2021

14th Dürer Competition

Problems, Solutions

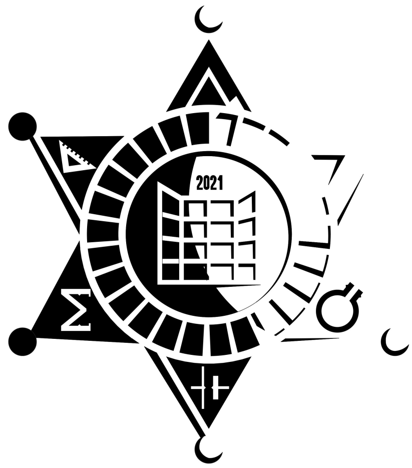
C, D, E, E+

Contents

Introduction – About the Dürer Competition

Plenty of mathematics contests are traditionally held in Hungary. From primary schoolers to university students, everybody can find a contest that fits their age and qualifications. These are mostly individual contests where the participants sit down in a room for a few hours, working on the problems quietly. However at the Dürer Competition, there are teams of 3 taking part. For the duration of the contest each team works together to solve the problems, so the contestants can experience the benefits of cooperative thinking. Our experience shows that the majority of students are happier and more relaxed than during an individual contest.

It is a very important goal for us to set interesting problems to show the beauty of mathematics and the joy of thinking to lots of students. We also wish to include as many original problems as possible. In each year, about 150 problems appear on the contest – of course not all can be original, but we invent most of the harder problems on our own.



The theme of the 14^{th} competition was the Wildwest.

At this point we definitely have to mention that the organising team traditionally consists of young people, mostly university students studying maths. This dates back to the early years of the competition, and ever since then, we can regularly welcome former competitors as new organisers. The success of the competition depends on this community, consisting of 30 to 70 people. Some of them have been organisers for 10 years already (and still take part enthusiastically, even alongside a full-time job) and some of them take important responsibilities as first-year undergraduates already.

This is the spirit in which we have been organising the contest for 14 years. The competition attracts more and more students and schools with each year. In the 2020–21 academic year, more than 600 Hungarian students competed in the high school maths categories.

This was the second year that we opened our two hardest categories for international competitors. Due to the pandemic the contest was held online with more than 150 students taking part in the regional round of the competition. To our satisfaction we attracted teams from four new countries: Bosnia and Herzegovina, Croatia, Indonesia and the United Kingdom.

Primary school students can take part in our competition in the following two categories: 5^{th} and 6^{th} grade students compete in *category A* while 7^{th} and 8^{th} grade students compete in *category B*. The contest is regional: it is organised in 6 cities in northeastern Hungary, but is open to anyone provided that they travel to one of the locations. (The problems of these two categories are not included in this booklet.)

Four categories are available for high school students:

- **Category C** is open to 9^{th} and 10^{th} graders who have never previously qualified for the final of any national math contest.
- **Category D** is open to 9^{th} to 12^{th} graders who are a bit more experienced, but do not come from a school that is outstanding in handling mathematical talents.

- **Category E** is open to 9^{th} to 12^{th} graders who already have good results from other contests, or come from a school outstanding in maths.
- **Category E$^+$** is designed for competitors who actively take part in olympiad training. In this category, most teams include some student who has taken part at an international olympiad (IMO, MEMO, EGMO, RMM, CMC), or is about to qualify for one in the same academic year.

We also organise the contest in physics (*category F*) and chemistry (*categories K* and K^+), but these are also omitted from this booklet.

For high schoolers (in categories C, D and E), the first round is an **online relay round** consisting of 9 problems. The answer to each question is an integer between 0 and 9999. Initially each team gets the first question only. They have three attempts to submit an answer – if they get it right, they score a set number of points, and can proceed to the next question. Each wrong attempt to a question reduces the possible score by 1, and after 3 wrong attempts the team must move on to the next question without scoring.

The second round is a traditional **olympiad-style contest**, where detailed proofs have to be given. The teams have 3 hours to solve 5 problems. The contest can be sat in the whole country, at about 20 locations.

Under normal conditions, the final round takes place in Miskolc. For high schoolers (categories C, D, E, E$^+$, F, K, K$^+$) we organise it on a weekend in early February from Thursday to Sunday. The first competition day is Friday, with the students working on five **olympiad-style problems** and a **game**. If a team thinks that they have found the winning strategy for the game, they can challenge us. If they can defeat us twice in a row, they get the maximal score for the problem. If they lose, they can still challenge us two more times for a partial score. On Saturday we hold a **relay round** consisting of 16 questions. The rules are similar to the online round. Rankings are based on a combined score from the two competition days.

At the weekend of the final, the students and teachers can participate in many educational and recreational activities, such as lectures, games and discussions about universities.

This year, due to the pandemic, the finals were held online with the help of the Discord platform. Based on the feedback from the students, it was very much enjoyed by them regardless.

The competition is expanded year after year. In 2021–22 we plan to introduce some changes in the marking of the international competition with involving more volunteers from other coutries. In the long run this would contribute to the establishment of the Dürer Competition as a well-known and renowned contest which is one of our main objectives.

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Publisher: Gábor Szűcs, The Joy of Thinking Foundation

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1 Problems

1.1 Online round

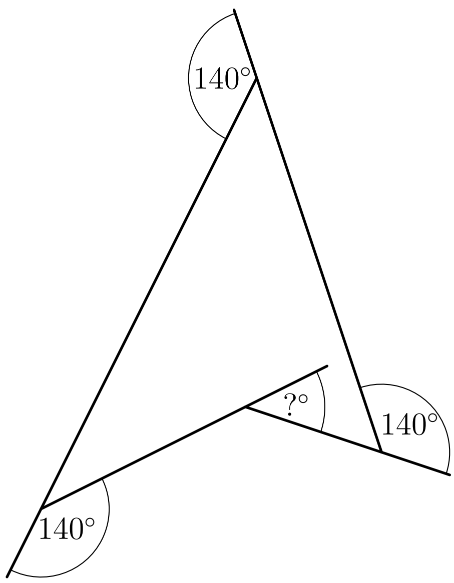
1.1.1 Category C

1. Timi completed the table below with the numbers $1, 2, 3, 4$ in a way that each column and each row contains each number exactly once. Which number can be written in the cell marked with x?

			1
	2		
		x	
1			4

(Solution)

2. How many degrees is the angle marked by ? on the picture?

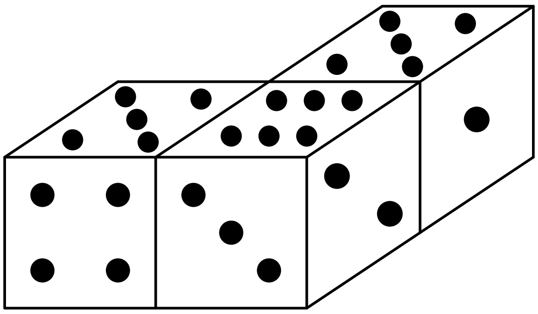


(Solution)

3. Kartal, Tarkal, Lartak and Raltak are good friends. They want to extend their social circle, but they only want to meet people whose name can be obtained from the word KARTAL by exchanging the consonants. What is the maximum number of new friends they can make if they do not want to have multiple people in the group sharing the same name?

(Solution)

4. What is the sum of digits on the non-visible sides of these three fair dice?
Remark: Each die contains the numbers from 1 to 6.



(Solution)

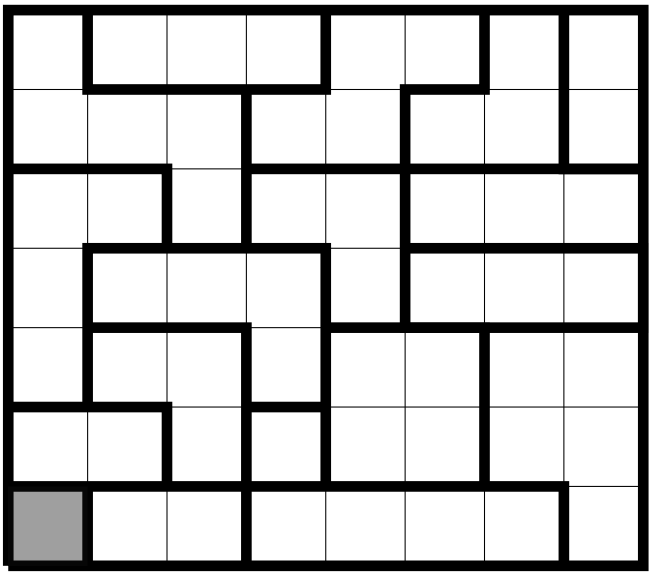
5. Grandma baked 91 cookies and distributed them equally among her grandchildren. Grandpa was saddened to find that not even a single cookie was left for him to taste. How many cookies did each grandchild get if we know that when Anne was just starting her second cookie, Tommy already finished his 9th one?

(Solution)

6. 2020 is such that the number of its digits, the sum of its digits and the product of its nonzero digits are all equal. How many positive integers less than 2020 satisfy this property? (If there is only one nonzero digit, the product is simply this number.)

(Solution)

7. A robot starts from the bottom-left cell of the following table, moving upward. At each step it either continues its journey to the cell in front of it or it turns 90° left or right. The robot stops when it reaches the starting cell again. How many times could the robot have turned on its journey if we know that it stepped on each cell exactly once and it stepped into the region bounded by the bold line only once?



(Solution)

8. A family (mother, father, grandma, little boy, little girl) travel in a 5-person car. Only the mother and the father can drive. The children cannot sit in the front, and the father does not want to sit next to the grandma. In how many way can they sit in the car? *In the car, there are two places in the front, 3 places in the back. Two places are neighboring, if they are in the same row next to each other.*

(Solution)

9. Gábor wrote down some 3-digit numbers in increasing order, so that the sums of their digits were in strictly decreasing order. What is the maximal number of numbers Gábor could have written?

(Solution)

10. Game: We place a rook on one square of a 8×8 chessboard. Two players take turns to move the rook. In one move, a player either moves any number of squares to the right, or any number of squares downwards. The player who reaches the bottom right corner of the board wins.

Beat the computer twice in a row in this game! Knowing the starting position, you can decide whether you want to take the role of the first or the second player.

(Solution)

1.1.2 Category D

1. Timi completed the table below with the numbers $0, 1, 2, 3, 4$ in a way that each column and each row contains each number exactly once. Which number can be written in the cell marked with x?

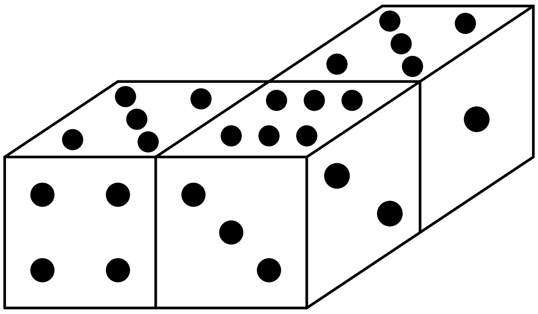
2	0		x	
	2	0		
		1	0	
			2	4
1	4			

(Solution)

2. Kartal, Tarkal, Lartak and Raltak are good friends. They want to extend their social circle, but they only want to meet people whose name can be obtained from the word KARTAL by exchanging the consonants. What is the maximum number of new friends they can make if they do not want to have multiple people in the group sharing the same name?

(Solution)

3. What is the sum of digits on the non-visible sides of these three fair dice?
Remark: Each die contains the numbers from 1 to 6.



(Solution)

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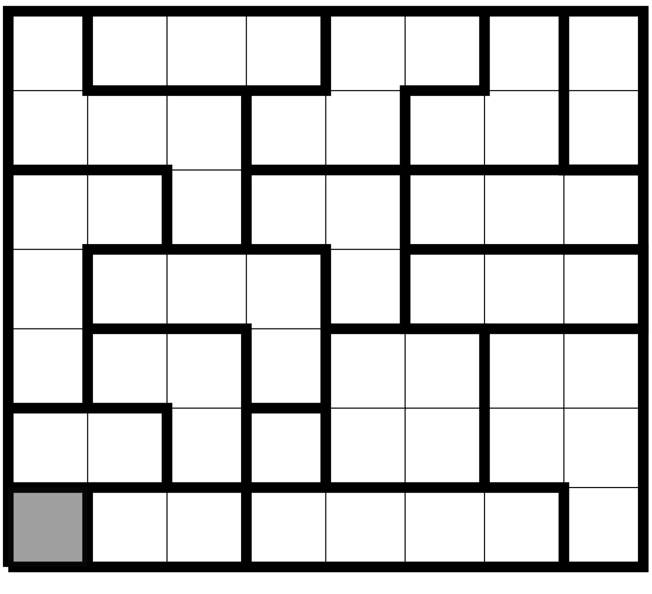
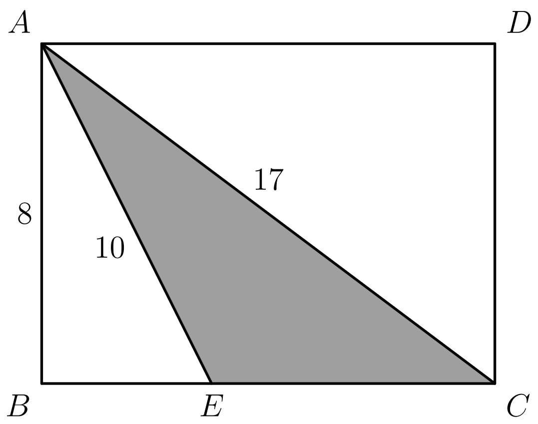


(Solution)

5. Ákos ordered some chicken nuggets and onion rings. Even though he asked for them in separate boxes, they arrived mixed together in one box, which made Ákos so annoyed that he ate one fifth of the chicken nuggets. This way one and a half times more onion rings remained than chicken nuggets. After this Kristóf ate 44 onion rings since he was not planning to see people for the rest of the day. This way the number of remaining onion rings was only $\frac{2}{5}$ times the number of chicken nuggets. How many chicken nuggets and onion rings did Ákos order in total?

(Solution)

6. The rectangle $ABCD$ has a side AB of length 8, and a side AC of length 17. E is an inner point of the side BC. What is the area of triange AEC, if the length of AE is 10?



(Solution)

7. It is a well-known fact that Süsü, the dragon has only one head, but all five of his siblings have more than one. One day the siblings of Süsü sit around a large table and each dragon tells to the others the average number of heads they have. The following sentences are said.

- You have 7 heads on average.
- You have 9 heads on average.
- You have 8 heads on average.
- You have 8 heads on average.
- You have 9 heads on average.

How many heads do the siblings of Süsü have in total?

(Solution)

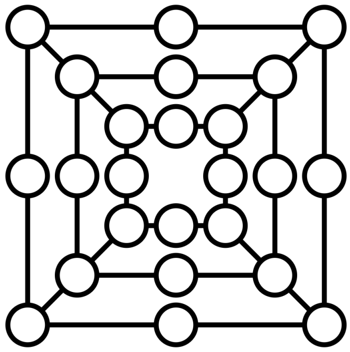
8. By using only two different kinds of number cards, Marvin created two positive integers. In how many possible ways could he do it, if the sum of the two numbers is 10000?
Note: Switching the two integers does not count as a separate case.

(Solution)

9. David used 9 sqares of side lengths $1, 4, 7, 8, 9, 10, 14, 15$ and 18 units to make a larger rectangle. How many units long is the shorter side of this rectangle?

(Solution)

10. Game: We have a modified Nine men's morris board. (See the picture.) Two players alternatingly place red and blue pieces on the board. (You cannot put a piece to a place that is already taken.) A player who can achieve three neighboring pieces in a line, wins. If every place is taken, but there are no three same-colored neighboring pieces anywhere, then the second player wins.
Beat the computer twice in a row in this game! You can decide whether you want to take the role of the first or the second player.



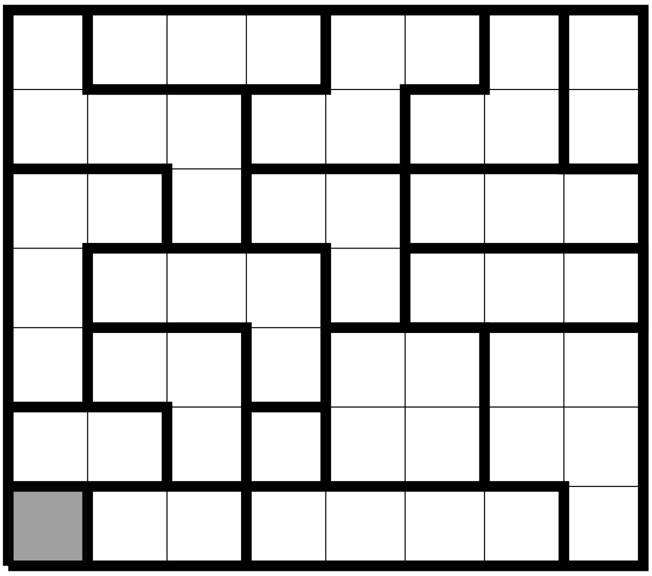
(Solution)

1.1.3 Category E

1. Grandma baked 91 cookies and distributed them equally among her grandchildren. Grandpa was saddened to find that not even a single cookie was left for him to taste. How many cookies did each grandchild get if we know that when Anne was just starting her second cookie, Tommy already finished his 9th one?

