E1. Albrecht is travelling in his car on the motorway at a constant speed. The journey is very long so Marvin who is sitting next to Albrecht gets bored and decides to calculate the speed of the car. He was a bit careless but he noted that at noon they passed milestone XY (where X and Y are digits), at 12:42 milestone YX and at 1pm they arrived at milestone X0Y. What did Marvin deduce, what is the speed of the car?

First solution: We know that it took 42 minutes for the car to travel $XY - YX$ miles and another 18 minutes to travel $X0Y - YX$ miles.

$XY = 10 \cdot X + Y$, $YX = 10 \cdot Y + X$ and $X0Y = 100 \cdot X + Y$ hence $XY - YX = (10 \cdot X + Y) - (10 \cdot Y + X) = 9(Y - X)$ and $(100 \cdot X + Y) - (10 \cdot Y + X) = 9(11X - Y)$ have the same ratio as 42 and 18 which is 7:3. We can divide through by 9 so $Y - X$ and $11X - Y$ have a ratio of 7:3. X and Y are digits thus $Y - X$ cannot be larger than 9 and $11X - Y \geq 11 - 9 = 2 > 0$ so $11X - Y$ is positive. Hence the only possible values that these expressions can have are 7 and 3.

$$Y - X = 7$$

$$11X - Y = 3$$

By adding the two equalities, we obtain $10X = 10$ so $X = 1$ and $Y = 8$.

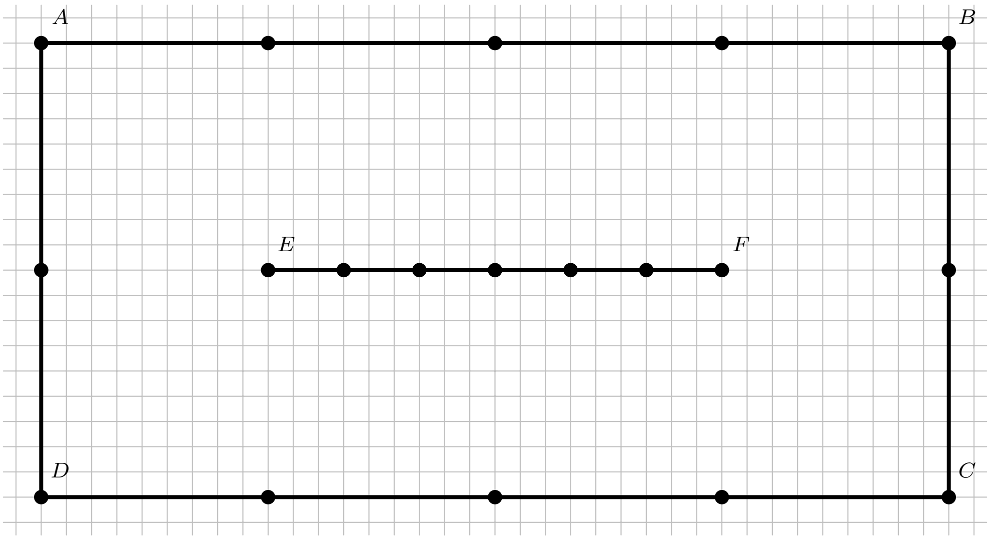
Therefore the car travelled $81 - 18 = 63$ miles in 42 minutes so the speed of the car is 90 miles per hour.

Second solution: We will now show a shorter solution. The distance covered between 12pm and 13pm is $X0Y - XY = X \cdot 90$ miles. Since $X = 1$, this means that Albrecht is travelling at 90 miles per h1our.

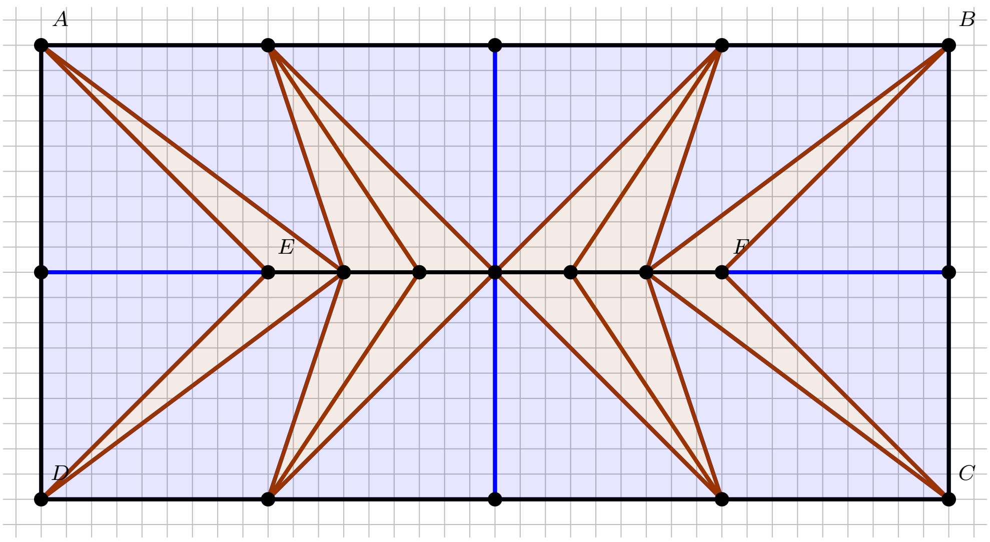
E2. The best part of grandma's 18 cm × 36 cm rectangle-shaped cake is the chocolate covering on the edges. Her three grandchildren would like to split the cake between each other so that everyone gets the same amount (of the area) of the cake, and they all get the same amount of the delicious perimeter too.
a) Can they cut the cake into three convex pieces like that?
b) The next time grandma baked this cake, the whole family wanted to try it so they had to cut the cake into six convex pieces this way. Is this possible?
c) Soon the entire neighbourhood has heard of the delicious cake. Can the cake be cut into 12 convex pieces with the same conditions?

