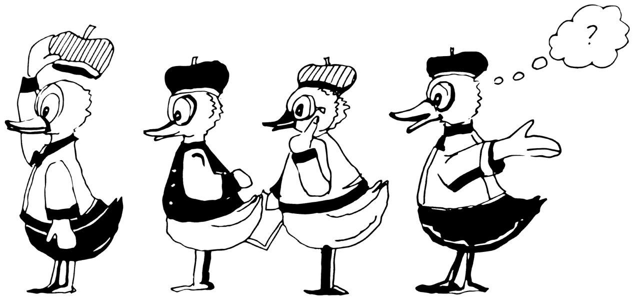
XV. DÜRER COMPETITION

Competition date: 12th November 2021.

MATHEMATICS

SOLUTIONS



E+ CATEGORY

E+1. We want to partition the integers $1, 2, 3, \ldots, 100$ into several groups such that within each group either any two numbers are coprime or any two are not coprime. At least how many groups are needed for such a partition?
We call two integers coprime if they have no common divisor greater than 1.

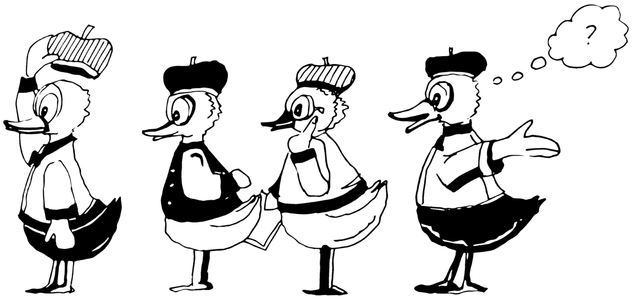
Solution: We can partition the numbers into 5 groups the following way: let us put into the first group all the numbers that are divisible by 2 and into the second group the numbers that are divisible by 3 and which are not in the first group. The third group will contain all the numbers that are divisible by 5, but are contained neither in the first nor in the second group. We put into the fourth group the numbers that are divisible by 7 and are not contained in any of the previous groups. And finally the last group will contain the rest of the numbers.

For the first, second, third and fourth group the condition holds, since any two numbers from the same group have a common divisor of $2, 3, 5$ or 7. We will prove that the fifth group contains only primes and the number 1, and therefore any two of them are coprime. This is true because the smallest prime divisor of any composite number $n \leq 100$ must be either $2, 3, 5$ or 7, since $11^2 > 100$. Therefore any composite number is contained in one of the first four groups.

Now we prove, that we cannot partition the numbers into fewer groups. Assume by contradiction, that we have partitioned them into 4 groups. Consider the numbers $2, 2^2, 2^3, 2^4, 2^5$. Using the pigeonhole principle, we get that there is a group which contains at least two of these integers, which means that in this group any two numbers must have a common divisor greater than 1. And since the only prime divisor of the numbers above is 2, every other number from that group must be divisible by 2. Hence there are 3 groups left.

Consider the numbers $3, 3^2, 3^3, 3^4$. In the same way as before one can show that there is a second group, in which every number is divisible by 3. So there are two more groups left.

Now take the numbers $5, 7, 5^2, 7^2, 5 \cdot 7$. Using pigeonhole principle we get that there is a group which contains at least 3 of the integers above. Examining all possible cases we get that there are only two possible ways to partition these numbers into two groups: $5, 5^2$ and $7, 7^2, 7 \cdot 5$ or $7, 7^2$ and $5, 5^2, 7 \cdot 5$. In both cases it is true that any two numbers from the same group must have a common divisor greater than 1, which leads to a contradiction, since for example we cannot put 1 into any of these groups. Therefore at least five groups are needed.



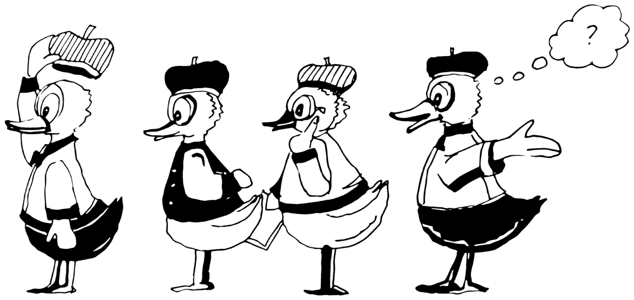
E+2. Determine all triangles that can be split into two congruent pieces by one cut. A cut consists of segments P_1P_2, P_2P_3, $\dots$, $P_{n-1}P_n$ where points $P_1, P_2, \dots, P_n$ are distinct, points P_1 and P_n lie on the perimeter of the triangle and the rest of the points lie in the interior of the triangle such that the segments are disjoint except for the endpoints.