(Solution)

2. A robot starts from the bottom-left cell of the following table, moving upward. At each step it either continues its journey to the cell in front of it or it turns 90° left or right. The robot stops when it reaches the starting cell again. How many times could the robot have turned on its journey if we know that it stepped on each cell exactly once and it stepped into the region bounded by the bold line only once?

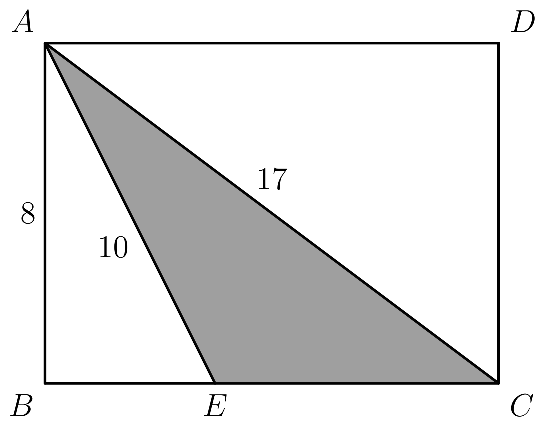


(Solution)

3. A family (mother, father, grandma, little boy, little girl) travel in a 5-person car. Only the mother and the father can drive. The children cannot sit in the front, and the father does not want to sit next to the grandma. In how many way can they sit in the car? *In the car, there are two places in the front, 3 places in the back. Two places are neighboring, if they are in the same row next to each other.*

(Solution)

4. The rectangle $ABCD$ has a side AB of length 8, and a side AC of length 17. E is an inner point of the side BC. What is the area of triange AEC, if the length of AE is 10?



(Solution)

5. It is a well-known fact that Süsü, the dragon has only one head, but all five of his siblings have more than one. One day the siblings of Süsü sit around a large table and each dragon tells to the others the average number of heads they have. The following sentences are said.

- You have 7 heads on average.
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How many heads do the siblings of Süsü have in total?

(Solution)

6. By using only two different kinds of number cards, Marvin created two positive integers. In how many possible ways could he do it, if the sum of the two numbers is 10000?
Note: Switching the two integers does not count as a separate case.

(Solution)

7. On a 10×10 chessboard, we want to move with a knight from the left bottom corner to the top right corner. In how many possible ways can we do this, if only those steps are allowed that move at least one column to the right and at least one row upwards? *A knight always moves two squares in one direction (up, down, right, left) and one square in a direction perpedicular to this.*

(Solution)

8. David used 9 sqares of side lengths $1, 4, 7, 8, 9, 10, 14, 15$ and 18 units to make a larger rectangle. How many units long is the shorter side of this rectangle?

(Solution)

9. There are six rooks around Edinborough's castle, and secret passages between the rooks. When king Arthur wanted to attack the castle, then a spy of him accounted about the existence of the passages. She did not know the whereabouts of the passages, but she knew that there is one rook from which there is one secret passage starting, two rooks from which there are two secret passage starting and three rooks from which there are three secret passage starting. By these informations Arthur draw all possible map of the passages. How many maps did he need to draw if we know that one passage connects exactly two rooks, and there is no two rooks with more than one passage between them.

(Solution)

10. Game: We have integers from 1 to 9. Two player alternatingly pick a number that was not chosen before. The player who has three numbers such that their sum is 15, wins. If all 9 numbers are already chosen, but no one has three numbers such that their sum is 15, then the second player wins.

Beat the computer twice in a row in this game! You can decide whether you want to take the role of the first or the second player.

(Solution)

1.2 Regional round

1.2.1 Category C

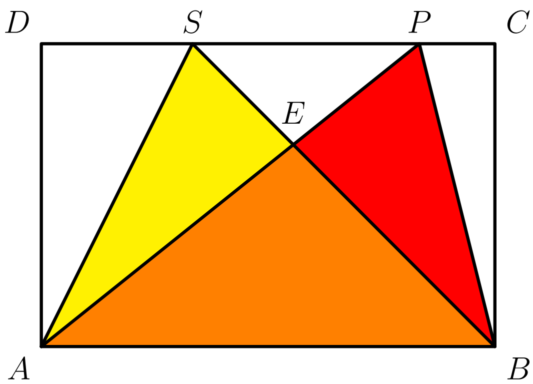
1. Andris noted that if he multiplies the number of years in his age, the age of his sister who is 8 years younger than him, and the age of his great-grandfather, the result will be 2020. How old is each family member?

(Solution)

2. Nine Indian tribes hold a vote to decide whether to go East or West to find new hunting grounds. Each tribe has nine members who determine the preference of the tribe by a simple majority vote. Then each tribe sends a representative, who come together under the Tall Pine Tree and announce the decision of their tribe. At the end they will go to whichever direction is chosen by more tribes.
a) Is it possible that more than half of the 81 Indians would like to go West, but they will still go East as a result of the vote?
b) At the end 5 tribes do not even show up at the Pine Tree and it also turns out that in each of the four other tribes only four members voted. Is it true that if none of the tribes had a draw in their local vote and the representatives decide to go West, the majority of the 16 Indians who voted indeed wanted to go that way?

(Solution)

3. Csenge has a yellow and a red foil on her rectangular window which look beautiful in the morning light. Where the two foils overlap, they look orange. The window is 80 cm tall, 120 cm wide and its corners are denoted by A, B, C and D in the figure. The two foils are triangular and both have two of their vertices at the two bottom corners of the window, A and B. The third vertex of the yellow foil is S, the trisecting point of side DC closer to D, whereas the third vertex of the red foil is P, which is one fourth on the way on segment SC, closer to C. The red region (i.e. triangle BPE) is of area 16 dm^2. What is the total area of the regions not covered by foil?



(Solution)

4. In the Wild West each banditry has 5 or 7 members. When some banditries had a boxing championship everbody played with everybody except the members of their own banditry. Every box match had a winner and a loser. At most how many gang members participated if we know that there were exactly 16 people who won at least one of their box matches?

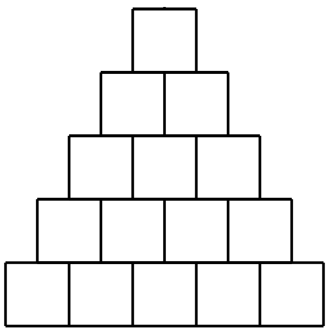
(Solution)

5. In his math test Albrecht had to solve the equation $\overline{abc} = \overline{ab} + \overline{bc} + \overline{ca}$, where a, b, c denotes digits. Unfortunately, he copied the problem badly, and he solved the problem $abc = ab + bc + ca$ among the positive integers. **a)** What was the solution of the original problem? **b)** What did Albrecht get? *Remark: the notation $\overline{xy}$ stands for the number that we get by writing the digits x, y next to each other, while xy denotes the product of x and y.*

(Solution)

1.2.2 Category D

1. We write a 0 or a 1 in each cell of the bottom row of the following pyramid. After that we fill the cells row by row moving upwards such that each cell contains the sum of the numbers in the two cells below. How many ways can we fill the bottom row that will result in an even number in the top cell?



(Solution)

2. Albrecht would like to partition the numbers $1, 2, \ldots, 12$ into two groups of size six so that each group contains a number that is the average of the other five.
a) Is this possible?
b) What if instead he would like to split the numbers $1, 2, \ldots, 18$ into three groups with this property?

(Solution)

$a! + b! + c! = 3^d$.

3. Find all quadruples (a, b, c, d) of positive integers that satisfies $a! + b! + c! = 3^d$.
Remark: $n!$ denotes the product of the first n numbers. that is, $n! = 1 \cdot 2 \cdot ... \cdot n$.

(Solution)

4. In the triangle ABC we have $30°$ at the vertex A, and $50°$ at the vertex B. Let O be the center of inscribed circle. Show that $AC + OC = AB$.

(Solution)

5. 9 university professors went to a scientific conference. Unfortunately, the lectures were boring, so from time to time they fall asleep, but any of them fall asleep at most 4 times. We also know that for any two professors there were a moment when both of them were sleeping. Show that there were a moment when at least 3 professors were sleeping.

(Solution)

1.2.3 Category E

1. Albrecht is travelling in his car on the motorway at a constant speed. The journey is very long so Marvin who is sitting next to Albrecht gets bored and decides to calculate the speed of the car. He was a bit careless but he noted that at noon they passed milestone XY (where X and Y are digits), at 12:42 milestone YX and at 1pm they arrived at milestone X0Y. What did Marvin deduce, what is the speed of the car?

(Solution)

2. The best part of grandma's 18 cm $\times$ 36 cm rectangle-shaped cake is the chocolate covering on the edges. Her three grandchildren would like to split the cake between each other so that everyone gets the same amount (of the area) of the cake, and they all get the same amount of the delicious perimeter too.
a) Can they cut the cake into three convex pieces like that?
b) The next time grandma baked this cake, the whole family wanted to try it so they had to cut the cake into six convex pieces this way. Is this possible?
c) Soon the entire neighbourhood has heard of the delicious cake. Can the cake be cut into 12 convex pieces with the same conditions?

(Solution)

3. The floor plan of a contemporary art museum is a (not necessarily convex) polygon and its walls are solid. The security guard guarding the museum has two favourite spots (points A and B) because one can see the whole area of the museum standing at either point. Is it true that from any point of the AB section one can see the whole museum?

(Solution)

4. Determine all triples of positive integers a, b, c that satisfy
a) $[a, b] + [a, c] + [b, c] = [a, b, c]$.
b) $[a, b] + [a, c] + [b, c] = [a, b, c] + (a, b, c)$.
Remark: Here $[x, y]$ denotes the least common multiple of positive integers x and y, and (x, y) denotes their greatest common divisor.

(Solution)

5. 21 bandits live in the city of Warmridge, each of them having some enemies among the others. Initially each bandit has 240 bullets, and duels with all of his enemies. Every bandit distributes his bullets evenly between his enemies, this means that he takes the same number of bullets to each of his duels, and uses each of his bullets in only one duel. In case the number of his bullets is not divisible by the number of his enemies, he takes as many bullets to each duel as possible, but takes the same number of bullets to every duel, so it is possible that in the end the bandit will have some remaining bullets.
Shooting is banned in the city, therefore a duel consists only of comparing the number of bullets in the guns of the opponents, and the winner is whoever has more bullets. After the duel the sheriff takes the bullets of the winner and as an act of protest the loser shoots all of his bullets into the air. What is the largest possible number of bullets the sheriff can have after all of the duels have ended?
Being someones enemy is mutual. If two opponents have the same number of bullets in their guns during a duel, then the sheriff takes the bullets of the bandit who has the wider hat among them.
Example: If a bandit has 13 enemies then he takes 18 bullets with himself to each duel, and they will have 6 leftover bullets after finishing all their duels.

(Solution)

1.2.4 Category $\mathbf{E}^+$

1. Determine all triples of positive integers a, b, c that satisfy
a) $[a,b] + [a,c] + [b,c] = [a,b,c]$.
b) $[a,b] + [a,c] + [b,c] = [a,b,c] + (a,b,c)$.
Remark: Here $[x,y]$ denotes the least common multiple of positive integers x and y, and (x,y) denotes their greatest common divisor.

(Solution)

2. 21 bandits live in the city of Warmridge, each of them having some enemies among the others. Initially each bandit has 240 bullets, and duels with all of his enemies. Every bandit distributes his bullets evenly between his enemies, this means that he takes the same number of bullets to each of his duels, and uses each of his bullets in only one duel. In case the number of his bullets is not divisible by the number of his enemies, he takes as many bullets to each duel as possible, but takes the same number of bullets to every duel, so it is possible that in the end the bandit will have some remaining bullets.
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Example: If a bandit has 13 enemies then he takes 18 bullets with himself to each duel, and they will have 6 leftover bullets after finishing all their duels.

(Solution)

3. Let k_1 and k_2 be two circles that are externally tangent at point C. We have a point A on k_1 and a point B on k_2 such that C is an interior point of segment AB. Let k_3 be a circle that passes through points A and B and intersects circles k_1 and k_2 another time at points M and N respectively. Let k_4 be the circumscribed circle of triangle CMN.
Prove that the centres of circles k_1, k_2, k_3 and k_4 all lie on the same circle.

(Solution)

4. Find all pairs of polynomials (p, q) with integer coefficients that satisfy the equation

$$p(x^2) + q(x^2) = p(x)q(x)$$

such that p is of degree n and has n nonnegative real roots (with multiplicity).

(Solution)

5. There are n distinct lines in three-dimensional space such that no two lines are parallel and no three lines meet at one point. What is the maximal possible number of planes determined by these n lines?
We say that a plane is determined if it contains at least two of the lines.

(Solution)

1.3 Final round – day 1

1.3.1 Category C

1. Eagleeye, the Indian chief, drives his 50 heads of cattle to the fair. At the fair he can do exchange and barter in the following way: he can exchange 5 heads of cattle to 3 pigs and 5 goats. If he gives 1 head of cattle and 1 goat he can get 2 pigs, and for 3 pigs and 3 goats he can get 2 heads of cattle.
a) Since Eagleeye wants to raise goats and pigs besides cattle he wants to barter in such a way that he can take home at least 20 animals from each kind. Can he do it? **b)** Show that it is possible that the Indian chief leaves with less than 15 animals from the fair.

(Solution)

2. Ludmilla has written down an 8-digit positive integer A into her booklet. She deleted one of its digits and recieved the 7-digit number B. Then she realized that the sum of the two numbers is 20210521; exactly the date of Dürer final. What could have been the first number that Ludmilla wrote down?

(Solution)

3. In the isoscele triangle ABC we have $AC = BC$. Let X be an arbitrary point of the segment AB. The line parallel to BC and passing through X intersects the segment AC in N, and the line parallel to AC and passing through BC intersects the segment BC in M. Let k_1 be the circle with center N and radius NA. Similarly, let k_2 be the circle with center M and radius MB. Let T be the intersection of the circles k_1 and k_2 different from X. Show that the angles $NCM\angle$ and $NTM\angle$ are equal.

(Solution)

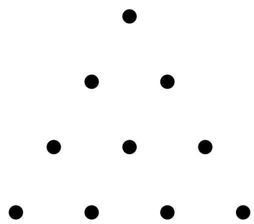
4. On Monday we will celebrate Grandma's 80-th birthday and the whole family will be there. For this occasion Grandma bought 3 liters apple juice that she would like to share among her grandchildren equally. Unfortunately, she is not certain that all her nine grandchildren will be there, or only eight of them. In the cupboard she finds 16 glasses, so she has the idea that she fills out the apple juices in such a way that no matter 8 or 9 grandchildren will participate the children could drink all apple juices and everybody gets the same amount of apple juice. **a)** How much apple juice should she fill into the glasses if only one child can drink from each glass due to hygienic reasons, but a child can drink from several glasses. **b)** Unfortunately, just before she started to fill out the apple juice she drops one of the glasses and the glass broke. Can she still solve the above problem with the remaining 15 glasses?

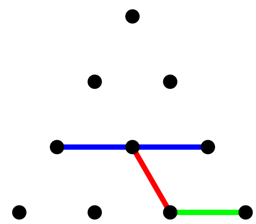
(Solution)

5. Csongor was very bored at the math class, in his boredom he started to scribe. First he writes down a natural number, and from that he starts to write the integers in increasing order. He has already written down 38 integers when he realizes that none of the numbers have sum of digits divisible by 11.
a) What was the number that Csongor first wrote down? **b)** Prove that no matter where he starts the numbers he won't be able to write down 39 numbers this way.

(Solution)

6. Game: In an Indian reservatory there are 10 totem poles arranged according to the left figure. Silent Stream and Red Fire used to play the following game: In turns they stretch ropes between two-two poles in such a way that every stretched rope is parallel to a side of the big triangle and no rope can go along a pole that is already touched by another rope. Furthermoe, if instead of a rope one can stretch out a straight line extension of the rope, then one should stretch out this extension. The one who cannot stretch out more ropes according to the rules loses.
Win two games in a row against the organizers! You can decide that you want to start or to be the second player. The figure on the right depicts the first three steps of a game. First Silent Stream stretches the blue rope, then Red Fire stretches the red one, finally Silent Stream stretches the blue one.





(Solution)

1.3.2 Category D

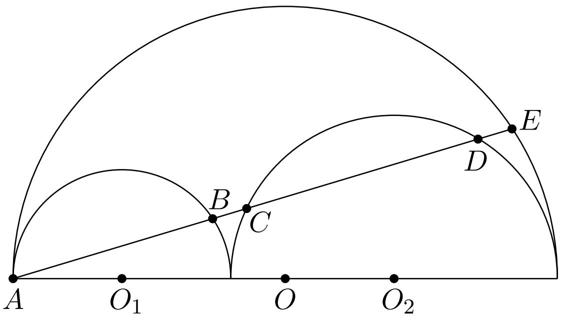
1. Eagleeye, the Indian chief, drives his 50 heads of cattle to the fair. At the fair he can do exchange and barter in the following way: he can exchange 5 heads of cattle to 3 pigs and 5 goats. If he gives 1 head of cattle and 1 goat he can get 2 pigs, and for 3 pigs and 3 goats he can get 2 heads of cattle.
a) Since Eagleeye wants to raise goats and pigs besides cattle he wants to barter in such a way that he can take home at least 20 animals from each kind. Can he do it? **b)** Show that it is possible that the Indian chief leaves with less than 15 animals from the fair.