First solution: First we show a solution for part **c)** , from which we can also get solutions to parts **a)** and **b)** .

Let E be a point which is at a distance of 9 cm from sides AB, CD and AD. Similarly let F be at a distance of 9 cm from sides AB, CD and BC. Also let us divide the perimeter into 12 equal parts and segment EF into 6 equal parts as shown in the picture:



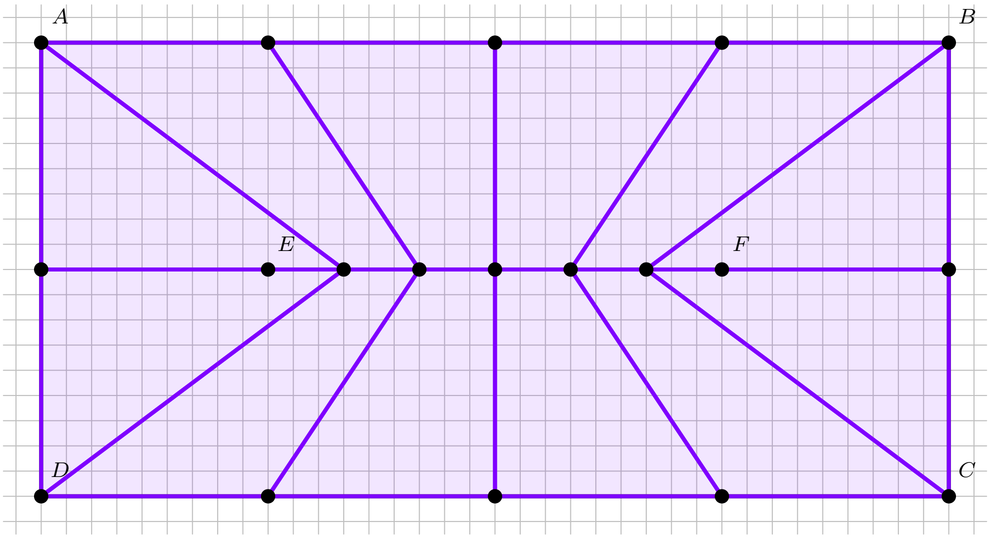
Now let's draw triangles in the following way:



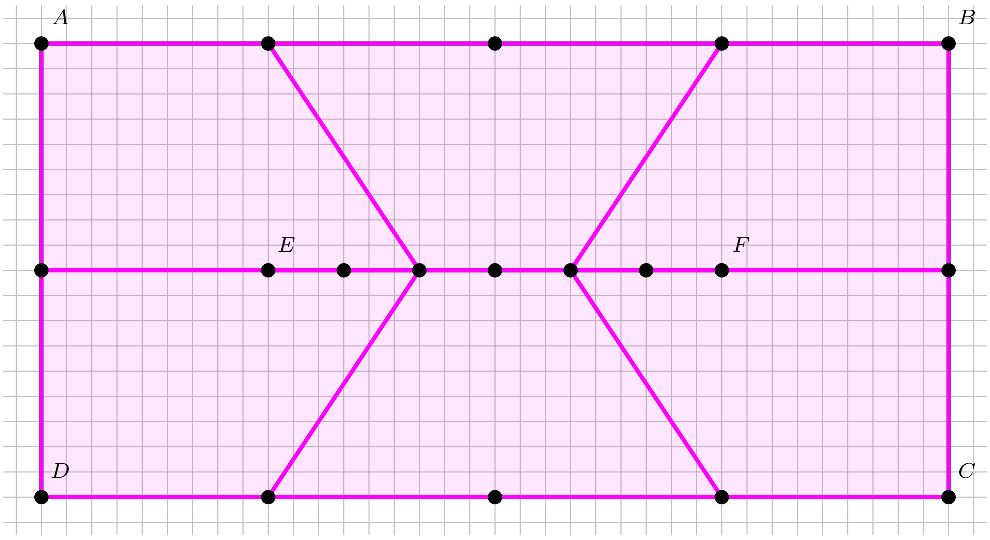
All blue triangles have a side of length 9 cm on the perimeter of the cake. And their corresponding altitude is 9 cm too, since the segment EF was defined in this way. So all blue triangles have equal area.

