XIV. INTERNATIONAL DÜRER COMPETITION

problems

20 November 2020

E+

1. Determine all triples of positive integers a, b, c that satisfy
a) $[a, b] + [a, c] + [b, c] = [a, b, c]$.
b) $[a, b] + [a, c] + [b, c] = [a, b, c] + (a, b, c)$.
Remark: Here $[x, y]$ denotes the least common multiple of positive integers x and y, and (x, y) denotes their greatest common divisor.

2. 21 bandits live in the city of Warmridge, each of them having some enemies among the others. Initially each bandit has 240 bullets, and duels with all of his enemies. Every bandit distributes his bullets evenly between his enemies, this means that he takes the same number of bullets to each of his duels, and uses each of his bullets only in one duel. In case the number of his bullets is not divisible by the number of his enemies, he takes as many bullets to each duel as possible, but takes the same number of bullets to every duel, so it is possible that in the end some bullets will remain by the bandit.
Shooting is banned in the city, therefore a duel consists only of comparing the number of bullets in the guns of the opponents, and the winner is the one who has more bullets. After the duel the sheriff takes the bullets of the winner to himself and as a protest the loser shoots all of his bullets into the air. What is the largest possible number of bullets by the sheriff after all of the duels have ended?
The enemy relations are mutual. If two opponents have the same number of bullets in their guns during a duel, then the sheriff takes the bullets of the bandit who has the wider hat among them.
Example: If a bandit has 13 enemies then he takes 18 bullets with himself to each duel, and 6 bullets remain by him in the end.

3. Let k_1 and k_2 be two circles that are externally tangent at point C. We have a point A on k_1 and a point B on k_2 such that C is an interior point of segment AB. Let k_3 be a circle that passes through points A and B and intersects circles k_1 and k_2 another time at points M and N respectively. Let k_4 be the circumscribed circle of triangle CMN.
Prove that the centres of circles k_1, k_2, k_3 and k_4 all lie on the same circle.

4. Let p and q be polynomials with integer coefficients such that p is of degree n and has n nonnegative real roots (counted with multiplicity). Find all pairs of polynomials (p, q) that satisfy the equation

$$p(x^2) + q(x^2) = p(x)q(x)$$

and also the conditions mentioned above.

5. We have n distinct lines in three-dimensional space such that no two lines are parallel and no three lines meet at one point. What is the maximal possible number of planes determined by these n lines?
We say that a plane is determined if it contains at least two of the lines.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every page.
Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XIV. Dürer Competition