(Solution)

2. How many solutions does the equation $n^3 - 2 = k!$ have if n and k are positive integers?
Remark: $k!$ denotes the products of the positive integers not bigger than k.

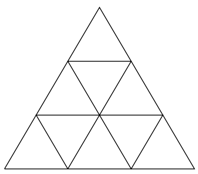
(Solution)

3. Given a semicirle with center O an arbitrary inner point of the diameter divides it into two segments. Let there be semicircles above the two segments as visible in the below figure. The line e passing through the point A intersects the semicircles in 4 points: B, C, D and E. Show that the segments BC and DE have the same length.



(Solution)

4. Charlie colors the fields of the triangular lattice of an equiangular triangle shape of side length 3 to red. Then he gave it to Piroska who colors further fields to red according to the following rule: if she finds a trangle of length 2 where out of the 4 fields 3 fields are already colored to red, then she colors the fourth field to red too. She repeats this coloring until all colorable fields are colored.



a) What is the largest N that no matter how Charlie starts to color N fields Piroska won't be able to color all remaining fields?
b) What is the smallest M such that no matter how Charlie colors M fields Piroska will be able to color the remaining fields.
c) Determine the above numbers N and M in case of a triangular lattice of an equiangular triangle shape of length n too.

(Solution)

5. Indians find those sequences of non-negative real numbers $x_0, x_1, \ldots$ mystical that satisfy $x_0 < 2021$, $x_{i+1} = \lfloor x_i \rfloor \{x_i\}$ for every $i \geq 0$, furthermore the sequence contains an integer different from 0. How many sequences are mystical according to the Indians?
Remark: in case of a real number x the $\lfloor x \rfloor$ denotes the integer part of x, that is the largest integer x that is not bigger than x. The number $\{x\}$ is the fracional part of x, that is, $x - \lfloor x \rfloor$.

(Solution)

6. Game: In an Indian reservatory there are 10 totem poles arranged according to the left figure. Silent Stream and Red Fire used to play the following game: In turns they stretch ropes between two-two poles in such a way that every stretched rope is parallel to a side of the big triangle and no rope can go along a pole that is already touched by another rope. Furthermoe, if instead of a rope one can stretch out a straight line extension of the rope, then one should stretch out this extension. The one who cannot stretch out more ropes according to the rules loses.
Win two games in a row against the organizers! You can decide that you want to start or to be the second player. The figure on the right depicts the first three steps of a game. First Silent Stream stretches the blue rope, then Red Fire stretches the red one, finally Silent Stream stretches the blue one.

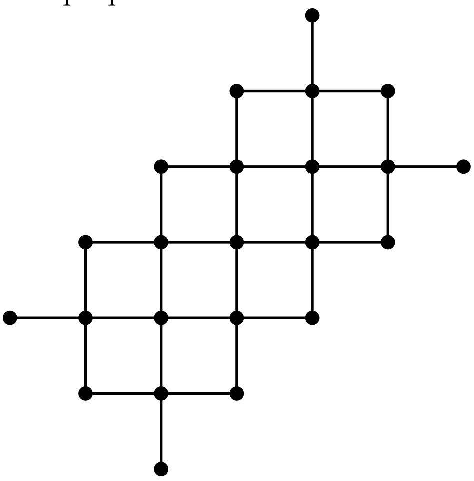
(Solution)

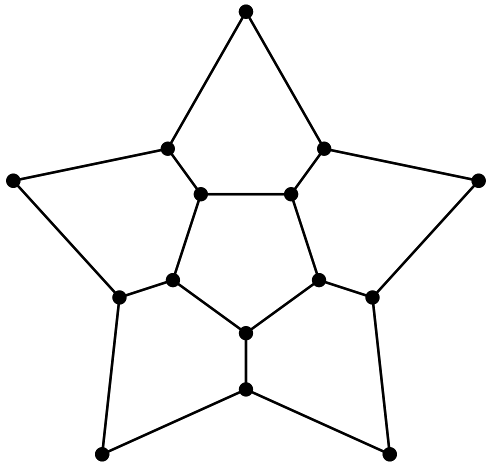
1.3.3 Category E

1. Show that if the difference of two positive cube numbers is a positive prime, then this prime number has remainder 1 after division by 6.

(Solution)

2. In the country of Óxisz the minister of finance observed at the end of the tax census that the sum of properties of any two neighboring city counted in dinar is divisible by 1000, and she also observed that the sum of properties of all cities is also divisible by 1000. What is the least sum of properties of all cities if the map of the cities looks as follows?

a) 

b) 

Remark: The cities may have non-integer properties, but it is also positive. On the map the points are the cities, and two cities are neighboring if there is a direct connection between them.

(Solution)

3. Let A and B different points of a circle k centered at O in such a way such that AB is not a diagonal of k. Furthermore, let X be an arbitrary inner point of the segment AB. Let k_1 be the circle that passes through the points A and X, and A is the only common point of k and k_1. Similarly, let k_2 be the circle that passes through the points B and X, and B is the only common point of k and k_2. Let M be the second intersection point of k_1 and k_2. Let Q denote the center of circumscribed circle of the triangle AOB. Let O_1 and O_2 be the centers of k_1 and k_2. Show that the points M, O, O_1, O_2, Q are on a circle.

(Solution)

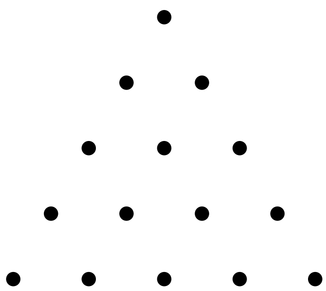
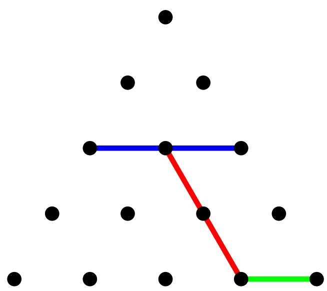
4. Indians find those sequences of non-negative real numbers $x_0, x_1, \ldots$ mystical that satisfy $x_0 < 2021$, $x_{i+1} = \lfloor x_i \rfloor \{x_i\}$ for every $i \geq 0$, furthermore the sequence contains an integer different from 0. How many sequences are mystical according to the Indians?
Remark: in case of a real number x the $\lfloor x \rfloor$ denotes the integer part of x, that is the largest integer x that is not bigger than x. The number $\{x\}$ is the fracional part of x, that is, $x - \lfloor x \rfloor$.

(Solution)

5. A torpedo set consists of 2 pieces of 1×4, 4 pieces of 1×3, 6 pieces of 1×2 and 8 pieces of 1×1 ships. **a)** Can one put the whole set to a 10×10 table so that the ships do not even touch with corners? (The ships can be placed both horizontally and vertically.) **b)** Can we solve this problem if we change 4 pieces of 1×1 ships to 3 pieces of 1×2 ships? **c)** Can we solve the problem if we change the remaining 4 pieces of 1×1 ships to one piece of 1×3 ship and one piece of 1×2 ship? (So the number of pieces are $2, 5, 10, 0$.)

(Solution)

6. Game: In an Indian reservatory there are 15 totem poles arranged according to the left figure. Silent Stream and Red Fire used to play the following game: In turns they stretch ropes between two-two poles in such a way that every stretched rope is parallel to a side of the big triangle and no rope can go along a pole that is already touched by another rope. Furthermoe, if instead of a rope one can stretch out a straight line extension of the rope, then one should stretch out this extension. The one who cannot stretch out more rope according to the rules loses.
Win two games in a row against the organizers! You can decide that you want to start or to be the second player. The figure on the right depicts the first three steps of a game. First Silent Stream stretches the blue rope, then Red Fire stretches the red one, finally Silent Stream stretches the blue one.

(Solution)

1.3.4 Category E$^+$

1. Let A and B different points of a circle k centered at O in such a way such that AB is not a diagonal of k. Furthermore, let X be an arbitrary inner point of the segment AB. Let k_1 be the circle that passes through the points A and X, and A is the only common point of k and k_1. Similarly, let k_2 be the circle that passes through the points B and X, and B is the only common point of k and k_2. Let M be the second intersection point of k_1 and k_2. Let Q denote the center of circumscribed circle of the triangle AOB. Let O_1 and O_2 be the centers of k_1 and k_2. Show that the points M, O, O_1, O_2, Q are on a circle.

(Solution)

2. Indians find those sequences of non-negative real numbers $x_0, x_1, \ldots$ mystical that satisfy $x_0 < 2021$, $x_{i+1} = \lfloor x_i \rfloor \{x_i\}$ for every $i \geq 0$, furthermore the sequence contains an integer different from 0. How many sequences are mystical according to the Indians?
Remark: in case of a real number x the $\lfloor x \rfloor$ denotes the integer part of x, that is the largest integer x that is not bigger than x. The number $\{x\}$ is the fracional part of x, that is, $x - \lfloor x \rfloor$.

(Solution)

3. On the evening of Halloween a group of n kids collected k bars of chocolate of the same type. At the end of the evening they wanted to divide the bars so that everybody gets the same amount of chocolate, and none of the bars is broken into more than two pieces. For which n and k is this possible?

(Solution)

4. A torpedo set consists of 2 pieces of 1×4, 4 pieces of 1×3, 6 pieces of 1×2 and 8 pieces of 1×1 ships. **a)** Can one put the whole set to a 10×10 table so that the ships do not even touch with corners? (The ships can be placed both horizontally and vertically.) **b)** Can we solve this problem if we change 4 pieces of 1×1 ships to 3 pieces of 1×2 ships? **c)** Can we solve the problem if we change the remaining 4 pieces of 1×1 ships to one piece of 1×3 ship and one piece of 1×2 ship? (So the number of pieces are $2, 5, 10, 0$.)

(Solution)

5. Let n be a positive integer. Show that every divisors of $2n^2 - 1$ gives a different remainder after division by $2n$.

(Solution)

6. Game: In an Indian reservatory there are 15 totem poles arranged according to the left figure. Silent Stream and Red Fire used to play the following game: In turns they stretch ropes between two-two poles in such a way that every stretched rope is parallel to a side of the big triangle and no rope can go along a pole that is already touched by another rope. Furthermoe, if instead of a rope one can stretch out a straight line extension of the rope, then one should stretch out this extension. The one who cannot stretch out more rope according to the rules loses.

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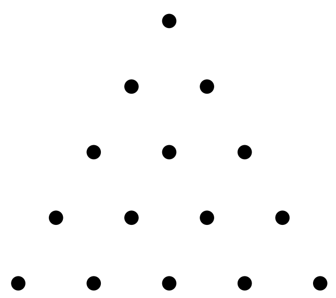
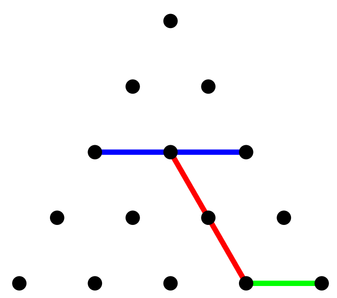




(Solution)

1.4 Final round – day 2

1.4.1 Category C

C-1. Kartal, Balint and Timi are playing with some cards. There are three cards on the table, showing one digit each and their sum is 17. First Kartal arranged the cards to make a 3-digit number, then Balint rearranged them to get another 3-digit number and finally Timi rearranged again to obtain a third one. At the end of this process they noted that none of the cards was at the same position in at least two of the numbers. What is the sum of the three numbers?

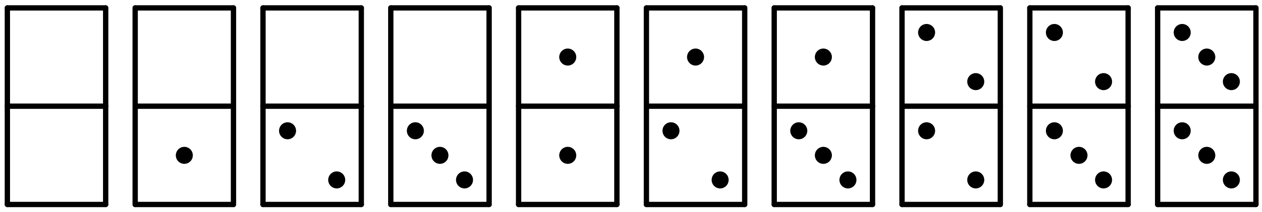
(3 points)

C-2. In Sixcountry there are 12 months, but each month consists of 6 weeks. The month are named the same way we do, from January to December, but in each month the weeks have different lengths. In the k-th month the weeks consist of 6^{k-1} days. What is the number of days of the spring (March, April, May together)? (3 points)

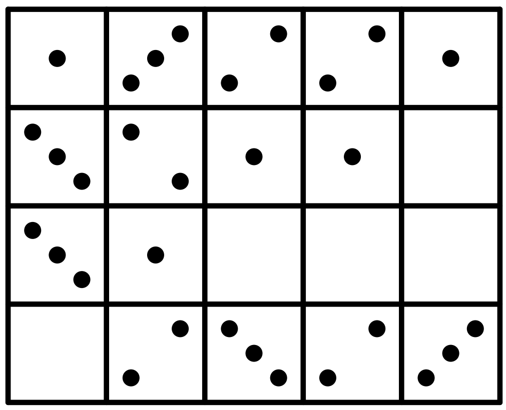
C-3. How many integers between 11 and 2021 have the property that each of its digits (except for the first one) is at least 3 larger than the previous one?

(3 points)

C-4. Dóra plays with the following domino set:



Dóra sets out the following figure. Some dominos are set horizontally, others are vertically.



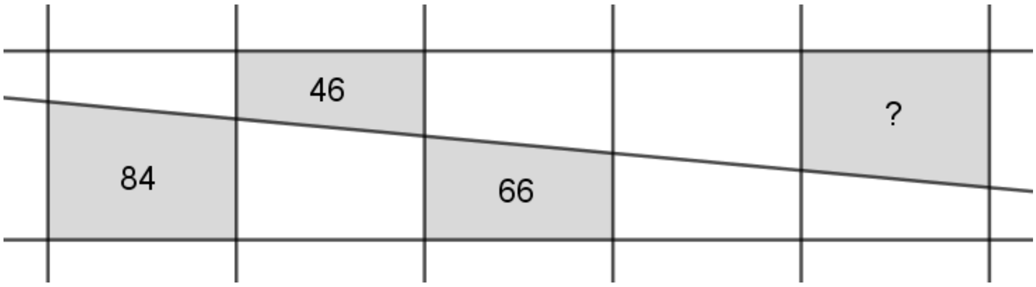
How many dominos are put vertically from the 10? *(The back of the dominos are empty.)*

(3 points)

C-5. In the evening some kids are playing a card game called *Moore* using two decks of playing cards (52 cards per deck). At the start of the game the dealer distributes all cards among the players as equally as possible (i.e. the number of cards of any two players differ by at most one). This was exactly 5 kids get one more card than the others.
At 10pm some kids go to bed and the others continue playing. After this the cards can be distributed equally among the remaining players. What is the number of players at the start of the evening?

(4 points)

C-6. The figure shows a line intersecting a square lattice. The area of some arising quadrilaterals are also indicated. What is the area of the region with the question mark?



(4 points)

C-7. What is the number of 4-digit numbers that contains exactly 3 different digits that have consecutive value? Such numbers are for instance 5464 or 2001.
Two digits in base 10 *are consecutive if their difference is* 1. (4 points)

C-8. Trapezoid $ABCD$ has bases AD and BC. Side AB is of length 13, AD is of length 14 and CD is of length 18. We also know that angle ADC is twice the size of angle ABC. What is the length of base BC?

(4 points)

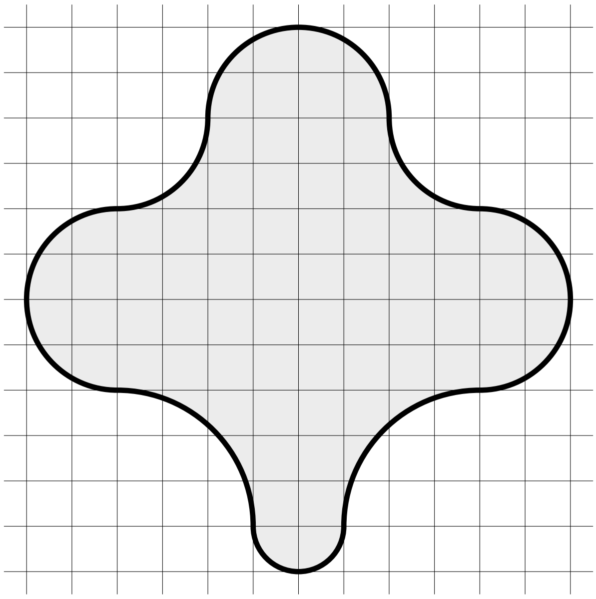
C-9. The Good Fairy ATM works as follows. At each use it multiplies our wealth by a and adds b forints. Unfortunately - like most good fairies - it only offers its services three times. We know that if we use the machine three times, we will have 21 forints more than 8 times our original wealth, no matter how much money we started with. How many forints will Dani have if he now has 2021 forints and he uses the ATM only once?