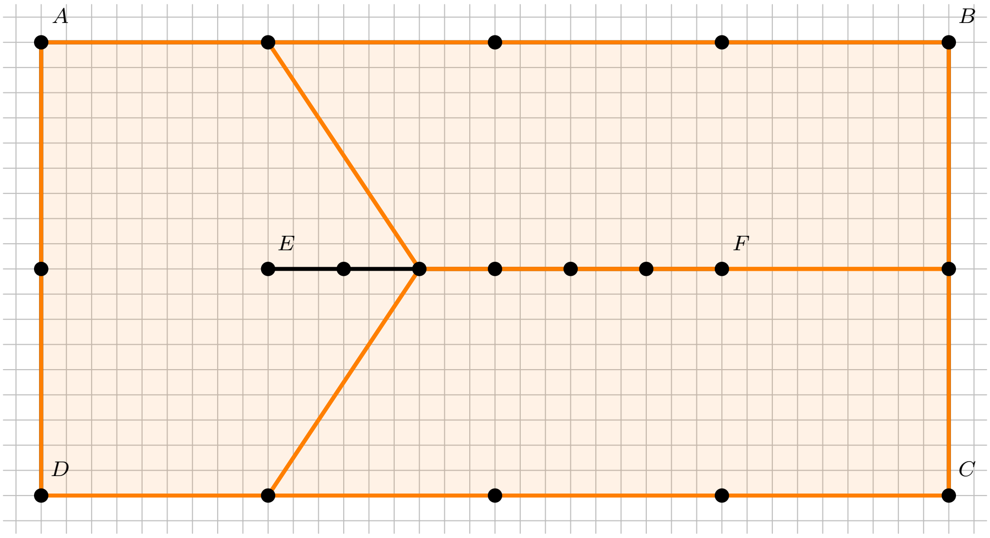
As we have divided segment EF into equal parts, there is a side length that appears in each of the red triangles. Also each red triangle has an altitude of 9 cm corresponding to it, so all red triangles have equal area too.

So if everybody gets a blue slice and a red one, then all will get equal shares of both the area and the chocolatey perimeter:

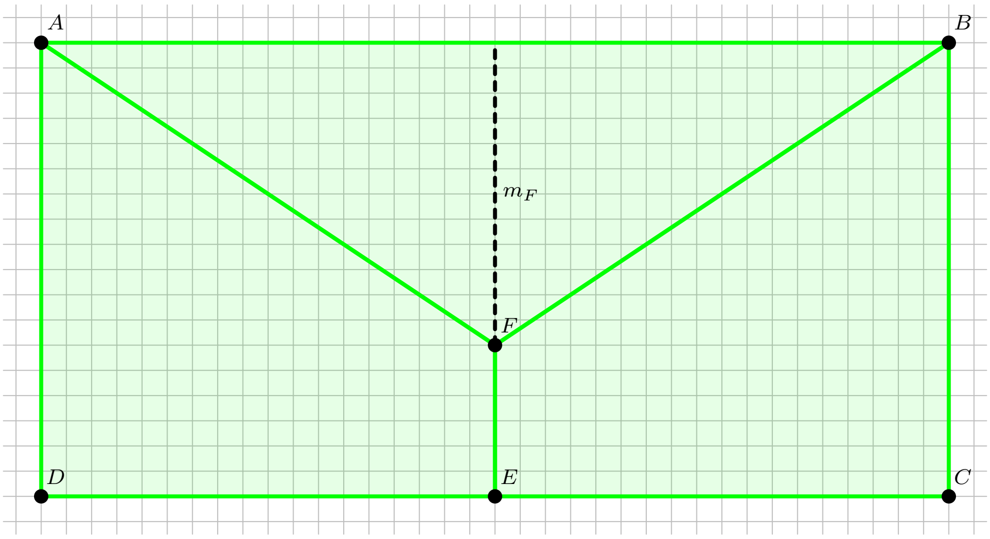


If everyone takes 2 blue and 2 red parts, then this gives a solution for part **b)** , and if everyone takes 4 blue and 4 red, we get a solution for **a)** :



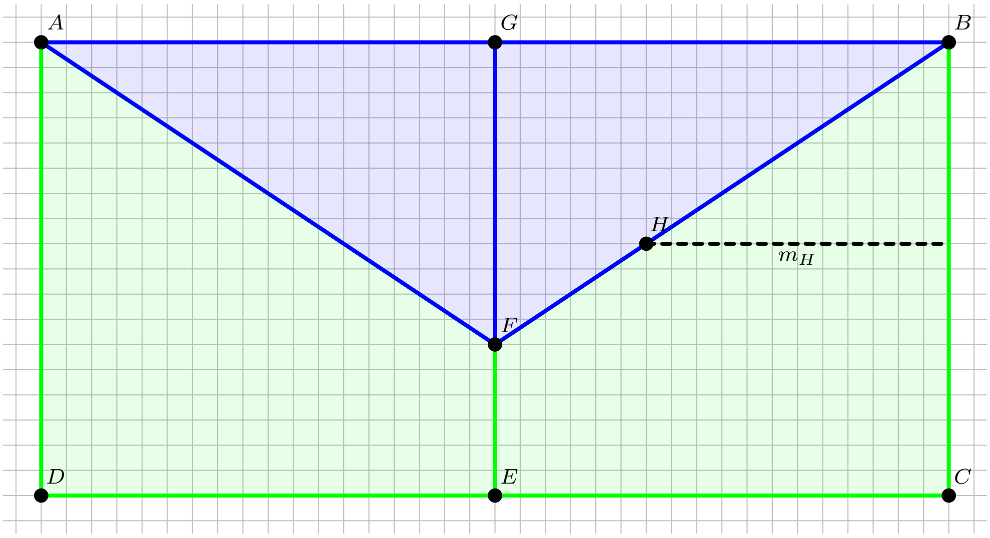


Second solution: a) The perimeter is of length $2(18+36) = 108$ cm, so everybody has to get a third of this, which is 36 cm. This is exactly the length of the longer side. Take the two endpoints of one of the longer sides (A, B) and the midpoint of the opposite side (E). These divide the perimeter into 3 equal parts. Let's find a point F inside the rectangle, which when connected to these three points, divides the rectangle's area into three equal parts as well:



