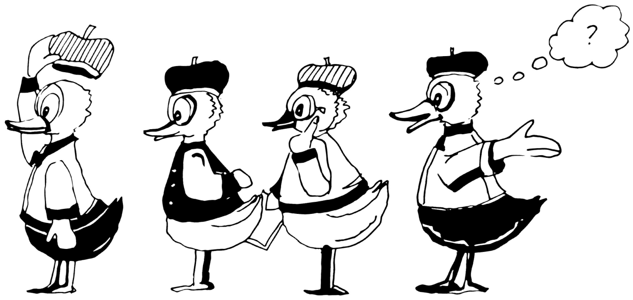
XV. DÜRER COMPETITION

Competition date: 12th November 2021.

MATHEMATICS
PROBLEMS



E+
CATEGORY

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1. We want to partition the integers $1, 2, 3, \ldots, 100$ into several groups such that within each group either any two numbers are coprime or any two are not coprime. At least how many groups are needed for such a partition?
We call two integers coprime if they have no common divisor greater than 1.

2. Determine all triangles that can be split into two congruent pieces by one cut. A cut consists of segments P_1P_2, P_2P_3, $\ldots$, $P_{n-1}P_n$ where points $P_1, P_2, \ldots, P_n$ are distinct, points P_1 and P_n lie on the perimeter of the triangle and the rest of the points lie in the interior of the triangle such that the segments are disjoint except for the endpoints.

3. a) A game master divides a group of 40 players into four teams of ten. The players do not know what the teams are, however the master gives each player a card containing the names of two other players: one of them is a teammate and the other is not, but the master does not tell the player which is which. Can the master write the names on the cards in such a way that the players can determine the teams? (All of the players can work together to do so.)

b) On the next occasion, the game master writes the names of 7 teammates and 2 opposing players on each card (possibly in a mixed up order). Now he wants to write the names in such a way that the players together cannot determine the four teams. Is it possible for him to achieve this?

c) Can he write the names in such a way that the players together cannot determine the four teams, if now each card contains the names of 6 teammates and 2 opposing players (possibly in a mixed up order)?

4. Let ABC be an acute triangle, and let F_A and F_B be the midpoints of sides BC and CA, respectively. Let E and F be the intersection points of the circle centered at F_A and passing through A and the circle centered at F_B and passing through B. Prove that if segments CE and CF have midpoints N and M, respectively, then the intersection points of the circle centered at M and passing through E and the circle centered at N and passing through F lie on the line AB.

5. Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be real numbers for which

$$\sum_{i=1}^{n} a_i^{2k+1} = 0$$

holds for all integers $0 \leq k < n$. Show that in this case, $a_i = -a_{n+1-i}$ holds for all $1 \leq i \leq n$.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XV. Dürer Competition