(5 points)

C-10. We call a positive integer *absolutely relative* if none of its digits is zero and any two of its digits are coprimes. How many 3-digit absolutely relative numbers are there?
Two positive integers are coprime if their greatest common divisor is 1.

(5 points)

C-11. Billy owns a nice piece of land in the Wild West, marked by a bold line in the map below. The border is formed from quarter-circles. The area of each little square on the map is 1 hectare. How many hectares is the total area of Billy's land?

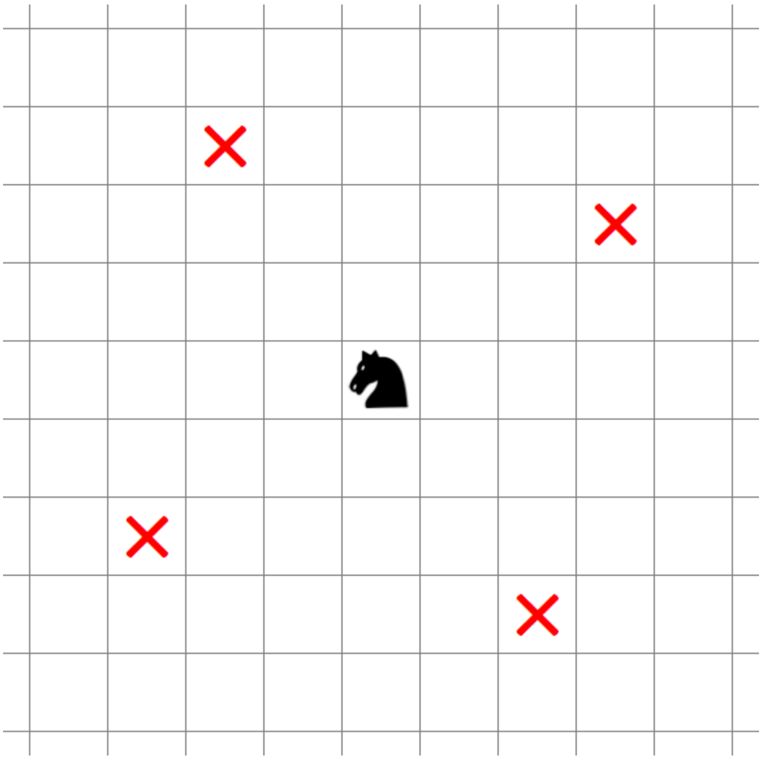


(5 points)

C-12. Billy let his herd freely. Enjoying their time the horses started to jump on the squares of a lattice of meadow that is infinite in both directions. Each horse can jump as follows: horizontally or vertically moves three, then turn to left and moves two. Naturally, under the jump a horse don't touch the ground.

The horses are standing on squares that no two can meet by such a jump. How many horses does Billy have if their number is the maximum possible?

The figure below shows where a horse can jump to. Notice that there 4 *places and not* 8 *like in chess.*



(5 points)

C-13. A digital clock displays the digits with dashes as follows.

The clock always displays the actual time with four digits from 00:00 to 23:59. For instance, at 09:45 one can see the following on the display.

How many minutechange happens in a day -including the change to midnight-, when the digital clock changes the state of at least 10 dashes (meaning that the dash changes from off to on or vice versa)? (6 points)

C-14. Jimmy's garden has right angled triangle shape that lies on island of circular shape in such a way that the corners of the triangle are on the shore of the island. When he made fences along the garden he realized that the length of the shortest side is 36 meter shorter than the longest side, and third side required 48 meter long fence. In the middle of the garden he built a house of circular shape that has the largest possible size. Jimmy measured the distance between the center of his house and the center of the island. What is the square of this distance?

(6 points)

C-15. Two teams of 3 are travelling to the Durer competition by tram. They found two empty rows opposite to each other, containing 5 seats each, and want to sit down there (everyone occupies exactly one seat). How many ways can they do this if members of different teams do not want to sit on neighbouring seats? Submit *one tenth* of the number of good configurations. *Two configurations are considered different if there is someone sitting at a different seat in the two konfigurations. Two seats are considered neighbouring if they are in the same row, next to each other.*

(6 points)

C-16. A date is called *rearranging* if we can obtain the year by arranging the two digits of the day and two digits of the month. E.g. 02/04/2004 is a rearranging date. Which year in the 21st century contains the most rearranging dates? If there are more than one such years, submit the latest one.
Note that 2100 is still in the 21st century.

(6 points)

1.4.2 Category D

D-1. How many integers between 11 and 2021 have the property that each of its digits (except for the first one) is at least 3 larger than the previous one?

(3 points)

D-2. Kartal, Balint and Timi are playing with some cards. There are three cards on the table, showing one digit each and their sum is 17. First Kartal arranged the cards to make a 3-digit number, then Balint rearranged them to get another 3-digit number and finally Timi rearranged again to obtain a third one. At the end of this process they noted that none of the cards was at the same position in at least two of the numbers. What is the maximal possible sum of the three numbers?

(3 points)

D-3. Find the number of integers n between 1 and 2021 such that $2^n + 2^{n+3}$ is a perfect square.

(3 points)

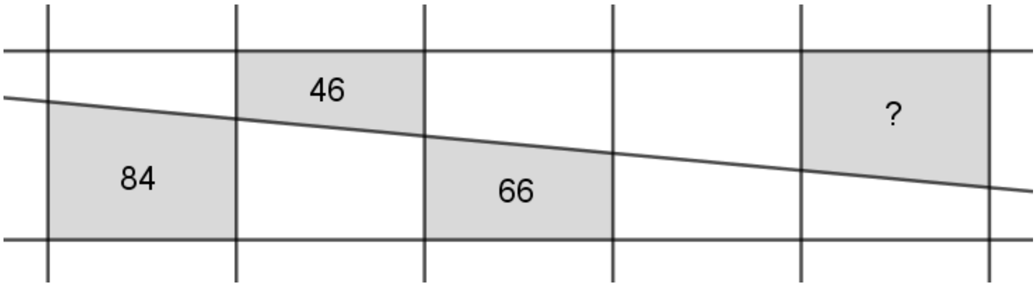
D-4. In the evening some kids are playing a card game called *Moore* using two decks of playing cards (52 cards per deck). At the start of the game the dealer distributes all cards among the players as equally as possible (i.e. the number of cards of any two players differ by at most one). This was exactly 5 kids get one more card than the others.
At 10pm some kids go to bed and the others continue playing. After this the cards can be distributed equally among the remaining players. What is the number of players at the start of the evening?

(3 points)

D-5. The Good Fairy ATM works as follows. At each use it multiplies our wealth by a and adds b forints. Unfortunately - like most good fairies - it only offers its services three times. We know that if we use the machine three times, we will have 21 forints more than 8 times our original wealth, no matter how much money we started with. How many forints will Dani have if he now has 2021 forints and he uses the ATM only once?

(4 points)

D-6. The figure shows a line intersecting a square lattice. The area of some arising quadrilaterals are also indicated. What is the area of the region with the question mark?



(4 points)

D-7. Given a right angled triangle ABC in which $ACB\sphericalangle = 90°$. Let D be an inner point of AB, and let E be an inner point of AC. It is known that $ADE\sphericalangle = 90°$, and that the length of the segment AD is 8, the length of the segment DE is 15, and the length of segment CE is 3. What is the area of triangle AB? (4 points)

D-8. On an 8×8 chessboard, a rook stands on the bottom left corner square.

We want to move it to the upper right corner, subject to the following rules: we have to move the rook exactly 9 times, such that the length of each move is either 3 or 4. (It is allowed to mix the two lengths throughout the "journey".) How many ways are there to do this?

In each move, the rook moves horizontally or vertically.

(4 points)

D-9. We call a positive integer *absolutely relative* if none of its digits is zero and any two of its digits are coprimes. How many 3-digit absolutely relative numbers are there?
Two positive integers are coprime if their greatest common divisor is 1.

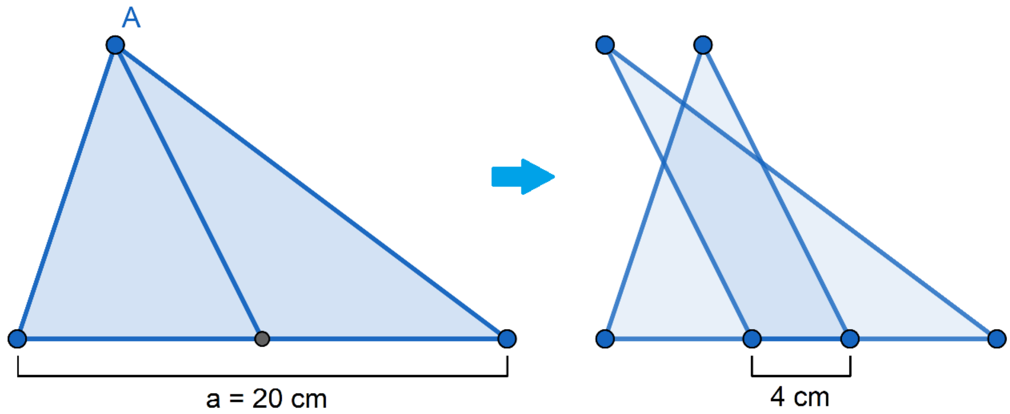
(5 points)

D-10. Greatgranny has 9 greatgrandchildren. Some members of her extended family are coming for a visit, so she made a lot (*really* a lot) of flapjack cookies to offer them to her greatgrandchildren. She noted that the number of cookies is not divisible by k so in case k of her greatgrandchildren come, she cannot distribute the cookies among them fairly. She immeadiately ate k of the cookies. The good news is that if not k, but any other number of her greatgrandchildren come to visit, she can distribute the remaining cookies among them equally.

For what values of $2 \leq k \leq 9$ can this happen? You should submit the sum of the appropriate values of k.

(5 points)

D-11. A triangle is given. Its side a is of length 20 cm, and its area is 125 cm^2. We cut the triangle into two parts at the median belonging to side a. Then we move the so-obtained two parts towards each other, such that the two segments of side a remain on the same line (i.e., the line initially occupied by side a). We move the two parts towards each other until we first reach a moment when the common part of the two segments is of length 4 cm. What is the area of the so-obtained shape in cm^2? *The so-obtained shape is the union of the two parts, which is a heptagon.*



(5 points)

D-12. Billy let his herd freely. Enjoying their time the horses started to jump on the squares of a lattice of meadow that is infinite in both directions. Each horse can jump as follows: horizontally or vertically moves three, then turn to left and moves two. Naturally, under the jump a horse don't touch the ground.

The horses are standing on squares that no two can meet by such a jump. How many horses does Billy have if their number is the maximum possible?

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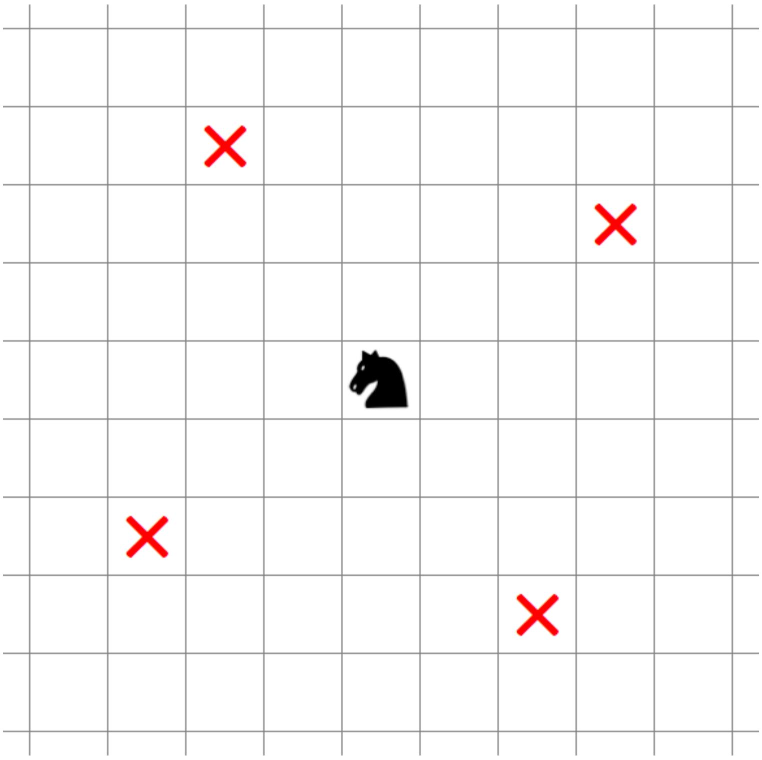


(5 points)

D-13. Japanese businessman Rui lives in America and makes a living from trading cows. On Black Thursday he was selling his cows for 2000 dollars each (the cows were of the same price), but after the financial crash there were huge fluctuations in the market and Rui was forced to follow them with his pricing. Every day he doubled, halved, multiplied by five or divided by five the price from the previous day (even if it meant he had to give change in cents). At the same time he managed to follow the Japanese superstition, so that the integer part of the price in dollars never started with digit 4.

On the day when Billy visited him to buy some cows the price of each cow was 80 dollars. What is the minimal number of days that could have passed since Black Thursday by then?

(6 points)

D-14. Two teams of 3 are travelling to the Durer competition by tram. They found two empty rows opposite to each other, containing 5 seats each, and want to sit down there (everyone occupies exactly one seat). How many ways can they do this if members of different teams do not want to sit on neighbouring seats? Submit *one tenth* of the number of good configurations. *Two configurations are considered different if there is someone sitting at a different seat in the two konfigurations. Two seats are considered neighbouring if they are in the same row, next to each other.*

(6 points)

D-15. A digital clock displays the digits with dashes as follows.

The clock always displays the actual time with four digits from 00:00 to 23:59. For instance, at 09:45 one can see the following on the display.

How many minutechange happens in a day -including the change to midnight-, when the digital clock changes the state of at least 8 dashes (meaning that the dash changes from off to on or vice versa)? (6 points)

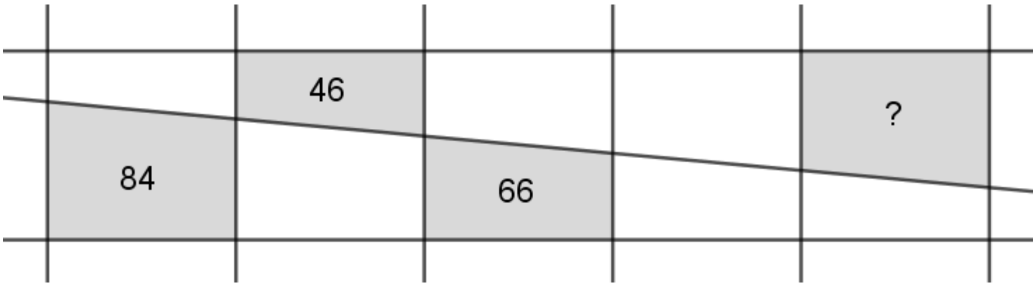
1.4.3 Category E

E-1. In Sixcountry there are 12 months, but each month consists of 6 weeks. The month are named the same way we do, from January to December, but in each month the weeks have different lengths. In the k-th month the weeks consist of 6^{k-1} days. What is the number of days of the spring (March, April, May together)? (3 points)

E-2. Find the number of integers n between 1 and 2021 such that $2^n + 2^{n+3}$ is a perfect square.

(3 points)

E-3. The figure shows a line intersecting a square lattice. The area of some arising quadrilaterals are also indicated. What is the area of the region with the question mark?



(3 points)

E-4. What is the number of 4-digit numbers that contains exactly 3 different digits that have consecutive value? Such numbers are for instance 5464 or 2001.
Two digits in base 10 *are consecutive if their difference is* 1. (3 points)

E-5. How many integers $1 \leq x \leq 2021$ make the value of the expression

$$\frac{2x^3 - 6x^2 - 3x - 20}{5(x-4)}$$

an integer?

(4 points)

E-6. Bertalan thought about a 4-digit positive number. Then he draw a simple graph on 4 vertices and wrote the digits of the number to the vertices of the graph in such a way that every vertex recieved exactly the degree of the vertex. In how many ways could he think about? *In a simple graph every edge connects two different vertices, and between two vertices at most one edge can go.* (4 points)

E-7. Jimmy's garden has right angled triangle shape that lies on island of circular shape in such a way that the corners of the triangle are on the shore of the island. When he made fences along the garden he realized that the length of the shortest side is 36 meter shorter than the longest side, and third side required 48 meter long fence. In the middle of the garden he built a house of circular shape that has the largest possible size. Jimmy measured the distance between the center of his house and the center of the island. What is the square of this distance?
(4 points)

E-8. John found all real numbers p such that in the polynomial $g(x)$ below, the quadratic term has coefficient 2021. What is the sum of all of these values p?

$$g(x) = (x - 1)^2(p + 2x)^2$$

(4 points)

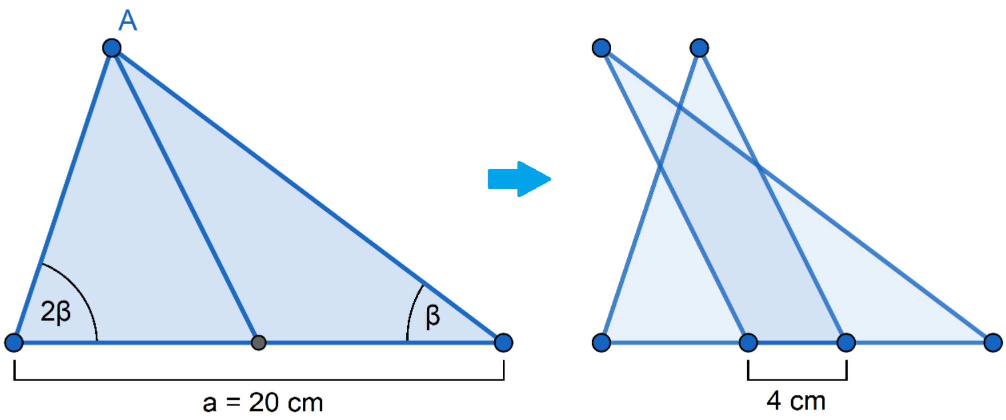
E-9. On an 8×8 chessboard, a rook stands on the bottom left corner square.