So that the areas on the left and right are equal, we look for such a point F on the perpendicular bisector of side AB. As the area of triangle ABF has to be equal to the third of the cake's area, it should be $18 \cdot 36/3 = 216$ cm^2, and we also know the length of AB, so we can determine the corresponding altitude: $m_F = \frac{2T_{ABF}}{AB} = \frac{2 \cdot 216}{36} = 12$ cm, which means $EF = 18 - 12 = 6$ cm.

b) From here, we can halve all three parts properly. For triangle ABF this is easy to do: segment FG will divide both the area and the chocolate part of triangle ABF:

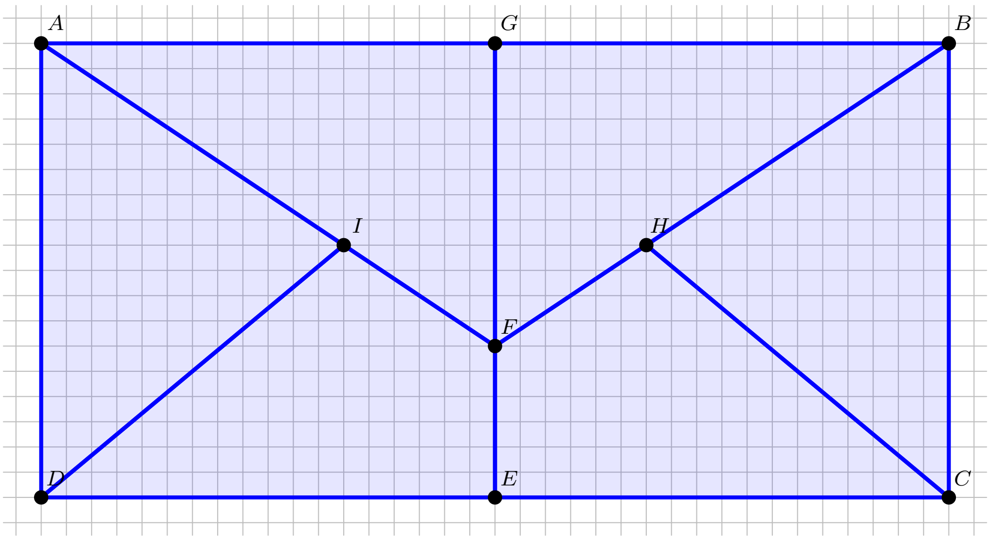


To halve quadrilateral $BCEF$, we need to cut it using a line from point C (so that the chocolate part is cut into two equal halves of length 18 cm each), and the remaining question is where we should take the point H so that CH cuts the area of $BCEF$ into half.

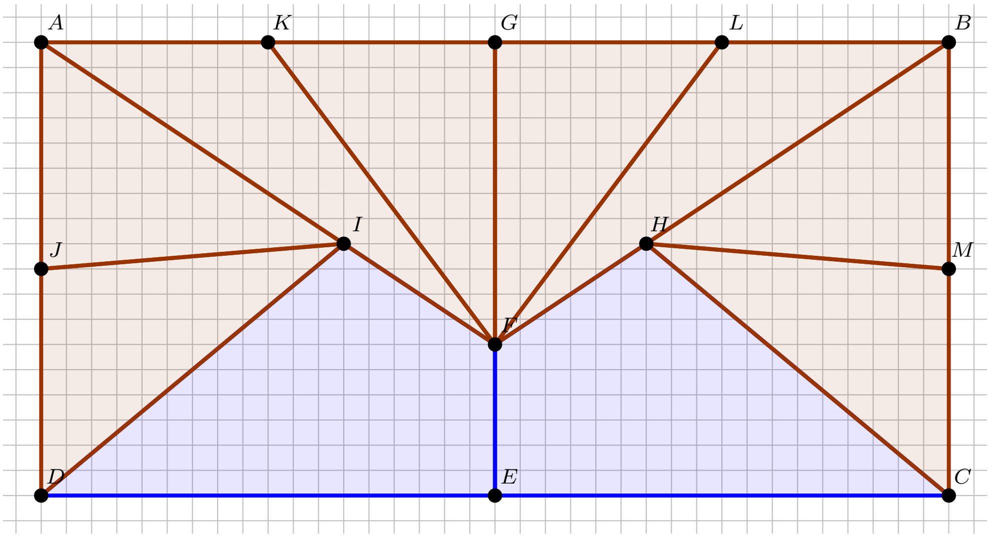
This is again easy to determine, since the area of HBC needs to be one-sixth of the rectangle (which is 108 cm^2), and we know the base BC (18 cm), so the corresponding altitude should be $m_H = \frac{2T_{HBC}}{BC} = \frac{2 \cdot 108}{18} = 12$ cm.

Actually this means that H will be the trisection point of segment BF lying closer to F, and also it is at a distance of $\frac{2}{3} \cdot GF = 8$ cm.

