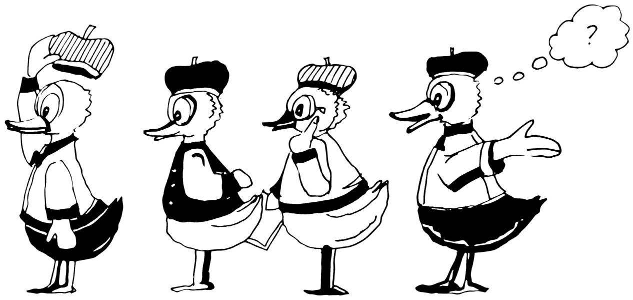
XV. DÜRER COMPETITION

Competition date: 12th November 2021.

MATHEMATICS

SOLUTIONS

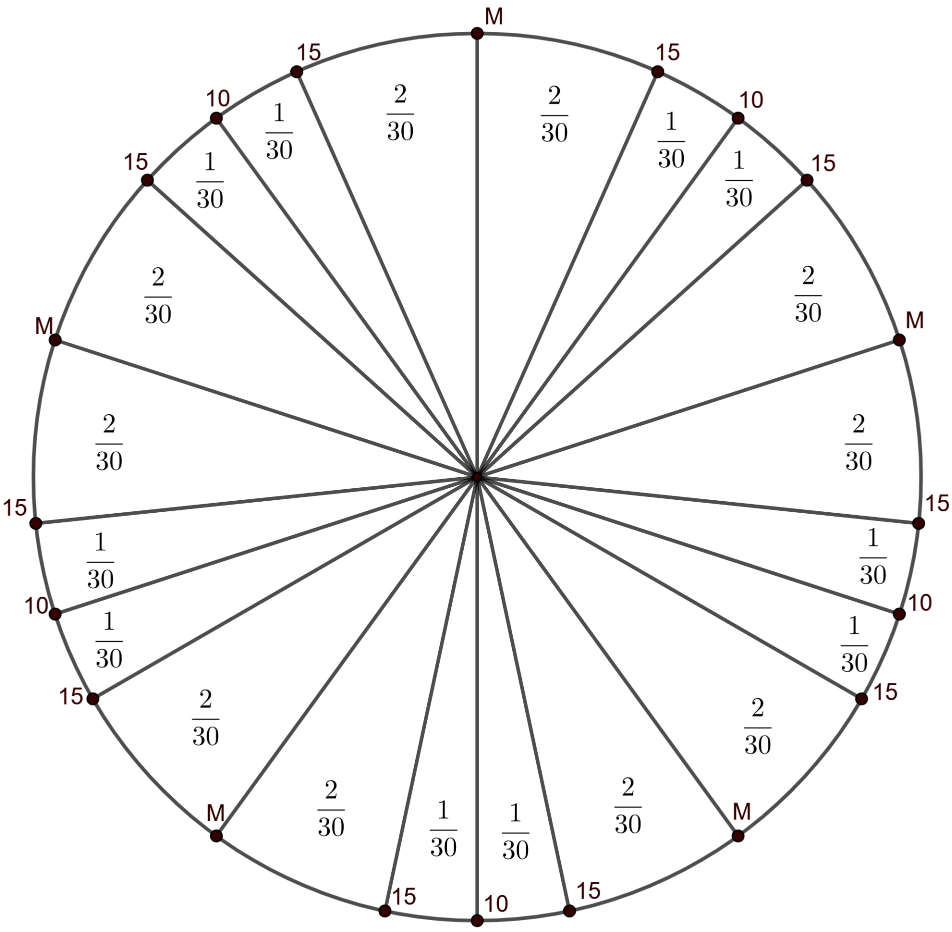


E

CATEGORY

E1. Dorothy organized a party for the birthday of Duck Mom and she also prepared a cylinder-shaped cake. Since she was originally expecting to have 15 guests, she divided the top of the cake into this many equal circular sectors, marking where the cuts need to be made. Just for fun Dorothy's brother Donald split the top of the cake into 10 equal circular sectors in such a way that some of the radii that he marked coincided with Dorothy's original markings. Just before the arrival of the guests Douglas cut the cake according to all markings, and then he placed the cake into the fridge.
This way they forgot about the cake and only got to eating it when only 6 of them remained. Is it possible for them to divide the cake into 6 equal parts without making any further cuts?