We want to move it to the upper right corner, subject to the following rules: we have to move the rook exactly 9 times, such that the length of each move is either 3 or 4. (It is allowed to mix the two lengths throughout the "journey".) How many ways are there to do this?

In each move, the rook moves horizontally or vertically.

(5 points)

E-10. A triangle is given. Its side a is of length 20 cm, and its area is 125 cm^2. It is also known that one of the angles lying on side a is twice as large as the other one. We cut the triangle into two parts at the median belonging to side a. Then we move the so-obtained two parts towards each other, such that the two segments of side a remain on the same line (i.e., the line initially occupied by side a). We move the two parts towards each other until we first reach a moment when the common part of the two segments is of length 4 cm. What is the area of the so-obtained shape in cm^2? *The so-obtained shape is the union of the two parts, which is a heptagon.*



(5 points)

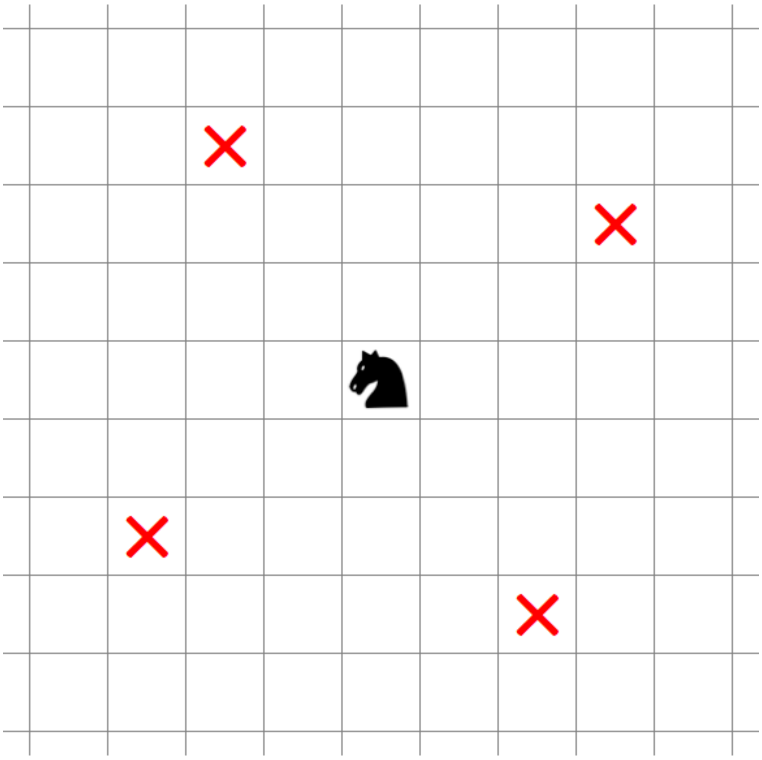
E-11. Japanese businessman Rui lives in America and makes a living from trading cows. On Black Thursday he was selling his cows for 2000 dollars each (the cows were of the same price), but after the financial crash there were huge fluctuations in the market and Rui was forced to follow them with his pricing. Every day he doubled, halved, multiplied by five or divided by five the price from the previous day (even if it meant he had to give change in cents). At the same time he managed to follow the Japanese superstition, so that the integer part of the price in dollars never started with digit 4.
On the day when Billy visited him to buy some cows the price of each cow was 80 dollars. What is the minimal number of days that could have passed since Black Thursday by then?

(5 points)

E-12. Billy let his herd freely. Enjoying their time the horses started to jump on the squares of a lattice of meadow that is infinite in both directions. Each horse can jump as follows: horizontally or vertically moves three, then turn to left and moves two. Naturally, under the jump a horse don't touch the ground.

The horses are standing on squares that no two can meet by such a jump. How many horses does Billy have if their number is the maximum possible?

The figure below shows where a horse can jump to. Notice that there 4 *places and not* 8 *like in chess.*



(5 points)

E-13. The trapezoid $ABCD$ satisfies $AB \parallel CD$, $AB = 70$, $AD = 32$ and $BC = 49$. We also know that $\angle ABC = 3 \cdot \angle ADC$. How long is the base CD?

(6 points)

E-14. How many functions $f : \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 16\}$ have the property that $f(f(x)) - 4x$ is divisible by 17 for all integers $1 \leq x \leq 16$?

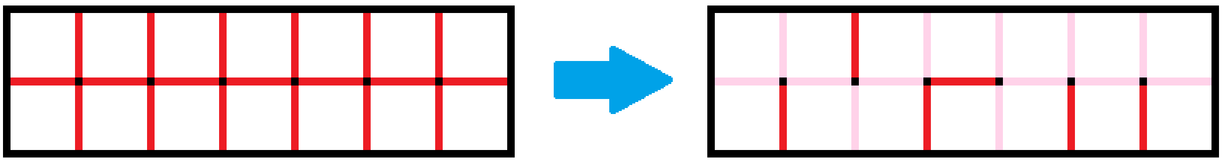
(6 points)

E-15. King Albrecht founded a family. In the family everyone has exacly 8 children. The only, but really important rule is that among the grandchildren of any person at most x can be named Bela. (None of Albrecht's children is called Bela.) For which x is it possible that after a certain time each newborn in the family has at least one direct ancestor in the Royal family called Bela.
No two of Albrecht's descendants (including himself) have a common child.

(6 points)

E-16. Consider a table consisting of 2×7 squares. Each little square is surrounded by walls (each internal wall belongs to two squares). We would like to remove some internal walls to make it possible to get from any square to any other one without crossing walls. How many ways can we do this while removing the minimal possible number of internal walls?

The figure shows a possible configuration, the remaining walls are marked in red, the removed ones are marked in light pink. Two configurations are considered the same if the same walls are removed.



(6 points)

1.4.4 Category E$^+$

E$^+$-1. Given a right angled triangle ABC in which $ACB\sphericalangle = 90°$. Let D be an inner point of AB, and let E be an inner point of AC. It is known that $ADE\sphericalangle = 90°$, and that the length of the segment AD is 8, the length of the segment DE is 15, and the length of segment CE is 3. What is the area of triangle AB?

(3 points)

E$^+$-2. In a french village the number of inhabitants is a perfect square. If 100 more people moved in, then the number of people would be 1 bigger than a perfect square. If again 100 more people moved in, then the number of people would be a perfect square again. How many people lives in the village if their number is the least possible? (3 points)

E$^+$-3. How many integers $1 \leq x \leq 2021$ make the value of the expression

$$\frac{2x^3 - 6x^2 - 3x - 20}{5(x-4)}$$

an integer?

(3 points)

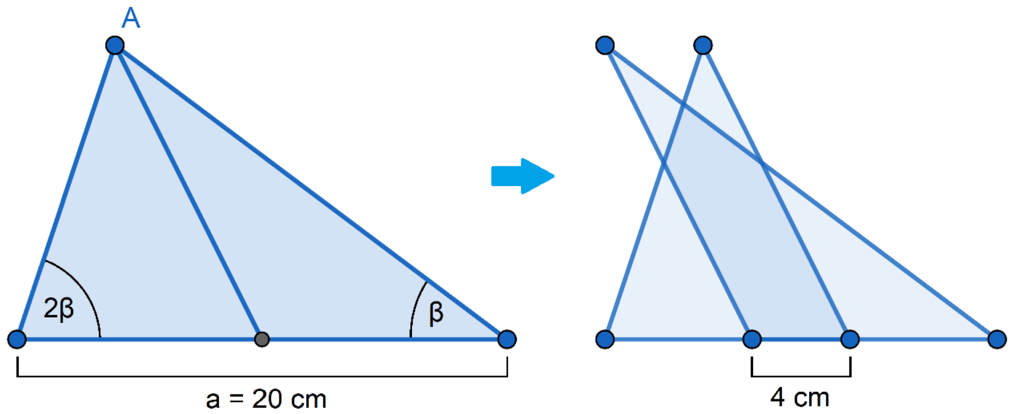
E$^+$-4. Bertalan thought about a 4-digit positive number. Then he draw a simple graph on 4 vertices and wrote the digits of the number to the vertices of the graph in such a way that every vertex recieved exactly the degree of the vertex. In how many ways could he think about? *In a simple graph every edge connects two different vertices, and between two vertices at most one edge can go.* (3 points)

E$^+$-5. Joe, who is already feared by all bandits in the Wild West, would like to officially become a sheriff. To do that, he has to take a special exam where he has to demonstrate his talent in three different areas: tracking, shooting and lasso throwing. He successfully completes each task with a given probability, independently of each other. He passes the exam if he can complete at least two of the tasks successfully. Joe calculated that in case he starts with tracking and completes it successfully, his chance of passing the exam is 32%. If he starts with successful shooting, the chance of passing is 49%, whereas if he starts with successful lasso throwing, he passes with probability 52%.

The overall probability of passing (calculated before the start of the exam) is $\frac{X}{1000}$. What is the value of X?

(4 points)

E$^+$-6. A triangle is given. Its side a is of length 20 cm, and its area is 125 cm^2. It is also known that one of the angles lying on side a is twice as large as the other one. We cut the triangle into two parts at the median belonging to side a. Then we move the so-obtained two parts towards each other, such that the two segments of side a remain on the same line (i.e., the line initially occupied by side a). We move the two parts towards each other until we first reach a moment when the common part of the two segments is of length 4 cm. What is the area of the so-obtained shape in cm^2? *The so-obtained shape is the union of the two parts, which is a heptagon.*



(4 points)

E$^+$-7. On an 8×8 chessboard, a rook stands on the bottom left corner square.

We want to move it to the upper right corner, subject to the following rules: we have to move the rook exactly 9 times, such that the length of each move is either 3 or 4. (It is allowed to mix the two lengths throughout the "journey".) How many ways are there to do this?

In each move, the rook moves horizontally or vertically.

(4 points)

E$^+$-8. Benedek wrote the following 300 statements on a piece of paper.

$$\begin{array}{ccccccc}
2 \mid 1! & & & & & & \\
3 \mid 1! & 3 \mid 2! & & & & & \\
4 \mid 1! & 4 \mid 2! & 4 \mid 3! & & & & \\
5 \mid 1! & 5 \mid 2! & 5 \mid 3! & 5 \mid 4! & & & \\
\vdots & & & & & & \\
24 \mid 1! & 24 \mid 2! & 24 \mid 3! & 24 \mid 4! & \cdots & 24 \mid 23! & \\
25 \mid 1! & 25 \mid 2! & 25 \mid 3! & 25 \mid 4! & \cdots & 25 \mid 23! & 25 \mid 24!
\end{array}$$

How many true statements did Benedek write down?
The symbol $\mid$ *denotes divisibility, e.g.* $6 \mid 4!$ *means that* 6 *is a divisor of number* $4!$.

(4 points)

E$^+$-9. Japanese businessman Rui lives in America and makes a living from trading cows. On Black Thursday he was selling his cows for 2000 dollars each (the cows were of the same price), but after the financial crash there were huge fluctuations in the market and Rui was forced to follow them with his pricing. Every day he doubled, halved, multiplied by five or divided by five the price from the previous day (even if it meant he had to give change in cents). At the same time he managed to follow the Japanese superstition, so that the integer part of the price in dollars never started with digit 4.
On the day when Billy visited him to buy some cows the price of each cow was 80 dollars. What is the minimal number of days that could have passed since Black Thursday by then?

(5 points)

E$^+$-10. Billy owns some really energetic horses. They are hopping around on points of the plane having two integer coordinates. Each horse can make the following types of jumps. They can hop to a point obtained from their current position via a translation by vector $(15, 9)$, $(-9, 15)$, $(-15, -9)$ or $(9, -15)$.

The horses are now standing on lattice points such that no two can meet by making jumps as above. What is the maximal possible number of horses Billy can own?

(5 points)

E$^+$-11. How many functions $f : \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 16\}$ have the property that $f(f(x)) \equiv 4f(x) \pmod{17}$ for all integers $1 \leq x \leq 16$? *Please submit your answer modulo* 10000.

(5 points)

E$^+$-12. The trapezoid $ABCD$ satisfies $AB \parallel CD$, $AB = 70$, $AD = 32$ and $BC = 49$. We also know that $\angle ABC = 3 \cdot \angle ADC$. How long is the base CD?

(5 points)

E$^+$-13. At a table tennis competition, every pair of players played each other exactly once.

Every boy beat thrice as many boys as girls, and every girl was beaten by twice as many girls as boys. How many competitors were there, if we know that there were 10 more boys than girls?

There are no draws in table tennis, every match was won by one of the two players.

(6 points)

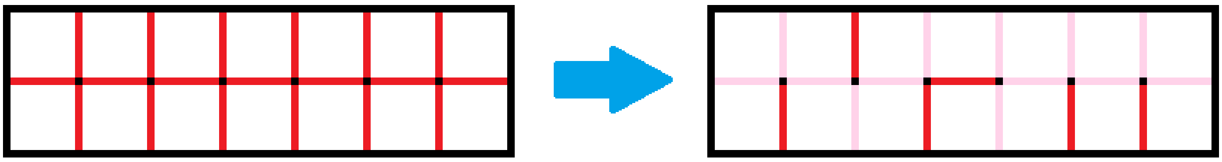
E$^+$-14. King Albrecht founded a family. In the family everyone has exacly 8 children. The only, but really important rule is that among the grandchildren of any person at most x can be named Bela. (None of Albrecht's children is called Bela.) For which x is it possible that after a certain time each newborn in the family has at least one direct ancestor in the Royal family called Bela.

No two of Albrecht's descendants (including himself) have a common child.

(6 points)

E$^+$-15. Consider a table consisting of 2×7 squares. Each little square is surrounded by walls (each internal wall belongs to two squares). We would like to remove some internal walls to make it possible to get from any square to any other one without crossing walls. How many ways can we do this while removing the minimal possible number of internal walls?

The figure shows a possible configuration, the remaining walls are marked in red, the removed ones are marked in light pink. Two configurations are considered the same if the same walls are removed.



(6 points)

E$^+$-16. The angles of a convex quadrilateral form an arithmetic sequence in clockwise order, and its side lengths also form an arithmetic sequence (but not necessarily in clockwise order). If the quadrilateral is not a square, and its shortest side has length 1, then its perimeter is $a + \sqrt{b}$, where a and b are positive integers. What is the value of $a + b$?

(6 points)

2 Solutions

2.1 Online round

2.1.1 Category C

1. Let's look at the fourth (last) row! The missing numbers must be 2 and 3, but if we place 2 in the 2nd column then that column will have two 2's, so this is not possible. So in this row, 2 must go in the third column and 3 must go in the second column. Using similar logic, in the fourth column the 3 goes in the 2nd row and the 2 goes in the 3rd row. So currently our table looks like this:

			1
	2		3
		x	2
1	3	2	4

Let's investigate the 2nd row. Here the numbers 1 and 4 must be placed in some order. However, using the previous process of logic, the 1 in the intersection of the 4th row and 1st column excludes the possibility of writing a 1 in the 1st column, so the 4 will go there, and the 1 will go in the 3rd column. Similarly we can fill in the 2nd column. Then our table looks like this:

	4		1
4	2	1	3
	1	x	2
1	3	2	4

Finally, let's look at the 3rd row. The 3 and 4 are missing. The entry in the 2nd row and 1st column excludes the possibility of writing the 4 in the first column, so it can only go in the position with the x. So we are done, $x = 4$.

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2. Let's look at the 140-degree external angles. An external angle and its corresponding internal angle always add up to 180°, so the corresponding internal angles are $180° - 140° = 40°$. We also know that in any quadrilateral, the sum of the internal angles is 360°. So we can compute the fourth angle of the quadrilateral (the reflex angle): it is $360° - 3 \cdot 40° = 240°$. This angle is the sum of a straight angle and the question-mark angle. So the question-mark angle is $240° - 180° = 60°$.

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3. The word KARTAL contains 4 consonants. We can write them in 24 orders as we can put 4 consonants to the first place, then having chosen the first one we can put 3 consonants to the second place, and after having chosen the first and second consonant we can choose from 2 consonants to the third place, and 1 to the fourth place. So we have $4 \cdot 3 \cdot 2 = 24$ choices for the consonants. However people with names Kartal, Tarkal, Lartak and Raltak cannot join to the company so they can have $24 - 4 = 20$ new friends.