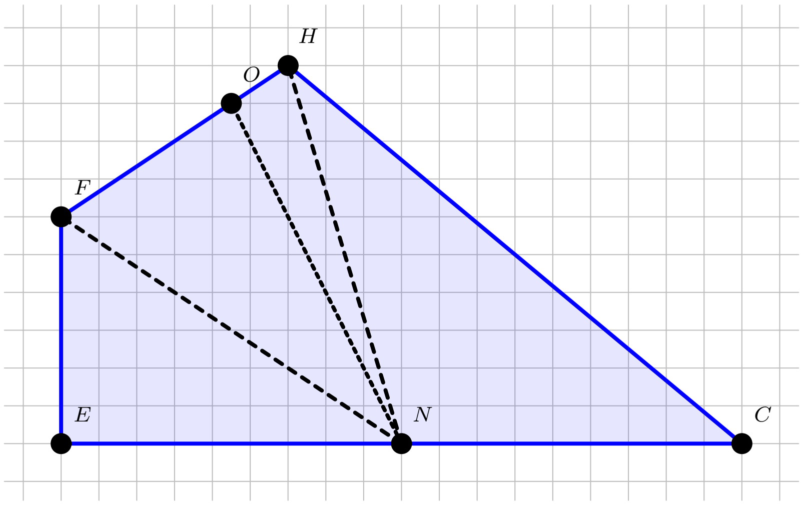
Symmetrically we can halve the quadrilateral $ADEF$ correctly using segment DI:



c) Again we will halve the parts obtained in part **b)**. It is easy to do this correctly for triangles DIA, AFG, GFB and BHC:. we just take their medians IJ, FK, FL and HM, which will halve the area of each triangle and also the opposite side (which is the chocolate part):



It remains to care about quadrilateral $CEFH$. If we can halve this correctly, then we can also do so for the symmetric $DEFI$. Let N be the midpoint of EC, and let's find the point O for which NO halves the area:



The areas of triangles NEF and NHC can be calculated, and we will see that both are less than 54 cm^2, which means that point O has to lie somewhere on segment FH - as we can see in the figure.

$T_{NEF} = \frac{NE \cdot EF}{2} = \frac{9 \cdot 6}{2} = 27\text{ cm}^2$.

In order to calculate the area of NHC, let's remind ourselves that H was at a distance of 8 cm from AB, which means that it is at a distance of 10 cm from EC. So $T_{NHC} = \frac{NC \cdot 10}{2} = \frac{9 \cdot 10}{2} = 45\text{ cm}^2$.

So we have to choose point O in such a way that $T_{OHN} = 54 - 45 = 9\text{ cm}^2$ and $T_{OHF} = 54 - 27 = 27\text{ cm}^2$. The ratio of the two areas is $1:3$, so if we take O to be the point on HF for which $HO : OF = 1 : 3$, then the two triangles will have equal altitudes from N and the bases are in a ratio of $1:3$ so the areas will have the right ratio.

Note: In fact we don't need to exactly determine where point O lies, it is just enough to prove that *there exists* an appropriate point.

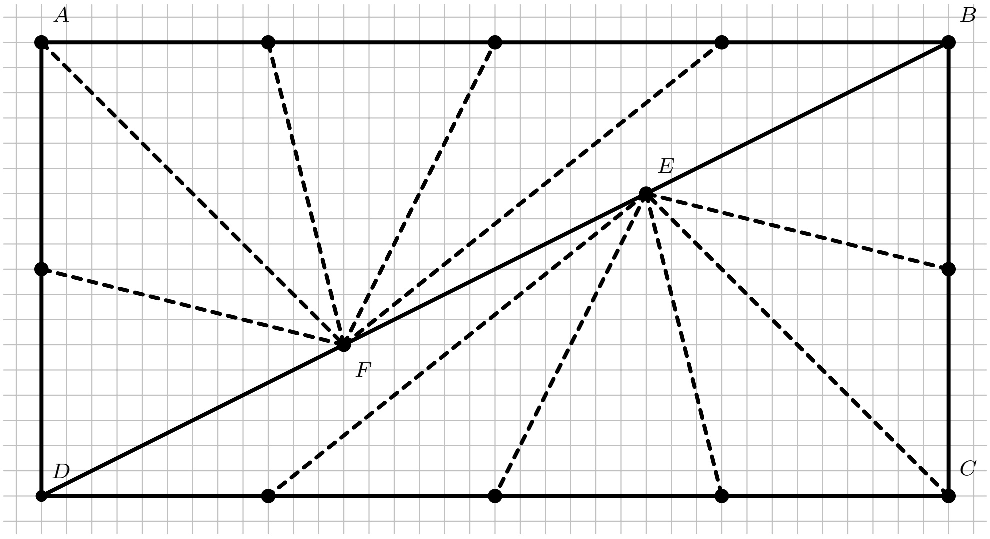
Let's imagine that we place our knife on ray NE, and we start rotating it in a clockwise direction so that one of its ends stays at N. (So our knife would traverse the segments in the figure in the order NE, NF, NO, NH, NC.)

So initially the left area is 0 and the right area is the whole 108 cm^2, and as we rotate the knife, the left area grows and the right area decreases gradually. At the end, the left area will be 108 cm^2, and the right area will be 0. As the area changed continuously, there had to be a position of the knife that cut the area exactly into half.