Solution: If ABC is an isosceles triangle, then the altitude corresponding to the the base of the triangle splits the triangle into two congruent right triangles.

If ABC is not isosceles, then we claim there is no such cut. Assume by contradiction that there is a suitable cut, consisting of $P_1, P_2, \dots, P_n$. If neither of P_1, P_n coincides with a vertex of the triangle, then one of the resulting polygons would have $n+1$ sides, while the other would have $n+2$ sides, which is a contradiction. Because of this we can assume $P_1 = A$ from now on.

The perimeter of the two resulting polygons is equal, and the segments AP_2, P_2P_3, ..., $P_{n-1}P_n$ are sides of both, thus $AB + BP_n = AC + CP_n$ has to hold. Furthermore, because of the congruence, the multiset of the side lengths of the polygons has to be same for both. They have $n-1$ coinciding sides, therefore this implies that $\{AB, BP_n\} = \{AC, CP_n\}$. However, we assumed that the triangle is not isosceles, so $AB \neq AC$, thus $AB = CP_n$ and $AC = BP_n$. From this we get the equation $AB + AC = BP_n + CP_n = BC$, which contradicts the triangle inequality.

Therefore there is a suitable cut if and only if the triangle is isosceles.



E+3. **a)** A game master divides a group of 40 players into four teams of ten. The players do not know what the teams are, however the master gives each player a card containing the names of two other players: one of them is a teammate and the other is not, but the master does not tell the player which is which. Can the master write the names on the cards in such a way that the players can determine the teams? (All of the players can work together to do so.)

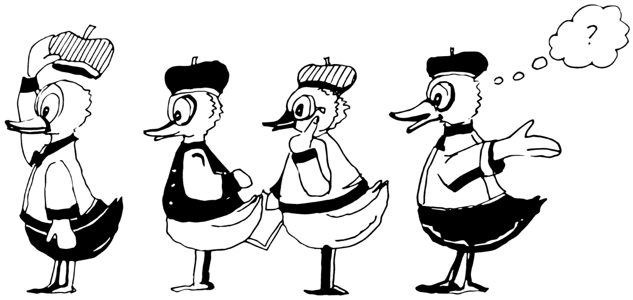
b) On the next occasion, the game master writes the names of 7 teammates and 2 opposing players on each card (possibly in a mixed up order). Now he wants to write the names in such a way that the players together cannot determine the four teams. Is it possible for him to achieve this?

c) Can he write the names in such a way that the players together cannot determine the four teams, if now each card contains the names of 6 teammates and 2 opposing players (possibly in a mixed up order)?

Solution: a) Yes, he can achieve this. The game master imagines that the 4 teams are sitting around two circular tables with 20 seats each. At the first table, the players in the first two teams sit alternatingly, and at the second table the players of the other two teasm sit alternatingly. The game master gives to each player the names of the two players immediately to their right at the same table. It is clear that everybody received the name of one teammate and one opposing player. Take any possible assignment of the 4 teams that satisfy the cards just described. If there are two players at the same table who are teammates and sit next to each other, then the player sitting immediately to their left would have received the names of two players on the same team, which is impossible. So nobody is in the same team as their neighbour to the right. So everybody must be in the same team as the second player to their right. This implies that a team must contain every other player from the same table. So the composition of the four teams is indeed unique.

b) No. Make everybody add their own name to their card, so their cards will contain 8 teammates and 2 opposing players. If two players are in the same team then their cards must contain at least 6 names in common, as both of them only have 2 teammates not appearing on their card, so their team has only at most 4 players not appearing on either of the two cards. However if two players are in different teams then there can be at most 4 players appearing on both of their cards, as if someone appears on both then he is not a teammate of at least one of the two players, and the two cards contain 4 such players in total. So any two players can determine (by comparing their cards) whether they are in the same team or not, so the players (all working together) can determine the teams.

c) Yes. Divide the players into 8 groups $A_1, A_2, \ldots, A_8$ each containing 5 players. Let the real teams be the union of groups A_{2i-1} and A_{2i} for $1 \leq i \leq 4$. For every i, let each person in group A_i get the names of everybody in the same group, and two people from group A_{i+1} and two from A_{i-1} (indices are considered modulo 8). It is clear that according to the real teams, everybody received the names of 6 teammates and 2 opposing players, however if the teams were the unions of groups A_{2i} and A_{2i+1} for $1 \leq i \leq 4$, then the same cards would also be appropriate. So the players cannot determine what the actual teams are.