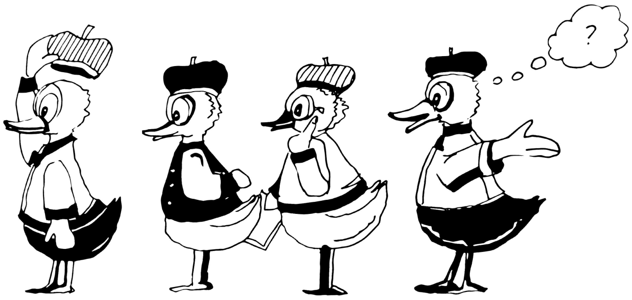
Solution:



Yes, they can. On the figure the cuts which are part of both splits are marked with an M, and the cuts which only appear in the 10 and 15 piece splits are marked with a 10 and 15 respectively. Note that 3 adjacent slices from the 15 piece split have the same size as 2 adjacent slices from the 10 piece split, as they are both one fifth of the cake, and hence there are 5 cuts appearing in both splits. For sake of simplicity, let's only consider the cake as a circle, and assume the area of the cake is 1. We will see, that all the slice sizes are multiples of $\frac{1}{30}$, so we will always count in one-thirtieths.

Let's look at two adjacent cuts marked by M. The circular sector between these two is split into 3 equal parts by the 15-cuts and the 10-cut splits the middle piece into two equal pieces. Hence between the two M cuts there are 2 slices with size $\frac{2}{30}$ and 2 with size $\frac{1}{30}$. The cake consists of 5 blocks like this, thus in total the cake has been split into 10 slices of size $\frac{2}{30}$, and 10 slices of size $\frac{1}{30}$.

