle BAC and the circle k. It is known that D is the midpoint of the arc BC not containing A. It also satisfies that $DB = DI = DC$. Let k_A be the circle with center D and radius DI. The radical axis of the circles k and k_A is the line BC. If we consider the point I a circle, then the radical axis of the circles I and k_A is the line that is perpendicular to the bisector of angle BAC in the point I. So the point F_A is the intersection of these two radical axis. This means that it is on the radical axes of the circles k and I. Then by symmetry the points F_B and F_C are incident to this line too, so the three points are collinear.

Second solution: Let e, f and g be the outer angle bisectors of the triangle ABC. These are perpendicular to the segments AI, BI and CI respectively since the latter segments are the angle bisectors. Thus the lines IF_A, IF_B and IF_C are parallel. The lines e, f and g close a triangle whose vertices are the centers of the escribed circles. Let these centers be E, F and G. Then the lines AE, BF and CG pass through the point I. Hence we can use the Desargues theorem to the triangles ABC and EFG: the intersections of the lines AB and EF, BC and FG, and CA and GE are collinear.

Let $\lambda \in [0, 1]$, and let E_λ, F_λ and G_λ be points obtained by enlarging the points E, F and G from I with ratio λ. Then we can apply the Desargues theorem to these points and the triangle ABC. The appropriate intersections of the triangles ABC and $E_\lambda F_\lambda G_\lambda$ are collinear. By taking the limit $\lambda = 0$ we get exactly the statement of the theorem.

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5. Let $G(V, E)$ be an arbitrary connected 3-regular graph on $2n$ vertices. In an orientation we call a vertex source if it has out-degree 3, and we call it a sink if has out-degree 0.

In any orientation the sum of the out-degrees is $3n \equiv n \pmod 2$ since this is the number of edges. For any orientation $\mathcal{O}$ let $A(\mathcal{O})$ be the set of vertices with odd out-degree, and let $B(\mathcal{O})$ be the set of vertices with even out-degree. Then $|A(\mathcal{O})| \equiv n \pmod 2$ and so $|B(\mathcal{O})| = 2n - |A(\mathcal{O})| \equiv n \pmod 2$. So if for some set B the parity of $|B|$ does not coincide with the parity of n, then there is no orientation $\mathcal{O}$ of G for which $B = B(\mathcal{O})$. On the other hand, we show that if $B \subset V$ satisfies $|B| \equiv n \pmod 2$, then there exists 2^{n+1} orientations $\mathcal{O}$ for which $B(\mathcal{O}) = B$.

Let us consider an arbitrary spanning tree of G, this is where we use that G is connected. The spanning tree contains $2n - 1$ edges, and so there are $n + 1$ edges outside the spanning tree. We can orient these $n + 1$ edges in 2^{n+1} ways. We show that for each partial orientation of these $n + 1$ edges we can uniquely extend it to an orientation $\mathcal{O}$ of G such that $B(\mathcal{O}) = B$. So take a partial orientation, and let us repeat the following steps.

Let us take a leaf of the tree and orient the corresponding edge in such a way that the parity of the out-degree should be consistent with the leaf vertex being in B or not. Note that this orientation is unique. Then delete this leaf vertex. We got a smaller tree, and so we can repeat the process till we get only an edge (u, v). Let us orient this edge in such a way that the parity of the out-degree of vertex v is consistent with v being in B or not. Then the parity of the out-degree of the other vertex will be good immediately since $|B| \equiv n \pmod 2$, and there is no orientation $\mathcal{O}$, where the parity of $B(\mathcal{O})$ does coincide with the parity of n. Hence we can uniquely extend any orientation of the edges outside the spanning tree to an orientation such that $B(\mathcal{O}) = B$. So there are 2^{n+1} such orientations.

Next observe that the difference of the number of sources and sinks of an orientation $\mathcal{O}$ only depends on the cardinality of $B(\mathcal{O})$. Suppose that in an orientation $\mathcal{O}$ there are x sources and y sinks, and $|B(\mathcal{O})| = b$. Then we have $b - y$ vertices with out-degree 2, and $2n - b - x$ vertices with out-degree 1. Since the sum of out-degrees is $3n$ we have

$$2n - b - x + 2(b - y) + 3x = 3n,$$

and so

$$x - y = \frac{n - b}{2}.$$

So $x - y$ only depends on b. So the number of orientations for which $x - y = 0$ is the number of orientations with $|B(\mathcal{O})| = n$. Since then $|B(\mathcal{O})| \equiv n \pmod 2$ we have 2^{n+1} orientations for each such set B. We can choose the set B in $\binom{2n}{n}$, so all together there are $2^{n+1}\binom{2n}{n}$ orientations with the same number of sources and sinks.