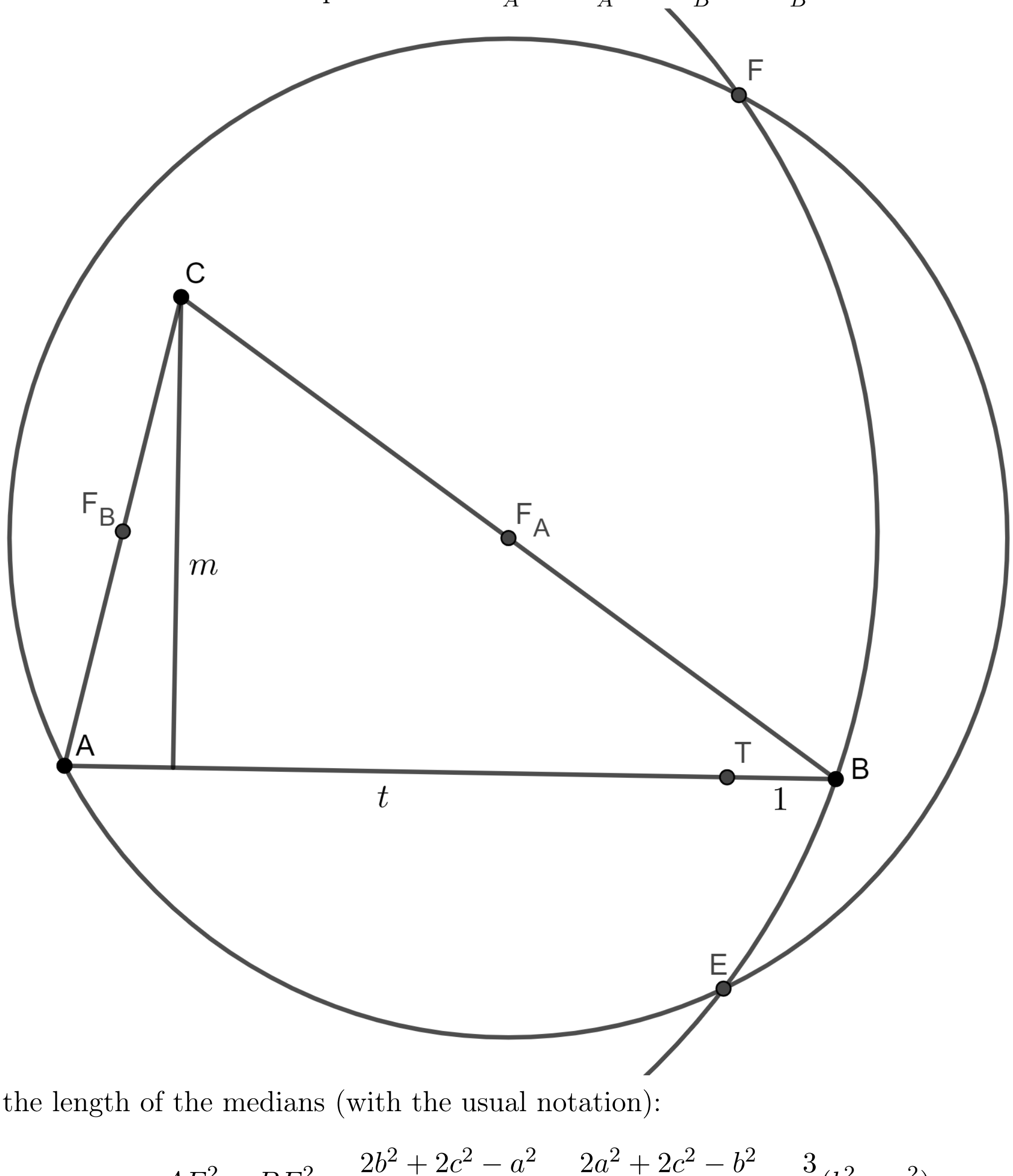
E+4. Let ABC be an acute triangle, and let F_A and F_B be the midpoints of sides BC and CA, respectively. Let E and F be the intersection points of the circle centered at F_A and passing through A and the circle centered at F_B and passing through B. Prove that if segments CE and CF have midpoints N and M, respectively, then the intersection points of the circle centered at M and passing through E and the circle centered at N and passing through F lie on the line AB.