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4. The sum of the numbers on a die is 21, so 63 in total on the three dice. On the visible sides there are $4 + 5 + 3 + 2 + 6 + 5 + 1 = 26$ dots, so on the non-visible sides, the sum of the numbers is $63 - 26 = 37$.

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5. Grandma shared the 91 cookies equally, so both the grandchildren and number of cookies received by a grandchild should be a divisior of 91. The divisiors of 91 are $1, 7, 13$, and 91. We know that the number of grandchildren is at least 2 and one grandchild ate at least 9 cookies. So a granchild either ate 13 or 91 cookies, but since there are at least two grandchildren it is only possible that a grandchild got 13 cookies.

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6.

Among 1-digit numbers only the 1 whose sum of digits and the product of non-zero digits is 1.

Let ab a 2-digit number. Then $a + b = 2$ is satisfied. Then either $a = 2$ and $b = 0$ or $a = 1$ and $b = 1$. Then the product of non-zero digits is 2 in the former case, and 1 in the latter case. Hence only 20 satisfies the conditions.

Let abc be a 3-digit number. In this case we have $a + b + c = 3$. Then the multiset $\{a, b, c\}$ should be one of the multisets $\{3, 0, 0\}$, $\{2, 1, 0\}$ és $\{1, 1, 1\}$. The product of the non-zero digits is 3 only in the case $\{3, 0, 0\}$. So the only 3-digit number satisfying the conditions is 300.

Finally let us consider the 4-digit numbers. Let $abcd$ be a 4-digit number satisfying the conditions. Then $a + b + c + d = 4$, and so the multiset $\{a, b, c, d\}$ must be one of the multisets $\{4, 0, 0, 0\}, \{3, 1, 0, 0\}, \{2, 2, 0, 0\}, \{2, 1, 1, 0\}, \{1, 1, 1, 1\}$. The products of non-zero digits are $4, 3, 4, 2, 1$ so only $\{4, 0, 0, 0\}$ and $\{2, 2, 0, 0\}$ can be the multiset. This leads to the numbers $4000, 2002, 2020, 2200$. Of these numbers only 2002 is smaller than 2020.

Hence there are 4 numbers satsifying the conditions: $1, 20, 300, 2002$.

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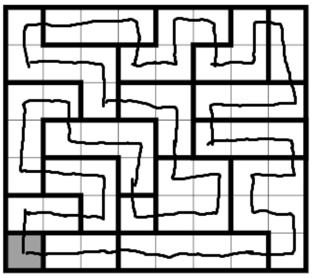
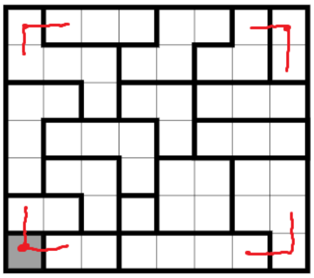
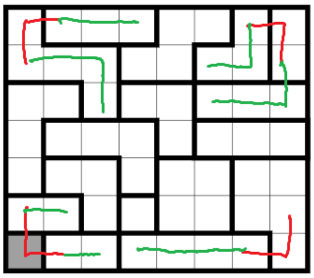
7. The route of the robot is the following.



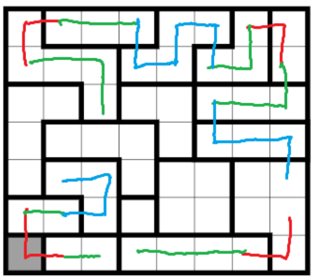
Hence the robot turned 35 times. Let us show how one can find it out. First at the corners the robot should go into the corner and then come out:



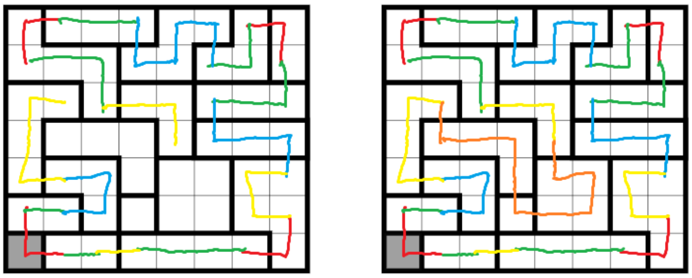
Then if we arrive to a part bordered by thick line where one can go only one ways (that is, there are two squares that has an odd number of neighbors in the part bordered by the thick line, then we go straight there. Let us call this parts *corridors*.



We can go the corridors on those squares that have an odd number of neighbors.



Then one can finish it easily as follows.



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8. Only mother or father can drive the car. If mother drives, only father or grandma can sit next to her, as the children cannot sit in front. No matter which one sits in front, the other one can sit in the back with the children in 6 ways. (There are 3 ways to choose who sits in the left-hand seat, and if this person is chosen, then the other two people can sit in the remaining 2 seats in 2 possible orders, so there are $3 \cdot 2 = 6$ ways for them to sit in the back.) So if mother drives, there are $2 \cdot 6 = 12$ possible ways for the passengers to sit in the car.

If father drives, only mother can sit next to him, as father cannot sit next to grandma, and the children cannot sit in the front. The ones sitting in the back can again do so in 6 possible orders.

So all in all, there are $12 + 6 = 18$ ways for the family to sit in the car.

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9. We claim that Gábor could have written at most 14 numbers. Here is a way that he could have written exactly 14:

number	199	279	287	295	339	347	355	363	371	406	414	422	430	501
sum of digits	19	18	17	16	15	14	13	12	11	10	9	8	7	6

Now we show that Gábor couldn't write more than 14 numbers. Suppose that $a_1, a_2, \ldots, a_{15}$ are three-digit numbers, with digit sums $d_1, d_2, \ldots, d_{15}$ respectively, and we have

$$a_1 < a_2 < \ldots < a_{15}$$
$$d_1 > d_2 > \ldots > d_{15}$$

If we decrease the hundreds digit of each number, then this decreases each digit sum by 1, and this also doesn't alter the order of the numbers, so the numbers will still be in increasing order, and their digit sums will remain in decreasing order. We can repeat this step until the first digit of a_1 becomes 1. So we can assume that $100 \leq a_1 \leq 199$.

The conditions imply that $d_{15} \leq d_1 - 14$, and since the first digit of a_1 is a 1, we have $d_1 \leq 19$. So this means $d_{15} \leq 5$. Let's distinguish cases depending on the value of d_{15}:

1. $d_{15} = 1$: Then a_{15} is a 3-digit number with digit sum 1, so $a_{15} = 100$, but then we cannot have $a_1 < a_{15}$, since a_1 is also a three-digit number.

2. $d_{15} = 2$: Then $a_{15} \le 200$, so $a_1, a_2, \ldots, a_{14}$ all fall between 100 and 199. Then the second digit of the the 14 numbers have to be a strictly monotone increasing sequence. If say the second digit of a_1 and a_2 would be the same, then they would only differ in the third digit and it would mean that it cannot occur that $a_1 < a_2$ and $d_1 > d_2$ at the same time. But the second digit can take at most 10 different values and we have 14 numbers. So this case is not possible.

3. $d_{15} = 3$: Then $a_{15} \le 300$. Hence some of the numbers $a_1, a_2, \ldots, a_{14}$ have first digit 1, the others have 2. Similarly to the previous case we can see that if the second digits have to form a strictly increasing sequence for those number that have the same first digit. This means that for such numbers the third digit have to decrease with at least 2 since the sum of the digits have to be a strictly decreasing sequence. This would mean that such a sequence have at most 5 members since the third digit is between 0 and 9. Then it means that at most 5 numbers can start with 1 and at most 5 numbers can start with 2. But this is at most 10 numbers, that is, less than 14.

4. $d_{15} = 4$: Then $a_{15} \le 400$. We also know that $d_1 \ge d_{15} + 14 = 18$, that is, $a_1 \ge 189$. By the argument of the previous cases we know that if a_3 starts with digit 1, then its second digit should be at least $8 + 2$ which cannot happen. Hence $a_3 \ge 200$. Similarly, if a_8's first digit is 2, then its third digit is at most the third digit of a_3 minus 10 which is not possible again. Hence $a_8 \ge 300$. The same argument gives that $a_{13} \ge 400$ that contradicts $a_{15} \le 400$.

5. $d_{15} = 5$: Then $a_{15} \le 500$. Since $d_1 \le 19 = 5+14$, this case can only occur if $d_1, d_2, \ldots, d_{15}$ have values $19, 18, 17, \ldots, 5$ in this order. Since $d_1 = 19$, we need to have $a_1 = 199$. So $a_2 \ge 200$, and $d_2 = 18$, whence $a_2 \ge 279$. If the first digit of a_5 is 2, then the second digit have to be at least 3 plus the second digit of a_2, this cannot occur. Thus $a_5 \ge 300$. This gives $a_{10} \ge 400$, otherwise the third digit of a_{10} would be at most the third digit of a_5 minus 10. Since $d_{10} = 10$, the last digit of a_{10} is at most 6. This gives that $a_{14} \ge 500$, otherwise the last digit of a_{14} would be at most the last digit of a_{10} minus 8. But this way $500 \le a_{14} < a_{15} \le 500$, contradiction.

We get a contradiction in each cases so one cannot give 15 numbers satisfying the conditions of the numbers. Hence Gábor could have written at most 14 numbers to the blackboard. Mindegyik esetben ellentmondást kaptunk, tehát nem adható meg

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10. Our strategy is the following: in each move, we move the rook to a square which is on the diagonal connecting the top-left and bottom-right corners. If the rook stands on this diagonal in the initial position, we choose to go second, otherwise we choose to go first.

We prove that if the rook doesn't stand on this diagonal then we can move onto the diagonal, and if it does stand on the diagonal, then we cannot make a move which stays on the diagonal.

As we can move any number of squares to the right or to the bottom, if we don't stand on the diagonal, then the rook is either to the left of it, in which case we can move to the right onto the diagonal, or it is upwards from it, in which case we can move downwards onto the diagonal.

If someone moves away to the bottom or to the right from the diagonal, then this move cannot bring the rook to a different square of the diagonal, since the diagonal only contains one square in each row and in each column. So the other player cannot move to the diagonal like this, so cannot win, as the winning cell is on the diagonal.

So it is guaranteed that we can always step onto the diagonal, and every time the opponent moves away from it, we can step back. The total number of squares that we need to move to the right and to the bottom to reach the corner decreases on every step, so eventually it will reach 0, and only we will move on the diagonal (never the opponent), so we will win.

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2.1.2 Category D

1. Consider the fourth column of the table. The numbers 1, 3 and 4 are missing from there. The bottom-most square cannot be filled with a 1 or 4, as the 5th row already contains both of these digits. So the 3 must go there. Now investigate the 4's appearing in the table. The third row must have a 4, but it can only go in the first empty square out of the three, since the 2nd and 5th columns already have a 4 in them. Similarly row 2 must also contain a 4, and it can only go in the 4th position, as the 1st and 5th columns already contain a 4. Now our table looks like this:

2	0		x	
	2	0	4	
4		1	0	
			2	4
1	4		3	

Considering the 4th column, we can see that Timi must have written 1 in the position marked with an x.

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2. For the solution, see Category C Problem 3.

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3. For the solution, see Category C Problem 4.

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4. For the solution, see Category C Problem 7.

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5. Let c and h be the number of chicken nuggets and onion rings. From the first condition we know that

$$\frac{3}{2}\cdot\left(c-\frac{1}{5}\right)=h$$
$$\frac{3}{2}\cdot\left(\frac{4}{5}c\right)=h$$
$$\frac{6}{5}c=h$$
$$6c=5h.$$

From the second condition we know that

$$h-44=\frac{2}{5}\cdot\left(c-\frac{1}{5}c\right)$$
$$5h-220=2\cdot\left(\frac{4}{5}c\right)$$
$$25h-1100=8c.$$

Let us substitute the condition $6c=5h$ into this equation.

$$30c-1100=8c$$
$$22c=1100$$
$$c=50$$

Then we find from the equation $\frac{6}{5}c=h$ that $h=\frac{6}{5}\cdot 50=60$, whence Ákos ordered $c+h=50+60=110$ chicken nuggets and onion rings together.

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6. During the solution, let $|XY|$ denote the length of segment XY. As $ABCD$ is a rectangle, we have $|AD|=|BC|$ and $|AB|=|DC|$. Using the Pythagorean theorem, we can determine the lengths of AD and BE:

$$|DC|^2+|AD|^2=|AC|^2,$$

which gives $|AD|=\sqrt{17^2-8^2}=15$. And

$$|AB|^2+|BE|^2=|AE|^2$$

gives $|BE|=\sqrt{10^2-8^2}=6$.

Now $T_{AEC}=\frac{|EC|\cdot|AB|}{2}$, since the area of the triangle is given by the formula $\frac{am_a}{2}$, where a is a side of the triangle and m_a is the corresponding altitude. And the altitude of triangle AEC corresponding to side EC is BA, since this is the segment from A perpendicular to the line EC. So the area of the triangle is $\frac{(15-6)(8)}{2}=36$.

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7. Let the number of heads f_1, f_2, f_3, f_4, f_5 in the order of speaking. The conditions of the problem are the following:

$$\frac{f_2+f_3+f_4+f_5}{4}=7$$
$$\frac{f_1+f_3+f_4+f_5}{4}=9$$
$$\frac{f_1+f_2+f_4+f_5}{4}=8$$
$$\frac{f_1+f_2+f_3+f_5}{4}=8$$
$$\frac{f_1+f_2+f_3+f_4}{4}=9$$

Let us add these equations together. Observe that each f_i appears exactly 4 times with a coeffient $\frac{1}{4}$ so in the sum the coefficient will be $4\times\frac{1}{4}=1$. Hence the sum of the five equations

$$f_1+f_2+f_3+f_4+f_5=7+9+8+8+9=41$$

which means that Süsü's 5 siblings have 41 heads together.

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8. In base 10, let the two numbers be $\overline{abcd}$ and $\overline{efgh}$. Let $\overline{abcd}\le\overline{efgh}$, which means that $abcd$ could possibly be a one-digit number. (So some of the variables a, b, c might not even denote a digit, i.e. might not actually belong to the number.) As $5000\le\overline{efgh}\le 9999$, the larger number must have four digits.

As $d+h$ is divisible by 10, the sum of the two digits in the ones' place must be 0 or 10. We can distinguish three cases depending on the values of d and h:

- Case 1: $d=h=0$,
- Case 2: $d=h=5$,
- Case 3: $d+h=10$ with $d\ne h$.

Let us investigate all of these three cases, one by one.

- Case 1: $d=h=0$. Apart from 0, we also have to use a positive digit, let it be k. The sum of the two numbers must be divisible by k, as all of their digits are divisible by k. So the sum - which is 10000 - can be written as $k\cdot s$, where s is a positive integer with at most four digits, such that each of the four digits of s is 0, 1 or 2 depending on how many of the two summands contained the digit k at that position. (For example $\overline{k0kk}+\overline{k0k}=1112\cdot k$, here $s=1112$.) $s|10000$, and as $k<10$, $s>1000$. Out of the divisors of 10000 greater than 1000, only 2000 is of this form, so $s=2000$ and $k=5$. This means that both numbers have 5 at the thousands place, and all their other digits are 0. This yields the sum $5000+5000$ which is a good solution, and the only solution in this case.

- Case 2: $d = h = 5$. One of the digits used is 5, and we have $\overline{abc}+\overline{efg} = 999$. So $c+g = 9$, which means $c \neq g$. There are two sub-cases:
  - Case 2a: one of c and g is 5, and the other is 4.
  - Case 2b: c does not denote a digit, so the smaller number is one-digit: $\overline{abcd} = d$.

  Now let us investigate these sub-cases (2a and 2b).
  - Case 2a: There is no carry at any position, so $a + e = b + f = c + g = 9$, and this can only happen if at all three positions (thousands, hundreds and tens), one of the digits is a 4 and the other is a 5. As $a < e$, we have $a = 4$ and $e = 5$, but for the hundreds and tens, the two digits can be in any order in the two numbers. So this gives a total $2 \cdot 2 = 4$ pairs of numbers: $4445 + 5555$, $4455 + 5545$, $4545 + 5454$, and $4555 + 5445$.
  - Case 2b: $\overline{abcd}$ is one-digit, $\overline{abc} = 0$ (but the 0's don't actually appear before d), and $\overline{efg} = 999$. Here the only good solution is $5 + 9995$.
- Case 3: d and h are two distinct positive digits with a sum of 10. Then $\overline{abc} + \overline{efg} = 999$, so $c + g = 9$ must be true in this case too. $c \neq g$, as their sum is odd. But c and g also cannot denote distinct digits, as then their sum would be 10. So c cannot denote a digit in this case either, so $\overline{abcd}$ is one-dgit and $\overline{efg} = 999$. One of the digits is necessarily 9, so the other is 1, and we get two further solutions: $1 + 9999$ and $9 + 9991$.

We have investigated all possible cases, so all in all there are 8 pairs of numbers satisfying the conditions of the problem.