This solution uses the idea of continuity, but as we have seen, the problem can be solved without it.

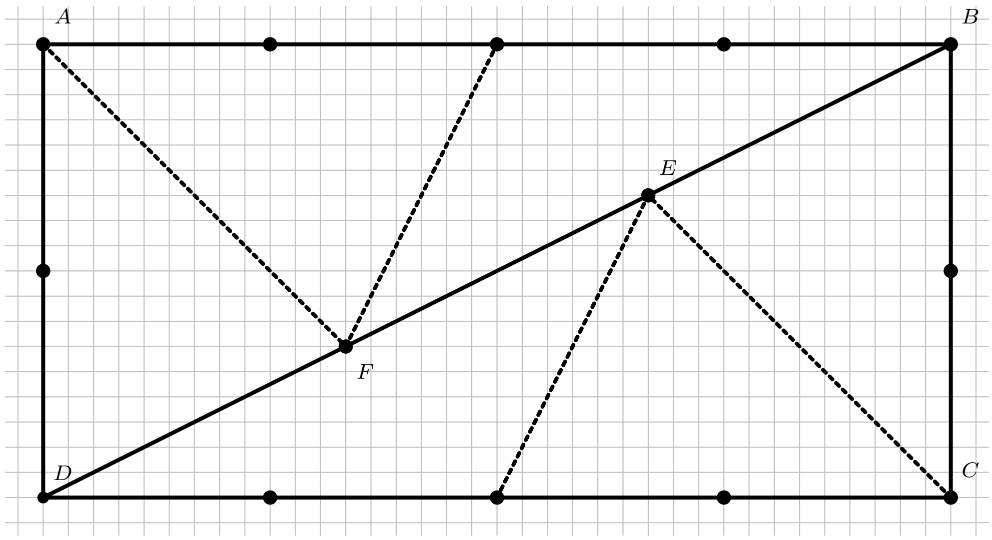
Third solution: Again we will give a correct slicing for part **c)** . Let's take points E and F on the diagonal so that they have an equal distance from sides BC and CD, and sides DA and AB respectively. In other words, E is on the bisector of angle BCD and F is on the bisector of angle DAB.

Again divide the perimeter into 12 equal parts, and connect the dividing points above diagonal BD with F, and those below it with E:



The sides of the triangles lying on the perimeter of the rectangle are all equal (so everybody gets equal chocolate), and the altitudes belonging to them will also be equal, since this is how we have chosen points E and F. So this slicing is correct.

If we only connect every other dividing point with F and E, then we get a new solution for part **b)**:



Unfortunately we can't get a new solution for part **a)** by combining parts in this construction because one of the shapes will always be concave.

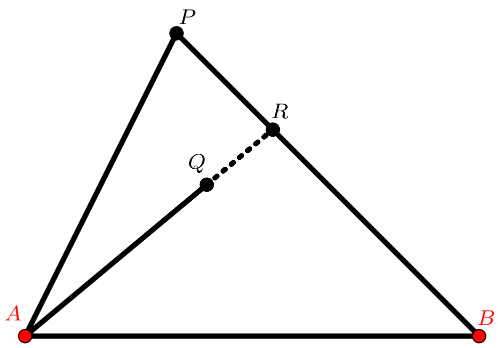
E3. The floor plan of a contemporary art museum is a (not necessarily convex) polygon and its walls are solid. The security guard guarding the museum has two favourite spots (points A and B) because one can see the whole area of the museum standing at either point. Is it true that from any point of the AB section one can see the whole museum?

Solution: Take an arbitrary point P of the museum. We are going to prove that every point of the interior and boundary of the triangle ABP is in the museum.

If X, a point of the museum, can be seen from point A (or B) then the whole section AX (or BX) is in the museum since if there was a point Y on AX that is not in the museum then X could not be seen from point X.

By this argument the sections AB, AP and BP (so the whole boundary of the ABP triangle) is in the museum.

Now we prove that every point Q inside of ABP is in the museum. Let R be the intersection of the lines AQ and BP. R is in the museum (since it is on the boundary of ABP) so R can be seen from A thus the whole AR section is in the museum thus Q is in the museum.



Therefore every point in the interior or on the boundary of ABP is a point of the museum. Hence P can be seen from every point C on the section AB since CP is in the museum.

This is true for every P in the museum so the whole museum can be seen from every point of the AB section.

E4. Determine all triples of positive integers a, b, c that satisfy
a) $[a, b] + [a, c] + [b, c] = [a, b, c]$.
b) $[a, b] + [a, c] + [b, c] = [a, b, c] + (a, b, c)$.
Remark: Here $[x, y]$ denotes the least common multiple of positive integers x and y, and (x, y) denotes their greatest common divisor.

First solution: a) We will show that there are no solutions. Observe that if p^k $(p, k \in Z^+)$ is a prime power that divides all three numbers, then if we divide the numbers by p^k, the equality still remains true, as all terms get divided by p^k. If we do this with all prime powers appearing in the prime factorization of the GCD $d = (a, b, c)$, then in the end we will have $(a, b, c) = 1$ and the equation will still hold. So in the remaining part of the solution we will assume $(a, b, c) = 1$.

