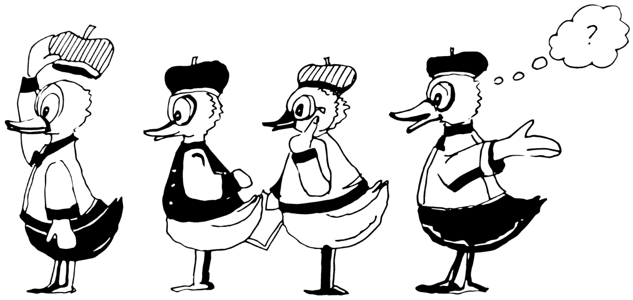
XV. DÜRER COMPETITION

Competition date: 12th November 2021.

MATHEMATICS

PROBLEMS



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CATEGORY

1. Dorothy organized a party for the birthday of Duck Mom and she also prepared a cylinder-shaped cake. Since she was originally expecting to have 15 guests, she divided the top of the cake into this many equal circular sectors, marking where the cuts need to be made. Just for fun Dorothy's brother Donald split the top of the cake into 10 equal circular sectors in such a way that some of the radii that he marked coincided with Dorothy's original markings. Just before the arrival of the guests Douglas cut the cake according to all markings, and then he placed the cake into the fridge.
This way they forgot about the cake and only got to eating it when only 6 of them remained. Is it possible for them to divide the cake into 6 equal parts without making any further cuts?

2. In the acute triangle ABC the circle through B touching the line AC at A has centre P, the circle through A touching the line BC at B has centre Q. Let R and O be the circumradius and circumcentre of triangle ABC, respectively. Show that $R^2 = OP \cdot OQ$.

3. Paraflea makes jumps on the plane, starting from the origin $(0,0)$. From point (x,y) it may jump to another point of the form $(x+p,\ y+p^2)$, where p is any positive real number. (The value of p may differ for each jump.)
a) Is there any point in quadrant I which cannot be reached by the flea? (Quadrant I contains points (x,y) for which x and y are positive real numbers.)
b) What is the minimum number of jumps that the flea must make from the origin so that it gets to the point $(100,1)$?

4. We want to partition the integers $1,2,3,\ldots,100$ into several groups such that within each group either any two numbers are coprime or any two are not coprime. At least how many groups are needed for such a partition?
We call two integers coprime if they have no common divisor greater than 1.

5. a) A game master divides a group of 12 players into two teams of six. The players do not know what the teams are, however the master gives each player a card containing the names of two other players: one of them is a teammate and the other is not, but the master does not tell the player which is which. Can the master write the names on the cards in such a way that the players can determine the teams? (All of the players can work together to do so.)

b) On the next occasion, the game master writes the names of 3 teammates and 1 opposing player on each card (possibly in a mixed up order). Now he wants to write the names in such a way that the players together cannot determine the two teams. Is it possible for him to achieve this?

c) Can he write the names in such a way that the players together cannot determine the two teams, if now each card contains the names of 4 teammates and 1 opposing player (possibly in a mixed up order)?

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XV. Dürer Competition