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9. Since the big rectangle can be decomposed into smaller squares the area of the rectangle is equal to the sum of areas of the squares. Hence the area of the rectangle is

$$\begin{aligned} T &= 1^2 + 4^2 + 7^2 + 8^2 + 9^2 + 10^2 + 14^2 + 15^2 + 18^2 = \\ &= 1 + 16 + 49 + 64 + 81 + 100 + 196 + 225 + 324 = \\ &= 1056. \end{aligned}$$

Assume that a and b are the lengths of the sides of the rectangle such that $a \leq b$. Then we know that the area of the rectangle is $T = ab = 1056$. We also know that a 18×18 square can be placed into the rectangle giving that $a \geq 18$ and $b \geq 18$.

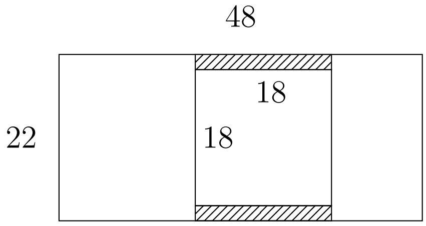
This implies that

$$18 \leq a \leq \sqrt{ab} = \sqrt{1056} < 33$$

Hence the possible values of a are $a = 22, 24, 32$ since these are the divisors of 1056 falling into this interval. The corresponding values of b are $b = 48, 44, 33$.

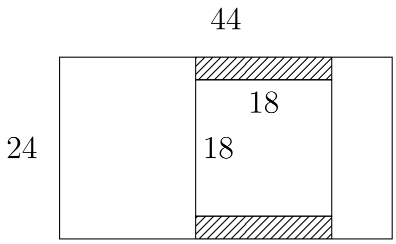
We claim that the rectangles of size 22×48 and 24×44 cannot be decomposed to the given squares, but the rectangle of size 32×33 can be decomposed.

Let us place the 18×18 square to the 22×48 rectangle. Let us extend the sides of the rectangle along the longer side (see the figure) whose sum of areas $4 \cdot 18 = 72$. Note that both rectangles have a side of length at most 4, so to cover them we can only use squeares of size 1×1 and 4×4 whose total area is $1 + 16 = 17 < 72$. Hence the two smaller rectangle cannot be covered by the given squares, and so 22×48 rectangle has no decomposition with the given squares.

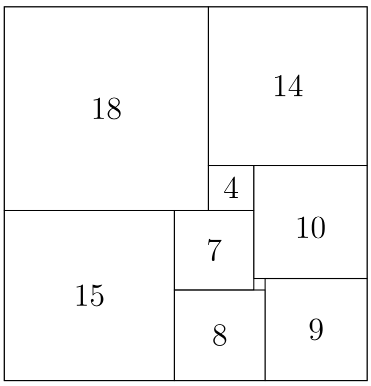


Similar argument works for the 24×44 rectangle. After placing the 18×18 rectangle, it yields two rectangles (one of them is possibly empty) whose total area is $6 \cdot 18 = 108$. Furthermore both rectangle have a side of length at most 6. So we can use only squares of side length 1 and 4 to cover the rectangles. But the total area of these squares is $1 + 16 = 17 < 108$, so there is no decomposition.

$1 + 16 = 17 < 108$,



On the other hand there is a decomposition of rectangle 32×33 to the given squares as follows.



So the length of the rectangles are 32 and 33, so the length of the shorter side is 32.

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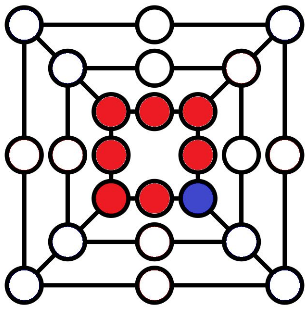
10. The starting player has a winning strategy. The first move can be any place which is in three lines.

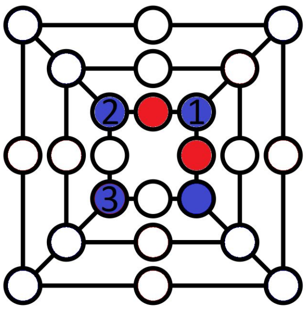
Now we show the winning strategy for one given starting place. Let the following blue piece be our starting move:



We will have two cases depending on whether the second player's first move is one of the places indicated red in the figure, or not.

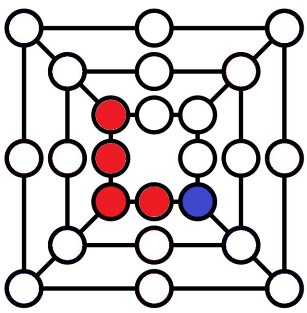
If it's not one of those places, then if we place the blue pieces on the figure below in the order of numbering, then there will be two options after we place the 1st and 2nd ones: either the opponent puts their piece between our current and our previous piece, or not, but if not, then they didn't make a line, since they only put one outside the innermost square. And then we win the game in the next move by filling in our gap.

So we can assume that they put their pieces in the indicated spaces after we placed our pieces 1 and 2.



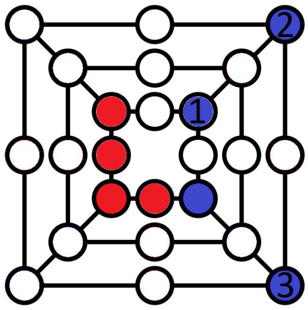
Then after we place our piece 3, the opponent still cannot win in the next move. But after their move, we can definitely win by creating a line, either by placing a piece between our pieces 2 and 3, or between piece 3 and the initial piece.

Now let's investigate the case when the opponent's first move was one of the places marked red on the topmost figure. Then without loss of generality we can assume that it was one of the following spaces (otherwise just reflect the board about the main diagonal).



Again, similarly to the previous case, we can give three spaces, where if we move in sequence, the opponent will always have to move between our current and our previous place, otherwise

we make a line there immediately with which we would win. And they cannot move anywhere else where they would win immediately.
After placing our 3rd stone, we make two lines consisting of two of our pieces and one empty place.
Here is a possible way to choose such spaces:



So we have shown that by making this first move, we can win no matter what our opponent replies.

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2.1.3 Category E

1. For the solution, see Category C Problem 5.

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2. For the solution, see Category C Problem 7.

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3. For the solution, see Category C Problem 8.

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4. For the solution, see Category D Problem 6.

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5. For the solution, see Category D Problem 7.

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6. For the solution, see Category D Problem 8.

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7. In each step, the knight moves two steps upwards and one to the right, or two to the right and one upwards. So if we look at the total number of squares that the knight still needs to travel upwards and to the right (so that it reaches the top-right corner), this number decreases by 3 on every move.

As the knight needs to move 9 squares upwards and 9 to the right, it will make $(9+9)/3 = 6$ moves overall. It needs to go upwards and to the right by the same amount, so it will move 2 squares upwards 3 times, and 2 squares to the right 3 times as well.

So to count the total number of possibilities for the knight to travel to the top-right corner, we just need to count the number of ways to select 3 out of the 6 moves, when the knight moves 2 steps to the right and 1 upwards. This can be done in $\binom{6}{3} = 20$ ways, so there are 20 possible ways for the knight to get to the top-right corner.

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8. For the solution, see Category D Problem 9.

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9. Let the vertices of a graph be the rooks, and the edges be the secret passages. Let u_1 be the degree 1 vertex, let v_1, v_2 be the degree 2 vertices and w_1, w_2, w_3 be the degree 3 vertices. Let $V = \{v_1, v_2\}$ and $W = \{w_1, w_2, w_3\}$. Let $S = V \cup \{u_1\} = \{u_1, v_1, v_2\}$.

We can choose the three rooks of degree 3 in $\binom{6}{3} = 20$ ways, and then we can choose the degree 1 vertex in 3 ways.

The sum of the degrees of the vertices in W is 9, the sum of the degrees of all other vertices is 5.

If there were 0 or 1 edge going inside W, then there would be 7 or 9 edges would go out from W, and this is more edges that can go out from S, so this is is not possible.

If there were 2 edges running inside W, then $9 - 4 = 5$ edges can go between W and S. This is exactly what S can accept. So there cannot be any edges inside S.

We can choose in 3 ways which two edges go inside W.

The two new edges would take away 2 edges from one vertex, and $1 - 1$ edges from two vertices. Let $w' \in W$ that is connected with the other two elements of W.

If w' is adjacent to u_1, then the elements of $W \setminus w'$ and V are all adjacent. There are $20 \cdot 3 \cdot 3$ such graphs.

If u_1 is not adjacent to w', then we can choose the neighbor of u_1 in two ways (any element of $W \setminus w'$). We can choose the neighbor of w' in two ways (any element of U), and the rest of the graph is determined. If say $w' = w_2$, u_1 is adjacent to w_1 and w_2 is adjacent to u_2, then the remaining edges should be (w_1, u_2), (w_3, u_1) (w_3, u_2). There are $20 \cdot 3 \cdot 3 \cdot 2 \cdot 2$ such graphs.

We still need to examine the case when all edges appear between W.

Then $9 - 6 = 3$ edges go between W and S, and so there is one edge can go inside S.

If the edge in S goes between v_1 and v_2, then the neighbor of u_1 is a vertex in W, and v_1, v_2 has one-one neighbor in W. In this case we can choose the neighbor w^* of u_1 in 3 ways and we have 2 choices to create a matching between V and $W \setminus w^*$. Hence there are $20 \cdot 3 \cdot 3 \cdot 2$ such graphs.

If the edge in S goes between u_1 and V, then we can choose the neighbor of u_1 in 2 ways, namely v_1 or v_2. If the neighbor of u_1 is v_1, then the other neighbor of this v_1 come from W, and the neighbors of v_2 are the remaining vertices from W. So we have 2 choices for the neighbor of u_1, three choices for the remaining neighbor of the neighbor of u_1. So we have $20 \cdot 3 \cdot 2 \cdot 3$ such graphs.

Alltogether we have $20\cdot3\cdot3+20\cdot3\cdot3\cdot2\cdot2+20\cdot3\cdot3\cdot2+20\cdot3\cdot2\cdot3 = 20\cdot3\cdot(3+12+6+6) = 60 \cdot 27 = 1620$ such graphs.

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10. The trick of the game is that it is equivalent with the tic-tac-toe game on the 3×3 magic square!

First let us collect the number triples of $1, 2, \ldots 9$ that satisfies that their sum is 15. We can simply list the triples by their smallest elements.

$$1+5+9 = 1+6+8 = 2+4+9 = 2+5+8 = 2+6+7 = 3+4+8 = 3+5+7 = 4+5+6$$

Now let us draw the 3×3 magic square with numbers $1, 2 \ldots 9$. We will use this table to describe the strategies of the players.

6	1	8
7	5	3
2	9	4

Now observe that this magic square contains the above triples in their rows, columns and diagonals. As a magic square in all rows, columns and diagonals have sum 15. So we see that the game described in the problem is the exact same as tic-tac-toe on this magic square with the twist that there is no draw: in case no one has a line in the tic-tac-toe game, then the second one wins. In the classical tic-tac-toe game it is possible to achieve at least a draw for the second player which means that there is a winning strategy for the second player in the game of the problem.

Here we do not detail the complete analysis of the game, only some main lines.

Probably the main line of the game is that the first player chooses the center of the square, that is, the number 5. In this case it is important to choose one of the corner elements $2, 4, 6, 8$. Once this is chosen the second player has to defend the lines or complete her own line. Some tipical games look as follows: (let the first player be blue, the second one is red):

5, 6, 9, 1 (defends line 1,5,9), 8 (defends line 1,6,8)

2 (defends line 1,5,9), 7 (defends line 2,7,8), 3 (defends line 3,5,7), 4.

Since there is no line, the second player wins this modified version of the tic-tac-toe.

In another game blue might try 8 as a second move:

$$5, 6, 8, 2 \text{ (defends line 2,5,8)}, 7 \text{ (defends line 2,6,7)}$$

$$3 \text{ (defends line 3,5,7)}, 9\ 1 \text{ (defends line 1,5,9)}, 4.$$

Again there is no line, the second player wins this modified version of the tic-tac-toe.

Remark: The film WarGames written by Lawrence Lasker and Walter F. Parkes and directed by John Badham, staring Matthew Broderick, Dabney Coleman, John Wood, and Ally Sheedy have a peculiar relevance concerning the tic-tac-toe game. In this film the military supercomputer learns that some games have no winner by playing tic-tac-toe with itself, and cancels a massive missile attack leading to a possible World War III. For further details see the Wikipedia page `https://en.wikipedia.org/wiki/WarGames`

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2.2 Regional round

2.2.1 Category C

1. The prime factorization of 2020 is $2020 = 2 \cdot 2 \cdot 5 \cdot 101$, so one of their ages is divisible by 101.

Andris or his sister cannot be 101 years old, because then their great-grandfather would be at most $2020/101 = 20$ years old, and he clearly should not be younger than his great-grandchildren. Therefore the age of the great grandfather is divisible by 101.

If the great-grandfather is 404 years old (yes, this would be suprising but no one said they are humans), then Andris and his sister are both at most $2020/404 = 5$ years old, therefore their age difference is less than 8 years.

If the great-grandfather is 202 years old (still impressive), then the product of Andris's age and the sister's age is 10, so they are either 1 and 10 years old, or 2 and 5 years old. None of these options can be correct, because the age difference is not 8.

Therefore the age of the great-grandpa is 101, and the product of the kids' ages is $2 \cdot 2 \cdot 5 = 20$.

We can write 20 as the product of two positive integers in the following ways: $1 \cdot 20$, $2 \cdot 10$, or $4 \cdot 5$. Only 2 and 10 satisfies the age difference requirement, therefore Andris is 10, his sister is 2 and the great-grandfather is 101 years old.

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2. a) Yes, it is possible. Here is an example.

Let the tribes be called A, B, C, D, E, F, G, H, I. In each of the tribes A, B, C, D, E, 5 members vote East and 4 members vote West. In the other tribes, everybody votes West. Then only $25 < \frac{81}{2}$ members voted East, but the Indians still go West.

b) Now the number of tribes is odd. The vote of the representatives was not a draw, so out of the 4 representatives, at least 3 voted West. For these tribes to vote West, in each of the tribes, at least 3 out of the 4 members had to vote West (otherwise the tribe would vote East, or the vote would be a tie, which was excluded in the problem). So overall, at least $3 \cdot 3 = 9$ of the Indians had to vote West, which is indeed more than half of all 16 Indians.

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3. Let us determine the distance of the points S and P from the vertices of the rectangle, Since S is a trisecting point we have $|DS| = \frac{120\text{ cm}}{3} = 40$ cm and $|SC| = 80$ cm. Since P is one fourth on the way on segment SC closer to C we get that $|PC| = \frac{80cm}{4} = 20$ cm.

Since ASD is a right angle triangle whose sides we know we can compute its area: $T_{ASD} = \frac{80\text{ cm}\cdot 40\text{ cm}}{2} = 1600\text{ cm}^2 = 16\text{ dm}^2$. Similarly we can compute the area of the triangle SBC: $T_{SBC} = \frac{80\text{ cm}\cdot 80\text{ cm}}{2} = 3200\text{ cm}^2 = 32\text{ dm}^2$.

Let us determine the areas of the triangles PSE and PBC. Let us use that the area of the triangle is the sum of the triangles PSE, BPE and PBC. Hence $T_{PSE}+T_{PBC} = T_{SBC}-T_{BPE} = 32\text{ dm}^2 - 16\text{ dm}^2 = 16\text{ dm}^2$.

Hence the total area of the white triangles is $T_{ASD} + (T_{PSE} + T_{PBC}) = 16\text{ dm}^2 + 16\text{ dm}^2 = 32\text{ dm}^2$.

Second solution: We can compute the area of the rectangle $ABCD$: $T_{ABCD} = 120\text{ cm} \cdot 80\text{ cm} = 9600\text{ cm}^2$. Our next goal is to determine the area of the triangle ABS. If T_{white} denote the total area of the white triangles, then $T_{ABCD} = T_{ABS} + T_{BPE} + T_{\text{white}}$. So by knowing T_{ABS} we would obtain T_{white}.

We use the following formula for the area of a trangle: $T = \frac{a\cdot m_a}{2}$, where a is the length of the side and m_a is the length of the corresponding altitude. Clearly, the altitude of the triangle ABS corresponding to the side AB is the length of AD, that is, 80 cm. Hence the area of ABS is $T_{ABS} = \frac{120\text{ cm}\cdot 80\text{ cm}}{2} = 4800\text{ cm}^2$, the half of the area of the rectangle $ABCD$.

Hence $T_{\text{white}} = T_{ABCD} - T_{ABS} - T_{PEB} = 96\text{ cm}^2 - 48\text{ dm}^2 - 16\text{ cm}^2 = 32\text{ dm}^2$.

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4. First let us study how many banditries can have a gang member that lost all his box matches. We claim that there is at most 1 such banditry. Since if a banditry has such a member, then the members of all other banditries won against that gang member. So there is at most banditry with such gang member.

Since 16 people won at least one of their matches, and at most one gang can have members who lost all their matches we surely know that $16 + 7 = 23$ is an upper bound for the number of participants. We show that there cannot be exactly 23 participants. This would mean that there is a banditry of 7 members that lost all their box matches, and so the remaining 16 members must be the union of all other banditries. But 16 cannot be decomposed to the sum of 5's and 7's. So this cannot happen.

So can have at most 22 participants. This can indeed occur: suppose we have three banditries of size 5 and a banditry of size 7. Furthermore, suppose that the banditry of size 7 has a member who beat everybody, but the other 6 members lost all their box matches. Then exactly $3 \cdot 5 + 1 = 16$ bandit has at least one box match winning, and 6 bandits lost all their matches.