Take a prime p that divides at least one of a, b and c. The equation is symmetric in the three variables so we can assume $p \mid a$. Then p divides $[a, b]$, $[a, c]$ and $[a, b, c]$ too, so by the equation it also divides $[b, c]$. So it divides either b or c (but not both, as otherwise $(a, b, c) = 1$ would not hold). So every prime divisor of a, b or c divides exactly two of the numbers.

Moreover, it is also true that if p divides both a and b, then it actually appears in the prime factorizations of a and b to the same power. We will prove this by contradiction. Suppose that p appears to the k-th power in a and to the l-th power in b, and WLOG suppose $k > l$. Then let us compute the p-power appearing in each of the terms of the equation: $[a, b]$, $[a, c]$ and $[a, b, c]$ all have p^k and $[b, c]$ has p^l (using the fact that p does not divide c). So any prime appears in the factorizations of two of the numbers, and it appears in both to the same power.

If we group all primes appearing in a, b or c depending on which two of the numbers they divide, we can write the numbers in the following form: $a = de$, $b = ef$ and $c = fd$, where d, e and f are pairwise coprime. (For example e is the product of prime powers dividing both a and b.) Then $[a, b] + [a, c] + [b, c] = def + def + def = 3def$, and $[a, b, c] = def$. But this is impossible since d, e and f are positive. So there are no solutions.

b) Again observe that if a prime divides all three numbers then we can again divide all three numbers by it, and the equation remains true. So for the moment, let us only consider the cases where we have $d = (a, b, c) = 1$.

Take a prime p that divides at least two numbers out of the three. By symmetry we can assume that p divides a and b. Then it will divide all of $[a, b]$, $[a, c]$, $[b, c]$ and $[a, b, c]$, so by the equation it must also divide (a, b, c). However $(a, b, c) = 1$ so this is impossible. So any prime can only divide at most one of the three numbers, so the three numbers are pairwise coprime. So the equation will be the following: $ab + ac + bc = abc + 1$.

By symmetry, we can assume that a is the smallest of the three numbers (it is not necessarily distinct from the other two, as there can be multiple smallest numbers). Consider three cases according to the possible values of a:

- If $a = 1$ then $b + c + bc = bc + 1$, so $b + c = 1$ which is impossible.
- If $a \geq 3$ then $ab + ac + bc \leq \frac{abc}{3} + \frac{abc}{3} + \frac{abc}{3} < abc + 1$, which is also impossible.
- Finally if $a = 2$ then $2b + 2c + bc = 2bc + 1$, which after reordering gives $bc - 2b - 2c + 1 = 0$, so $(b - 2)(c - 2) = 3$. As b and c are positive integers, the only possible solution is $a = 2$, $b = 3$ and $c = 5$.

So up to reordering of the variables, the only solution with $(a, b, c) = 1$ is $a = 2$, $b = 3$ and $c = 5$, and we can check that this is indeed a solution.

Now let us turn to the cases where $d = (a, b, c) > 1$. As dividing everything by d gives a good solution where the GCD of the three numbers is 1, the original numbers must be (up to reordering) of

the form $a = 2d$, $b = 3d$ and $c = 5d$ for some positive integer d. We can check that these are all correct solutions indeed, as $[a, b] + [a, c] + [b, c] = 6d + 10d + 15d = 31d$ and $[a, b, c] + (a, b, c) = 30d + d = 31d$ too.

Second solution: a) We know that $[a, b] \mid [a, b, c]$ and similarly $[b, c]$ and $[c, a]$ also divide $[a, b, c]$. So there exist positive integers p, q and r such that $[a, b, c] = p[a, b] = q[b, c] = r[c, a]$. Divide the equation in the problem by $[a, b, c]$ to get

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

The positive integer solutions of this equation (up to reordering) are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

Here is a possible way to derive this: suppose WLOG that $p \le q \le r$. If $p = 1$, the LHS would be greater than 1 so this is impossible. If $p > 3$ then the LHS is at most $\frac{3}{4}$ so this is also impossible. If $p = 3$ then the LHS can only be 1 if $p = q = r = 3$, which is indeed a solution. Finally if $p = 2$ then we have $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. Reordering this, we get $2r + 2q = qr$ so $qr - 2r - 2q + 4 = 4$, and this factorizes as $(q - 2)(r - 2) = 4$. This gives two possible solutions for q and r: $(q, r) = (3, 6)$ and $(q, r) = (4, 4)$.

Now note that if p, q and r are not pairwise coprime, then we can reach a contradiction: assume that a prime d divides both p and q. Then $d \mid \frac{[a,b,c]}{[a,b]}$ means that d appears in c to a higher power than in a, and $d \mid \frac{[a,b,c]}{[b,c]}$ means that d appears in a to a higher power than in c. And this is impossible.

So p, q and r must be pairwise coprime, but none of the solutions $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$ satisfy this. So there is no solution to the original equation.

b) Let us start similarly to part **a)**: as $(a, b, c) \mid [a, b, c]$, we have a positive integer s with $[a, b, c] = s(a, b, c)$. Also observe that (a, b, c) divides the pairwise LCM's of the three numbers (e.g. $(a, b, c) \mid a \mid [a, b]$), so if we define p, q and r in the same way as above, then p, q and r will all be divisors of s. Rearranging the equation and dividing by $[a, b, c]$ now gives

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{s} = 1$$

As the equation is still symmetric in a, b and c, we can assume that $p \le q \le r$.