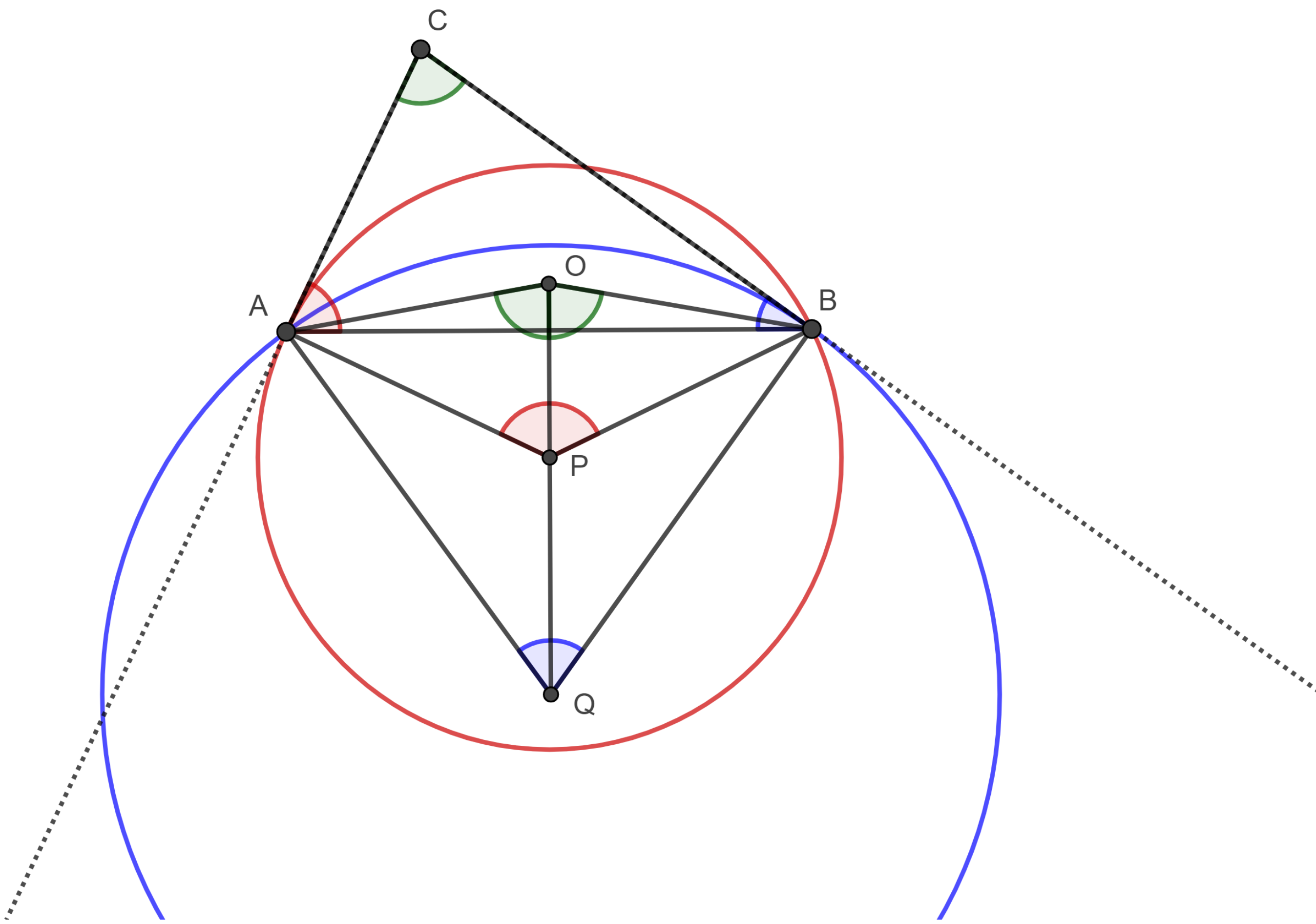
If 6 people want to share the cake fairly, everyone has to get $\frac{5}{30}$ of the cake. This is indeed achievable: 4 people get 2 big (with size $\frac{2}{30}$), and 1 small slices (with size $\frac{1}{30}$), while the remaining two people get 3 small, and one big slices.



E2. In the acute triangle ABC the circle through B touching the line AC at A has centre P, the circle through A touching the line BC at B has centre Q. Let R and O be the circumradius and circumcentre of triangle ABC, respectively. Show that $R^2 = OP \cdot OQ$.

Solution: Let the centre of the circumcircle be O, denote the angles of the triangle by α, β, γ and let k_1 and k_2 be the circles defined in the problem, tangent to sides BC and AC.

Since points O, P, Q all lie on the perpendicular bisector of segment AB, they are collinear.

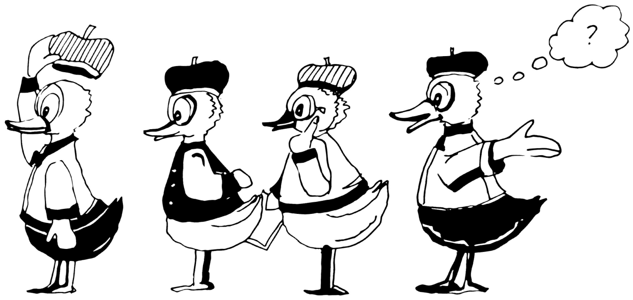


We will show that triangles AQO and PAO are similar. In this case because of the similarity, $\frac{OQ}{AO} = \frac{AO}{OP}$ holds and since O is the centre of the circumcircle, $AO = R$ from which we get the desired equality.

Now we will show that both of the triangles AQO and PAO are similar to triangle ABC. For this we will check that all of them have the same angles.

Using the inscribed angle theorem for circle k_2 and angle $\angle BAC$ we get that the inscribed angle belonging to chord BA is equal to α, therefore $\angle BPA = 2\alpha$. Similarly we get that in circle k_1, $\angle BQA = 2\beta$. Now by the inscribed angle theorem $\angle BOA = 2\gamma$.