2. solution We give a second proof that for every $B \subset V$, $|B| \equiv n \pmod 2$ there are exactly 2^{n+1} orientations $\mathcal{O}$ with $B(\mathcal{O}) = B$. The rest of the solution as the same as the first one.

Claim 1: For all $A \subset V$ with even cardinality there exists a spanning subgraph of G in which exactly the verticess of A have odd degree.
Proof: Let us first pair up the elements of A. For each pair there is a path in G connecting the two vertices since G is connected. Take such a path for each pairs. Let H be the graph obtained from G by putting those edges into H that are contained in exactly odd number of paths. This graph satisfies the requirements since exactly the vertices of A have odd degree in a union of paths, and when we keep the edges appearing odd times, the parity of the vertices will not change.

Claim 2: If $A, B \subset V$ such that $|A|$ and $|B|$ has the same parity, then the number of orientations of G where exactly the vertices of A have even out-degree is the same as the number of orientations of G where exactly the vertices of B have even out-degree.
Proof: Let C be the symmetric difference of A and B. Then $|C|$ is even, so there is a spanning subgraph $H \subset G$ in which exactly the elements of C have odd degree. Let S_A denote the set of orientations of G in which exactly the elements of A have even out-degree. Let S_B be defined similarly. We can define a map from S_A to S_B by reversing the edges of H. This map changes the parity of the elements of C thus it will indeed map to S_B. Since this map is injective we get that $|S_A| \le |S_B|$. By symmetry (or applying the map twice) we get that $|S_A| = |S_B|$.

In any orientation the sum of the out-degrees is $3n \equiv n \pmod 2$ since this is the number of edges. For any orientation $\mathcal{O}$, let $A(\mathcal{O})$ be the set of vertices with odd out-degree and let $B(\mathcal{O})$ be the set of vertices with even out-degree. Then $|A(\mathcal{O})| \equiv n \pmod 2$ and so $|B(\mathcal{O})| = 2n - |A(\mathcal{O})| \equiv n \pmod 2$. So if for some set B the parity of $|B|$ does not coincide with the parity of n, then there is no orientation $\mathcal{O}$ of G for which $B = B(\mathcal{O})$.

There are 2^{3n} orientations and there are 2^{2n-1} sets B for which $|B| \equiv n \pmod 2$ and we know that we have the same number of orientations for each such set with the property that exactly the vertices of B have even out-degree. So there must be 2^{n+1} orientations for each such set B.

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6. We call a position *winning* if one can win the game by moving to that position, that is, starting from that position, the second player has a winning strategy.

The only valid position where one cannot make a legal move is where each pile contains exactly one disk, therefore $(1, 1, 1, 1)$ is a winning position.

Moreover, a position (odd, odd, odd, odd) is also winning, since from this one the opposition can only reach a position of the form $(even, odd, odd, odd)$, which means that we can make a legal move by going to a position of the form (odd, odd, odd, odd). It also follows that a position of the form $(even, odd, odd, odd)$ is not winning, since one can go from that to a winning position. Similarly, one can see that a position of the form $(even, even, odd, odd)$ is also not winning.

So it remains to see how to handle cases with at most one odd pile. Starting from such a pile, if a player decides to divide an even pile into two odd ones then he ends up in a position that is not winning since the opposition could reach a position (odd, odd, odd, odd) and win the game. We introduce the *modified game* where such moves (starting from a position with at most one odd pile and dividing an even pile into two odd ones) are forbidden. It is clear that a position is winning in the original game if and only if it is winning in the modified game.

First we claim that in the modified game plays started from the positions $(2a, 2b, 2c, 2d)$ and $(2a, 2b, 2c, 2d-1)$ correspond to each other. What we mean by that is that if one can make a legal move starting from one of these positions then we can make one in the other as well so that the resulting positions are either the same or correspond to each other. Checking this fact is left to the reader. It then easily follows that $(2a, 2b, 2c, 2d)$ is a winning position if and only if $(2a, 2b, 2c, 2d - 1)$ is a winning position.

For the final step, we use the original game to solve the modified one. A position of the form $(2a, 2b, 2c, 2d)$ (or $(2a, 2b, 2c, 2d-1)$) of the modified game correspond to the position (a, b, c, d) of the original game, in the same sense as above. To see this, note that if in the modified game one can divide an even pile into two even ones, then a division can be made from (a, b, c, d) as well to obtain corresponding positions, and vice versa.