If $p = 1$ then as $r \mid s$, we have $r \le s$ so $\frac{1}{r} - \frac{1}{s} \ge 0$, so since $\frac{1}{q} > 0$, the LHS is greater than 1, a contradiction.

If $p \ge 3$, then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{s} < 3 \cdot \frac{1}{3} + 0$, so the LHS is smaller than 1.

So we must have $p = 2$. This yields

$$\frac{1}{q} + \frac{1}{r} - \frac{1}{s} = \frac{1}{2}$$

Suppose $q = 2$. Then we get $\frac{1}{r} - \frac{1}{s} = 0$, so $r = s$. This means $(a, b, c) = [c, a]$, so as $(a, b, c) \le a \le [c, a]$, we have $(a, b, c) = a = [c, a]$. This gives $a = c$. Then it is easy to see that the original equation reduces to $[a, b] + a = (a, b)$, which is impossible as $[a, b] + a > a \ge (a, b)$.

Suppose $q \ge 4$. Then $\frac{1}{q} + \frac{1}{r} - \frac{1}{s} < 2 \cdot \frac{1}{4} + 0 = \frac{1}{2}$, again a contradiction.

So we must have $q = 3$. Then we get

$$\frac{1}{r} - \frac{1}{s} = \frac{1}{6}$$

Here $r \ge q = 3$, and also $r \le 5$, since $r \ge 6$ would mean that $\frac{1}{r} - \frac{1}{s} < \frac{1}{6}$. So $r \in \{3, 4, 5\}$, and in each case s can be determined. We get the following three possibilities for the quadruple p, q, r, s:

$$(p; q; r; s) \in (2; 3; 3; 6), (2; 3; 4; 12), (2; 3; 5; 30)$$

The observation in part **a)** that p, q and r are pairwise coprime is true in this case as well, so this only leaves the case $(p; q; r; s) = (2; 3; 5; 30)$. This means

$$[a, b, c] = 2[a, b] = 3[b, c] = 5[c, a] = 30(a, b, c)$$

So $[a, b, c] = 2[a, b]$ means that the exponent of the prime 2 in c is higher than in both a and b. So in $[a, b, c]$, 2 appears to a higher power than in (a, b, c). Similarly 3 and 5 also appear to a higher power in $[a, b, c]$ than in (a, b, c). So $\frac{[a,b,c]}{(a,b,c)} \geq 2 \cdot 3 \cdot 5 = 30$.

Since we have equality here, the only possible case is that the powers of 2 in a and b are the same, and the power of 2 in c is one larger than that. The analogue is true for powers of 3 and 5 too (but of course they have to be the largest in b and a respectively). Also every prime appearing in $[a, b, c]$ other than 2, 3 or 5 has to appear in (a, b, c) to an equal power as in $[a, b, c]$. So these primes appear in all of a, b and c to the same power.

To summarize, we have $a = 2^x 3^y 5^{z+1} k$, $b = 2^x 3^{y+1} 5^z k$ and $c = 2^{x+1} 3^y 5^z k$ for some natural numbers $x, y, z \geq 0$ and a positive integer k that is coprime to 2, 3 and 5.

Taking out the common term $d = 2^x 3^y 5^z k$, this means that $a = 5d$, $b = 3d$ and $c = 2d$ where d can be any positive integer.

So the solution is that up to reordering, $(a; b; c) = (5d; 3d; 2d)$ where d is an arbitrary positive integer. In the original equation we can check that these are indeed solutions: $15d + 10d + 6d = 30d + d$.

E5. 21 bandits live in the city of Warmridge, each of them having some enemies among the others. Initially each bandit has 240 bullets, and duels with all of his enemies. Every bandit distributes his bullets evenly between his enemies, this means that he takes the same number of bullets to each of his duels, and uses each of his bullets only in one duel. In case the number of his bullets is not divisible by the number of his enemies, he takes as many bullets to each duel as possible, but takes the same number of bullets to every duel, so it is possible that in the end some bullets will remain by the bandit.
Shooting is banned in the city, therefore a duel consists only of comparing the number of bullets in the guns of the opponents, and the winner is the one who has more bullets. After the duel the sheriff takes the bullets of the winner to himself and as a protest the loser shoots all of his bullets into the air. What is the largest possible number of bullets by the sheriff after all of the duels have ended?
The enemy relations are mutual. If two opponents have the same number of bullets in their guns during a duel, then the sheriff takes the bullets of the bandit who has the wider hat among them.
Example: If a bandit has 13 enemies then he takes 18 bullets with himself to each duel, and 6 bullets remain by him in the end.

Solution: If there is one main villain, who is the enemy of everyone else, and the others are all friends with each other, then there are 20 duels in total, and to each of them the winner has taken all of his bullets with himself, so in this case the sheriff has $20 \cdot 240 = 4800$ bullets in the end.

We claim that it is not possible that the sheriff has a larger number of bullets by himself after the duels. Consider the bandit, who has the most enemies, if there are more bandits with this property then we consider the bandit who has the narrowest hat among them. This bandit looses each of his duels, so none of his 240 bullets end up by the sheriff. The other bandits have $20 \cdot 240 = 4800$ bullets in total, so the sheriff cannot take more than 4800 bullets to himself.

***Note**: You can read two more solutions for this problem in the E+ answer booklet (at problem 2).*