Since line OQ is the perpendicular bisector of segment AB, it bisects the angles mentioned, meaning that $\angle OPA = \alpha$, $\angle OQA = \beta$ and $\angle QOA = \gamma$. Now in triangle PAO we know that two of the angles are α and γ, this means that the third angle is β. Similarly in triangle AQO the third angle is α, thus we have proven that triangles AQO and PAO are similar.



E3. Paraflea makes jumps on the plane, starting from the origin $(0,0)$. From point (x,y) it may jump to another point of the form $(x+p,\ y+p^2)$, where p is any positive real number. (The value of p may differ for each jump.)
a) Is there any point in quadrant I which cannot be reached by the flea? (Quadrant I contains points (x,y) for which x and y are positive real numbers.)
b) What is the minimum number of jumps that the flea must make from the origin so that it gets to the point $(100,1)$?

Solution:

a) It can't reach all points of the first quadrant, for example, it can't reach the point $(1,2)$. Assume by contradiction that the flea can reach this point. Then for all of its jumps, $p \leq 1$ has to hold. This means $p^2 \leq p$, so for all the jumps the y coordinate can increase by at most as much as the x coordinate. Because of this, using only jumps with $p \leq 1$, the flea can only reach points for which $x \geq y$. However, this is not true for the point $(1,2)$, hence it is unreachable.

b) Assume that the flea reaches the point $(100,1)$ in n steps, using the numbers $p_1, p_2, \ldots, p_n$. Then we have

$$\sum_{i=1}^{n} p_i = 100 \qquad \text{and} \qquad \sum_{i=1}^{n} p_i^2 = 1.$$

Using the AM-QM inequality for p_i and squaring both sides, we get

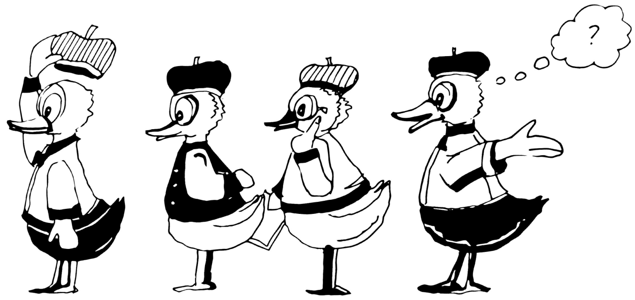
$$\frac{\sum_{i=1}^{n} p_i^2}{n} \geq \left(\frac{\sum_{i=1}^{n} p_i}{n}\right)^2.$$

Therefore,

$$\frac{1}{n} = \frac{\sum_{i=1}^{n} p_i^2}{n} \geq \left(\frac{\sum_{i=1}^{n} p_i}{n}\right)^2 = \frac{10000}{n^2},$$

thus $n \geq 10000$. But this lower bound is also achievable: if $n = 10000$, and $p_i = \frac{1}{100}$ for all i, then the sum of p_i is indeed 100, and the sum of their squares is $10000 \cdot \frac{1}{10000} = 1$.

Hence the flea needs at least 10000 jumps to get to the point $(100,1)$.



E4. We want to partition the integers $1, 2, 3, \ldots, 100$ into several groups such that within each group either any two numbers are coprime or any two are not coprime. At least how many groups are needed for such a partition?
We call two integers coprime if they have no common divisor greater than 1.

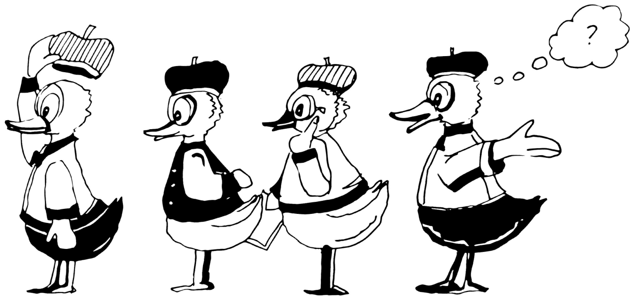
Solution: We can partition the numbers into 5 groups the following way: let us put into the first group all the numbers that are divisible by 2 and into the second group the numbers that are divisible by 3 and which are not in the first group. The third group will contain all the numbers that are divisible by 5, but are contained neither in the first nor in the second group. We put into the fourth group the numbers that are divisible by 7 and are not contained in any of the previous groups. And finally the last group will contain the rest of the numbers.