Solution: Firstly we will show that line FE is perpendicular to line AB and that the reflection of point C by line AB also lies on this line.

Segment F_AF_B is a midsegment in triangle ABC therefore it is parallel to AB. Since E and F are reflections of each other by line F_AF_B, it means that FE is perpendicular this line, therefore also to AB.

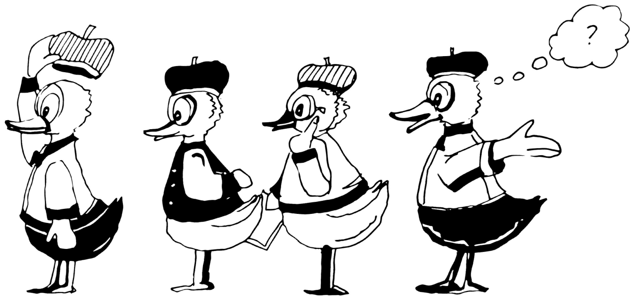
Denote the circle with centre F_A passing through A by k_A, the circle with centre F_B passing through B by k_B. Now since line FE is the power line of circles k_A and k_B, it is enough to show that the power of the reflection of the foot of the height from C in triangle ABC is the same in both circles.

Without loss of generality we can assume that BT is of length 1. Then if $AT = t$ and the height at C is m then we would like to prove that $TF_A^2 - AF_A^2 = TF_B^2 - BF_B^2$.



By the length of the medians (with the usual notation):

$$AF_A^2 - BF_B^2 = \frac{2b^2 + 2c^2 - a^2}{4} - \frac{2a^2 + 2c^2 - b^2}{4} = \frac{3}{4}(b^2 - a^2).$$



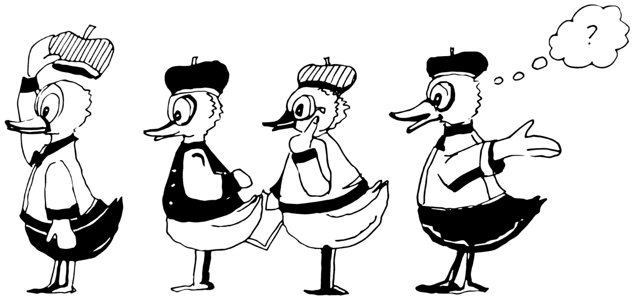
Using the Pythagorean theorem:

$$TF_A^2 - TF_B^2 = m^2/4 + (t/2-1)^2 - m^2/4 - (t-1/2)^2 = \frac{3}{4}(1-t^2),$$

$$\frac{3}{4}(b^2-a^2) = \frac{3}{4}(1+m^2-m^2-t^2).$$

Thus indeed EF is the line perpendicular to AB passing through the reflection of C to the perpendicular bisector of AB.

Using this result now in triangle CEF we get that the points of intersections mentioned in the problem will lie on the line perpendicular to EF, containing the reflection of the feet of the height from C by line EF. Clearly, this is line AB therefore we have finished the proof.



E+5. Let $a_1 \le a_2 \le \ldots \le a_n$ be real numbers for which

$$\sum_{i=1}^{n} a_i^{2k+1} = 0$$

holds for all integers $0 \le k < n$. Show that in this case, $a_i = -a_{n+1-i}$ holds for all $1 \le i \le n$.

Solution:

Let's take the $n \times n$ matrix M whose rows are numbered from 0 to $n-1$ and its k-th row is $(a_1^{2k}, a_2^{2k}, \ldots, a_n^{2k})$.