Using this fact it is straightforward to check weather a position of the form $(2a, 2b, 2c, 2d)$ or $(2a, 2b, 2c, 2d-1)$ is winning in the modified game (and hence, in the original game). Therefore the strategy can also be found.

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2.4 Final round – day 2

2.4.1 Category C

#	ANS	Problem	P
C-1	13	In a hotel, the rooms are numbered from 2 to 27	3p
C-2	51	Picur wanted to make a strangely shaped piece of chocolate	3p
C-3	8	Grandma has four grandsons:	3p
C-4	22	Andrew has written down 4 positive integers	3p
C-5	18	How many ways are there to tile	4p
C-6	18	How many triangles are there in the figure?	4p
C-7	18	In a movie theatre	4p
C-8	39	A pirate captain and his assistant	4p
C-9	76	4 horses ran a race	5p
C-10	3600	What is the largest possible k	5p
C-11	665	There are red and blue balls in a bag, 729 in total	5p
C-12	30	We have written 25! as a product	5p
C-13	1944	$ABCD$ is a square of side length 108 cm	6p
C-14	3335	What is the smallest integer k	6p
C-15	4976	Soma has a tower of 63 bricks	6p
C-16	9084	Positive integers a, b and c	6p

2.4.2 Category D

#	ANS	Problem	P
D-1	13	In a hotel, the rooms are numbered from 2 to 27	3p
D-2	18	How many ways are there to tile	3p
D-3	15	What number should we put in place of the question mark	3p
D-4	69	When returning from her holidays,	3p
D-5	39	A pirate captain and his assistant	4p
D-6	300	We have a positive integer n, whose sum of digits is 100	4p
D-7	9	The hexagon $ABCDEF$ has all angles equal	4p
D-8	2028	The square in the diagram has side length 78	4p
D-9	30	We have written 25! as a product	5p
D-10	2628	6 horses ran a race	5p
D-11	3335	What is the smallest integer k	5p
D-12	12	Santa Claus plays a guessing game with Marvin	5p
D-13	4976	Soma has a tower of 63 bricks	6p
D-14	9084	Positive integers a, b and c	6p
D-15	472	An integer n is called k-nice	6p
D-16	4824	In a movie theatre	6p

2.4.3 Category E

#	ANS	Problem	P
E-1	18	How many ways are there to tile	3p
E-2	15	What number should we put in place of the question mark	3p
E-3	27	Ann wrote down all the perfect squares	3p
E-4	300	We have a positive integer n, whose sum of digits is 100	3p
E-5	9	The hexagon $ABCDEF$ has all angles equal	4p
E-6	1650	We build a modified version of Pascal's triangle	4p
E-7	12	Santa Claus plays a guessing game with Marvin	4p
E-8	6	The integers 1, 2, 3, 4, 5 and 6	4p
E-9	1384	On a piece of paper, we write down all positive integers	5p
E-10	4976	Soma has a tower of 63 bricks	5p
E-11	420	The convex quadrilateral $ABCD$	5p
E-12	1920	We have a white table with 2 rows and 5 columns	5p
E-13	42	In triangle ABC we inscribe a square	6p
E-14	1560	How many ways are there to fill in the 8 spots	6p
E-15	4824	In a movie theatre	6p
E-16	145	Dora has 8 rods with lengths	6p

2.4.4 Category E$^+$

#	ANS	Problem	P
E$^+$-1	300	We have a positive integer n, whose sum of digits is 100	3p
E$^+$-2	9	The hexagon $ABCDEF$ has all angles equal	3p
E$^+$-3	1650	We build a modified version of Pascal's triangle	3p
E$^+$-4	12	Santa Claus plays a guessing game with Marvin	3p
E$^+$-5	7282	On a piece of paper, we write down all positive integers	4p
E$^+$-6	9084	Positive integers a, b and c	4p
E$^+$-7	880	There are red and blue balls in an urn	4p
E$^+$-8	4976	Soma has a tower of 63 bricks	4p
E$^+$-9	420	The convex quadrilateral $ABCD$	5p
E$^+$-10	1920	We have a white table with 2 rows and 5 columns	5p
E$^+$-11	42	In triangle ABC we inscribe a square	5p
E$^+$-12	1560	How many ways are there to fill in the 8 spots	5p
E$^+$-13	259	In the game of Yahtzee	6p
E$^+$-14	4824	In a movie theatre	6p
E$^+$-15	283	The function f is defined on positive integers	6p
E$^+$-16	455	Dora has 10 rods with lengths	6p