For the first, second, third and fourth group the condition holds, since any two numbers from the same group have a common divisor of $2, 3, 5$ or 7. We will prove that the fifth group contains only primes and the number 1, and therefore any two of them are coprime. This is true because the smallest prime divisor of any composite number $n \leq 100$ must be either $2, 3, 5$ or 7, since $11^2 > 100$. Therefore any composite number is contained in one of the first four groups.

Now we prove, that we cannot partition the numbers into fewer groups. Assume by contradiction, that we have partitioned them into 4 groups. Consider the numbers $2, 2^2, 2^3, 2^4, 2^5$. Using the pigeonhole principle, we get that there is a group which contains at least two of these integers, which means that in this group any two numbers must have a common divisor greater than 1. And since the only prime divisor of the numbers above is 2, every other number from that group must be divisible by 2. Hence there are 3 groups left.

Consider the numbers $3, 3^2, 3^3, 3^4$. In the same way as before one can show that there is a second group, in which every number is divisible by 3. So there are two more groups left.

Now take the numbers $5, 7, 5^2, 7^2, 5 \cdot 7$. Using pigeonhole principle we get that there is a group which contains at least 3 of the integers above. Examining all possible cases we get that there are only two possible ways to partition these numbers into two groups: $5, 5^2$ and $7, 7^2, 7 \cdot 5$ or $7, 7^2$ and $5, 5^2, 7 \cdot 5$. In both cases it is true that any two numbers from the same group must have a common divisor greater than 1, which leads to a contradiction, since for example we cannot put 1 into any of these groups. Therefore at least five groups are needed.

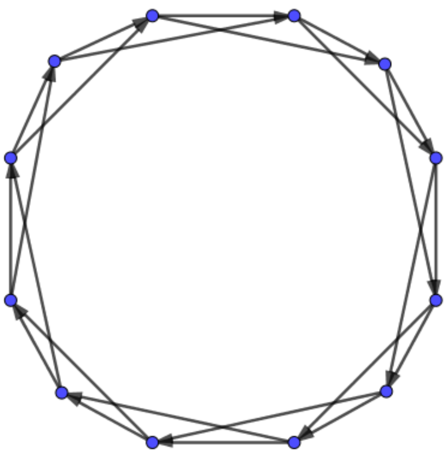


E5. **a)** A game master divides a group of 12 players into two teams of six. The players do not know what the teams are, however the master gives each player a card containing the names of two other players: one of them is a teammate and the other is not, but the master does not tell the player which is which. Can the master write the names on the cards in such a way that the players can determine the teams? (All of the players can work together to do so.)

b) On the next occasion, the game master writes the names of 3 teammates and 1 opposing player on each card (possibly in a mixed up order). Now he wants to write the names in such a way that the players together cannot determine the two teams. Is it possible for him to achieve this?

c) Can he write the names in such a way that the players together cannot determine the two teams, if now each card contains the names of 4 teammates and 1 opposing player (possibly in a mixed up order)?

Solution: a) Yes, he can achieve this, for example if he imagines that the players are arranged on a circle, such that the players from team A and team B alternate, and he gives to each player the names of the next two players on their right. This method works since if two neighbouring players were teammates, then the player immediately to the left of them would have received the names of two players in the same team, so it would not be possible that one of them is his teammate and the other is not. This means that along the circle, the players from the two teams must alternate. (In the figure, the points correspond to the players, and there is an arrow pointing from player A to player B if A received B's name.)