We multiply this matrix M by the vector $(a_1, a_2, \ldots, a_n)$. The resulting vector's elements are $(a_1 + a_2 + \ldots + a_n)$, $(a_1^3 + a_2^3 + \ldots + a_n^3)$, and so on, up until $(a_1^{2n-1} + a_2^{2n-1} + \ldots + a_n^{2n-1})$

We know that all these values are 0, from the assumptions in the problem. We can assume that the vector $(a_1, a_2, \ldots, a_k)$ is not the zero vector, as otherwise the statement of the problem is trivially true. Hence there is a non-zero vector v, for which $M \cdot v = \mathbf{0}$, so M's rows are linearly dependent.

Therefore there is an integer $0 < m \le n-1$, for which the m-th row of M is a linear combination of the rows $0, 1, \ldots, m-1$.

Let's prove that for all integers $k \ge 0$, $\sum_{j=1}^{n} a_j^{2k+1} = 0$ holds by induction.

According to the problem statement the induction hypothesis holds for $0 \le k < n$. Let's assume that for some $r \ge n$ we already know that for all $0 \le k < r$, $\sum_{j=1}^{n} a_j^{2k+1} = 0$ is true, and we want to prove that $\sum_{j=1}^{n} a_j^{2r+1} = 0$.

Let's take the vector $(a_1^{2(r-m)+1}, a_2^{2(r-m)+1}, \ldots, a_n^{2(r-m)+1})$. If we multiply this vector by the zeroth, first, ..., $(m-1)$-th row of matrix M, we get the values $\sum_{j=1}^{n} a_j^{2(r-m)+1}$, $\sum_{j=1}^{n} a_j^{2(r-m+1)+1}$, ... and $\sum_{j=1}^{n} a_j^{2(r-1)+1}$ respectively. According to our inductive hypothesis all of these values are 0.

The m-th row of matrix M can be written as a linear combination of the first m. Hence if we multiply $(a_1^{2(r-m)+1}, a_2^{2(r-m)+1}, ..., a_n^{2(r-m)+1})$ by the m-th row of the matrix, then the result is a linear combination of the previous products, thus this is 0 as well.

Therefore $\sum_{j=1}^{n} a_j^{2r+1} = 0$, and we have proven the inductive step.

Let $T = \max(|a_j|)$, and if not all a_j have absolute value T, let the second largest absolute value be R.

As $\frac{R}{T} > 1$, there exists a large enough k, so that $(\frac{R}{T})^{2k+1} < \frac{1}{2n}$.

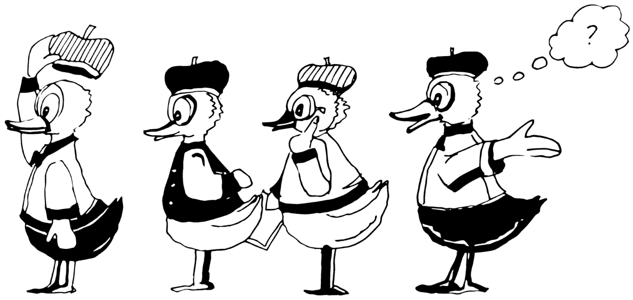
Among the a_j-s with absolute value T, let there be s with the value T and h with the value $-T$. Let's assume $s \neq h$. Then $0 = \sum_{j=1}^{n} a_j^{2k+1} = (s-h)T^{2k+1} + \sum_{|a_j| \le R} a_j^{2k+1}$.

On the other hand, for our sufficiently large k, $\left| \sum_{|a_j| \le R} a_j^{2k+1} \right| \le nT^{2k+1}(\frac{R}{T})^{2k+1} \le \frac{1}{2}T^{2k+1}$.

Using $|s-h| \ge 1$, we have

$$\left| \sum_{j=1}^{n} a_j^{2k+1} \right| = \left| (s-h)T^{2k+1} + \sum_{|a_j| \le R} a_j^{2k+1} \right| \ge \frac{T^{2k+1}}{2},$$

which is a contradiction, as this value should be 0.



Hence $s = h$, thus for the ordering $a_1 \leq a_2 \leq \ldots \leq a_n$, we have $a_1 = a_2 = \ldots = a_s = -a_{n-s+1} = -a_{n-s+2} = \ldots = -a_n$, so for these elements $a_i = -a_{n-i+1}$ holds. Removing these elements, the condition $\sum_{j=s+1}^{n-s} a_j^{2k+1} = 0$ will still hold for all k, as the removed terms had a sum of 0.

This way we have less elements in the sum, and by induction on n, we can prove that $a_i = -a_{n-i+1}$ holds for all i.