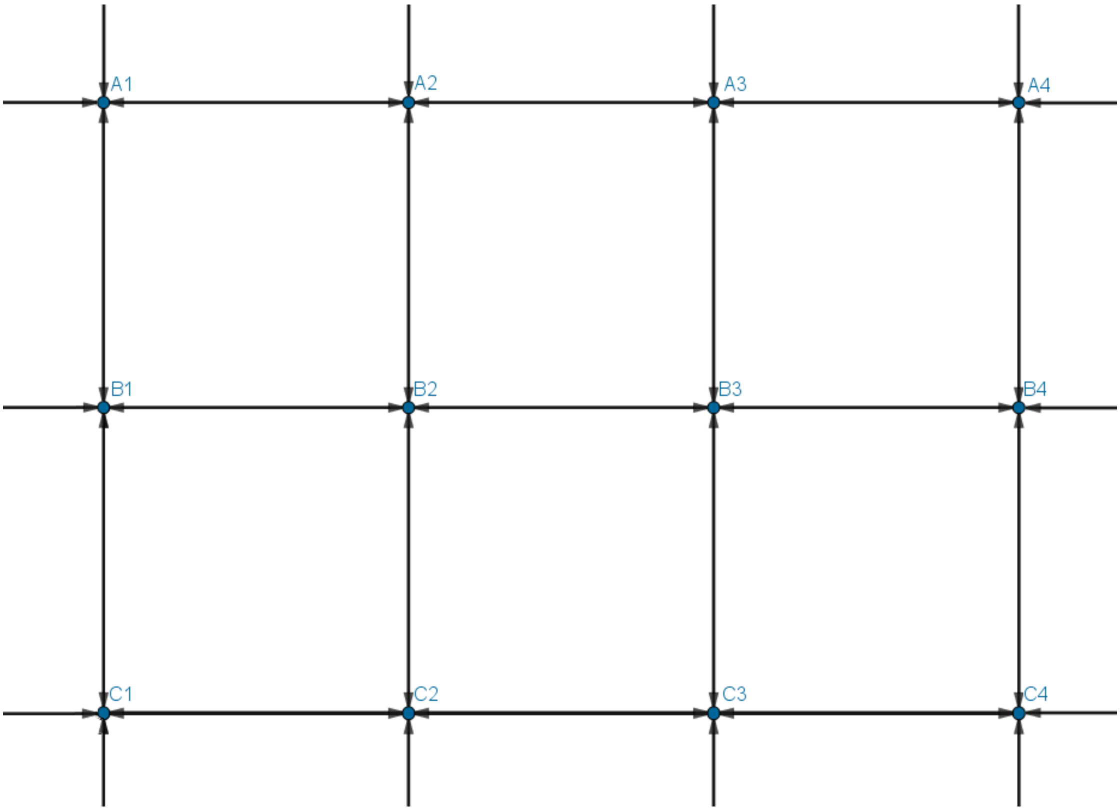


b) Yes, he can achieve this. Let the players be A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3 and C_4. Then the game master gives a card to each player containing the names of those players who either:

- have the same number, or
- have the same letter and a number differing by 1 or 3.

For example, C_3 receives the names of A_3, B_3, C_2 and C_4; and player A_1 receives the names of B_1, C_1, A_2 and A_4. Then the two teams can be as follows: one team contains those players whose number is 1 or 2, and the other contains those whose number is 3 or 4. But alternatively, one team could contain those players whose number is 1 or 4, and the other could contain those with 2 or 3.

(In the following figure, arrows leaving the figure on the left come back on the right and vice versa, and arrows leaving on the top come back on the bottom, and vice versa.)



c) No, he cannot achieve this. We know that each player is a teammate of themselves, so if everyone added his own name to his card, then among the 6 players on his card, 5 would be teammates, and 1 would be an opponent. So everybody would have one teammate not appearing on his card. If two players are in the same team, then after they have added their own names, their cards contain at least 4 names in common (since in their team, there is 1 player not appearing on one of the cards, and 1 not appearing on the other card. If these two non-appearing names are different then the two cards contain at least 4 names in common, and if they are the same then they contain at least 5). However if two players are not on the same team, then their cards can only contain at most 2 names in common, since each of them has 1 player who is not his teammate (and this player can appear on the other player's card as the other player's teammate). So if any two players compare their cards (after adding their own name) then depending on the number of common names, they can determine if they are teammates. So the players (working all together) can determine the two teams.