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5. a) a, b and c all appear as the first digit of a number with many digits, so their value is at least 1. The value of $\overline{abc}$ is $100a+10b+c$, and similarly $\overline{ab} = 10a+b$, $\overline{bc} = 10b+c$, and $\overline{ca} = 10c+a$. Let us write into the original equation: $(100a+10b+c) = (10a+b)+(10b+c)+(10c+a)$.

After rearrangement we get that $89a = b+10c$. The values of b and c are at most 9, whence $b+10c \leq 9+10\cdot 9 = 99$. Hence $a \leq 2$, otherwise $89a \geq 89\cdot 2 = 178 > 99$. So $a = 1$ and $b+10c = 89$. Since $10c$ is divisible by 10 the remainder of b after division by 10 must be as the remainder of 89 after division by 10, that is, $b = 9$ and consequently $c = 8$.

The values $a = 1, b = 9, c = 8$ is indeed solution of the original equation as $198 = 19+98+81$.

b) Observe that we can permute a, b, c without changing the equation. So we can assume that $a \leq b \leq c$. Having these solutions we can get all solutions of Albrecht's problem by permuting a, b, c.

a, b and c are all positive integers so their products abc is not 0, and so we can divide with it. Then we get that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$.

We can solve this equation by checking the possible values of a and b since there are not too many possibilities. First let us give some estimate to a. Since we assumed that $a \leq b \leq c$ we get that

$$1 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{1}{a}+\frac{1}{a}+\frac{1}{a} = \frac{3}{a}$$

implying that $a \leq 3$. If $a = 1$, then

$$1 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{1}{1} = 1$$

would be a contradiction. Hence $a = 2$ or $a = 3$.

Hence $a = 2$ or $a = 3$. Let us consider the following cases.

1. case: $a = 2$. By substituting this into the equation we get that $\frac{1}{b}+\frac{1}{c} = \frac{1}{2}$ Now let us estimate b similarly to a. By $b \leq c$ we get that $\frac{1}{2} = \frac{1}{b}+\frac{1}{c} \leq \frac{1}{b}+\frac{1}{b} = \frac{2}{b}$. Hence $b \leq 4$. On the other hand, $\frac{1}{2} = \frac{1}{b}+\frac{1}{c} > \frac{1}{b}$ giving that $b > 2$. Hence b is 3 or 4. If $b = 3$, then the equation $\frac{1}{2} = \frac{1}{b}+\frac{1}{c} = \frac{1}{3}+\frac{1}{c}$ gives $c = 6$. If $b = 3$, then the equation $\frac{1}{2} = \frac{1}{b}+\frac{1}{c} = \frac{1}{4}+\frac{1}{c}$ gives $c = 4$.

2. case: $a = 3$. Since $a \leq b \leq c$ we get that $1 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{a} = 1$. So we should have equality everywhere giving that $a = b = c$. This means that $a = b = c = 3$.

We have checked all cases. Albrect's problem has the solutions $(a, b, c) = (2, 3, 6)$, a $(2, 4, 4)$ and $(3, 3, 3)$ and their permutations. With their permutations we get 10 different solutions.

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2.2.2 Category D

1. Let us fill the bottom row of the pyramid with the variables a, b, c, d, e in that order. Each variable takes tha value 0 or 1. Then going up row by row let us fill the small squares with the sum of the expressions below (according to the problem). For instance, the rightmost element of the middle row contains the expression $(c+d)+(d+e) = c+2d+e$. In the topmost we square we write the expression $a+4b+6c+4d+e$. *Observe that the coefficient of each variable* *is exactly the number of routes with down-left and down-right steps from the topmost square to the square in the bottom row contaning the given variable.*

The expression $a + 4b + 6c + 4d + e$ is even if and only if $a + e$ is even since the coefficients of b, c and d is even. The sum $a + e$ is even if and only if both a and e are 0 or both of them are 1.

Since a and e can take 2 values and we can choose the values b, c, d independently in 2 ways. Hence we have $2 \cdot 2^3 = 16$ fillings such that the topmost element is even.

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2. **a)** One possible solution is the following: let one group consist of the numbers $1, 2, 3, 5, 7, 12$, and the other one consist of $4, 6, 8, 9, 10, 11$. Then 5 and 8 are equal to the average of the other five members of their group: $\frac{1+2+3+7+12}{5} = 5$ and $\frac{4+6+9+10+11}{5} = 8$.

b) If the numbers in a group a, b, c, d, e, f and f is the average of the other 5 numbers, then $5f = a + b + c + d + e$. Hence $6f = a + b + c + d + e + f$, so the sum is even. This means that if we can split thhe numbers $1, 2, \ldots, 18$ into three groups with the required property, then their sum is even. But $1 + 2 + ... + 18 = \frac{18 \cdot 19}{2} = 171$ is odd. Contradiction, there is no proper splitting.

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3. The equation does not change if we permute the numbers a, b, c so we can asusume without loss of generality that $a \geq b \geq c$.

Let us first study the equation in terms of parity, that is, divisibility by 2. 3^d is always odd. Hence $a! + b! + c!$ must be odd too. This can occur in two different ways. Either $a!$, $b!$ and $c!$ are all odd, or exactly two of them are even and one of them is odd.

Let us first consider the first case. If $a \geq 2$, then $a!$ will contain the factor 2, thereby it will be even. Hence if $a!, b!$ are $c!$ all odd, then $a = b = c = 1$. In this case the equation has a solution with $d = 1$.

In what follows we study the case when exactly one of $a!$, $b!$ and $c!$ is odd, and two of them is even. According to the previous argument the odd number must be $1! = 1$. Since we assumed that c is the smallest number among a, b, c we get that $c = 1$.

Now let us study the equation in terms of remainder modulo 3. Naturally 3^d is divisible by 3 as $d > 0$. If $b \geq 3$, then both $a!$ and $b!$ are divisible by 3 as 3 occurs in the product. In this case $a! + b! + c!$ would give 1 as a remainder after division by 3. Hence $b < 3$ and since $b!$ is even we get that $b \geq 2$. Hence $b = 3$.

So currently we have the equation $a! + 2! + 1! = 3^d$, that is, $a! + 3 = 3^d$. We immediately get that $d \geq 2$. After rearranging the equation we get that $a! = 3^d - 3 = 3 \cdot (3^{d-1} - 1)$. Since $d \geq 2$ we get that $3 \cdot (3^{d-1} - 1)$ is divisible by 3, but not divisible by 9. If a is at least 6, then $a!$ contains both 3 and 9 among the factors so it would be divisible by 9. Hence a is at most 5 and it is at least 3 otherwise $a! + 3$ is not divisible by 3. Hence a is either $3, 4, 5$. $a = 3$ gives a good solution with $d = 2$ as $3! + 2! + 1! = 9 = 3^2$. $a = 4$ also gives a good solution woth $d = 3$ as $4! + 2! + 1! = 27 = 3^3$. For $a = 5$ we get that $5! + 2! + 1! = 123$ which is not a power of 3.

Hence the solutions of the equation:

I. $a = b = c = d = 1$

II. $a = 3, b = 2, c = 1, d = 2$
III. $a = 4, b = 2, c = 1, d = 3$
Since we only assumed that $a \geq b \geq c$ for the easier case analysis in case II and III we get $6-6$ solutions after permuting a, b, c.

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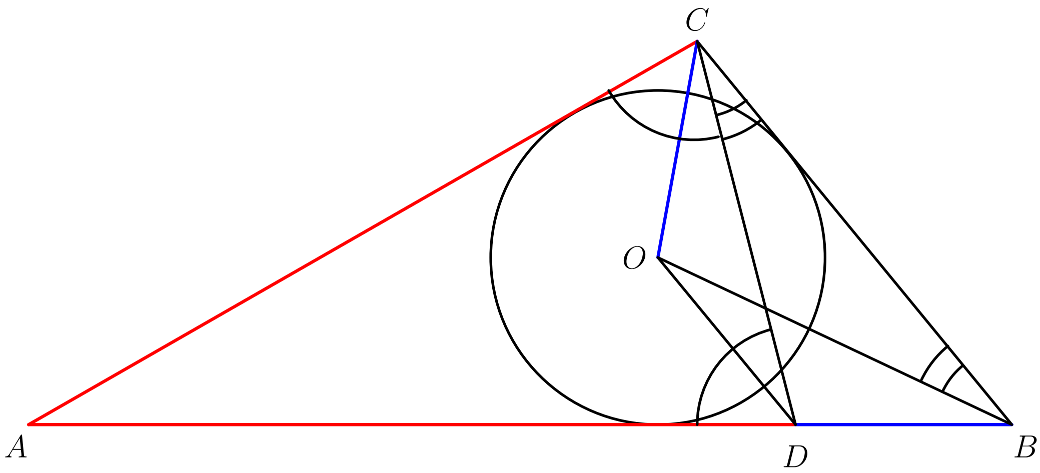
4. In the trinagle ABC we have $BCA\sphericalangle = 100°$. We know that OC is the bisector of the angle at C, whence $BCO\sphericalangle = 50°$. Let D be the intersection of the bisector of the angle BCO and the line AB. Then the triangles BCO and CBD are congruent since their side BC is common, and their angles are common. From this congruence we get that $OC = DB$.

We know that $DCA\sphericalangle = DCO\sphericalangle + OCA\sphericalangle = 75°$. We also get that $ADC\sphericalangle = 180° - DCA\sphericalangle - DAC\sphericalangle = 75°$. Hence the triangle ADC is an isoscele triangle implying that $AC = AD$.

Let us put together the previous observations.

$$|AB| = |AD| + |DB| = |AC| + |OC|$$

This is exactly what we wanted to prove.

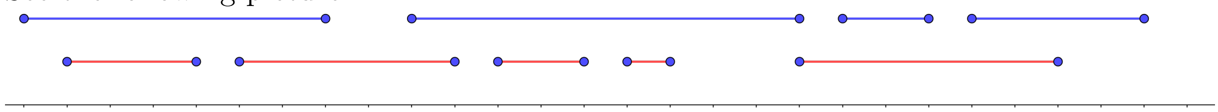


(Back to problems)

5. We will prove it indirectly. Let us assume for contradiction that in any moment, at most two professors were asleep. Write the sleep times as intervals. The starting point of the interval is when a professor falls asleep, and the endpoint is when he wakes up. We will color the intervals red and blue in a greedy way according to their starting points. Color the interval with the earliest starting point blue, and after that, for each interval, if its starting point is in another, already colored interval, then use the opposite color for this new interval. If not, we can choose an arbitrary color.

Since in any point of time, at most two professors are asleep, each starting point can be included in at most one other interval, we can color all the intervals with this method, and only intervals of the opposite color intersect. Intervals of the same color are pairwise disjoint - if two intervals intersected, then the starting point of the second one would be in the first interval. But we colored them in a way that this cannot occur.

See the following picture:



Let p be the number of red segments and k is the number of blue segments. Let us call the intervals between blue segments "blue gaps" and, so there are $k-1$ blue gaps in total. One red interval intersects one more blue interval than the number of blue gaps that it fills completely. Since the red intervals are pairwise disjoint, only one red interval can fill a blue gap completely.

Thus the number of intersecting (red interval, blue interval) pairs is at most $p+k-1$, because for each red interval, the number of (red interval, blue interval) pairs it can participate in is at most *1+the number of the blue gaps it fills.*

Two people share a moment where they sleep at the same time, if and onyl if their intervals intersect. Therefore at most $p+k-1$ professor-pairs can sleep at the same time.

There are 9 professors and each pair of them has a moment when they both sleep. This means $9 \cdot 8/2 = 36$ pairs. (Pick the first professor in 9 possible ways, the second in 8 possible ways, and the order does not matter.) Therefore $36 \leq p+k-1$. But all 9 professors sleep at most 4 times, so there are at most $4 \cdot 9 = 36$ sleep intervals. Thus $p+k \leq 36$. This means $36 \leq p+k-1 \leq 36-1 = 35$, which is a contradiction. So we proved that there was a moment when at least three professors were asleep.

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2.2.3 Category E

1. We know that it took 42 minutes for the car to travel $XY - YX$ miles and another 18 minutes to travel $X0Y - YX$ miles.

$XY = 10 \cdot X + Y$, $YX = 10 \cdot Y + X$ and $X0Y = 100 \cdot X + Y$ hence $XY - YX = (10 \cdot X + Y) - (10 \cdot Y + X) = 9(Y - X)$ and $(100 \cdot X + Y) - (10 \cdot Y + X) = 9(11X - Y)$ have the same ratio as 42 and 18 which is 7:3. Dividing both sides by 9 we get that $Y - X$ and $11X - Y$ have a ratio of 7:3. X and Y are digits thus $Y - X$ cannot be larger than 9 and $11X - Y \geq 11 - 9 = 2 > 0$ so $11X - Y$ is positive. Hence the only possible values that these expressions can have are 7 and 3.

$$Y - X = 7$$
$$11X - Y = 3$$

By adding the two equalities we obtain $10X = 10$ so $X = 1$ and $Y = 8$.

Therefore the car travelled $81 - 18 = 63$ miles in 42 minutes so the speed of the car is 90 miles per hour.

Second solution: We will now show a shorter solution. The distance covered between 12pm and 13pm is $X0Y - XY = X \cdot 90$ miles. Since $X = 1$, this means that Albrecht is travelling at 90 miles per h1our.

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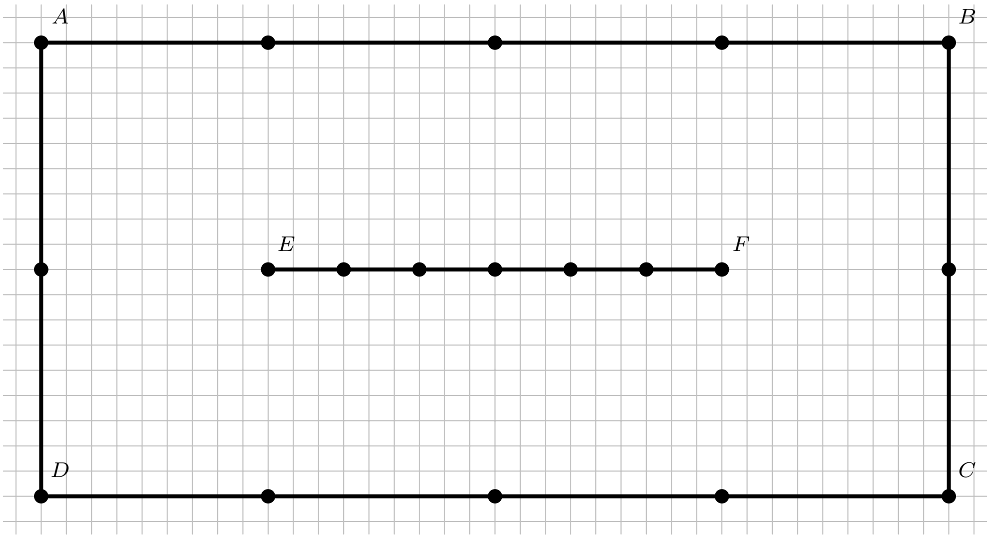
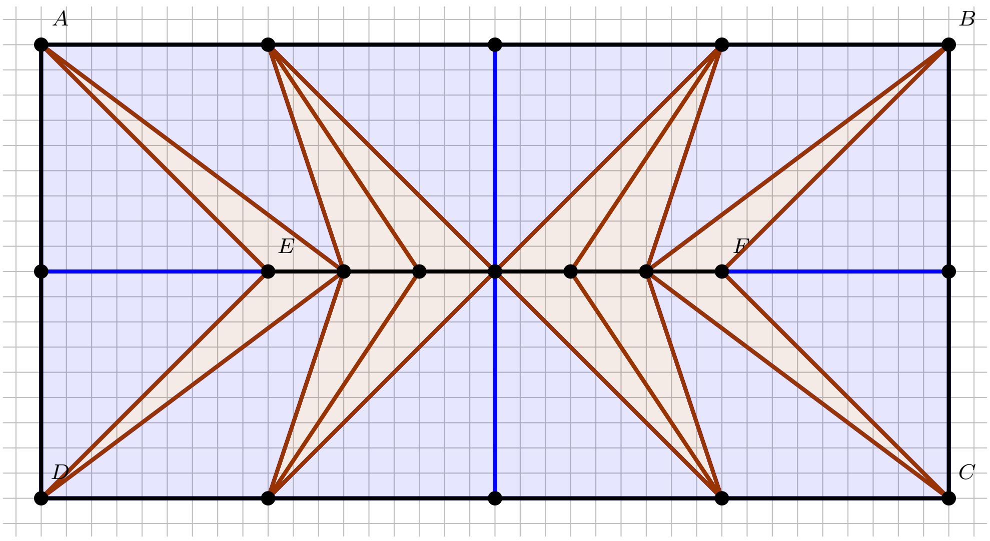
2. First we show a solution for part **c)** , from which we can also get solutions to parts **a)** and **b)** .

Let E be a point which is at a distance of 9 cm from sides AB, CD and AD. Similarly let F be at a distance of 9 cm from sides AB, CD and BC. Also let us divide the perimeter into 12 equal parts and segment EF into 6 equal parts as shown in the picture:



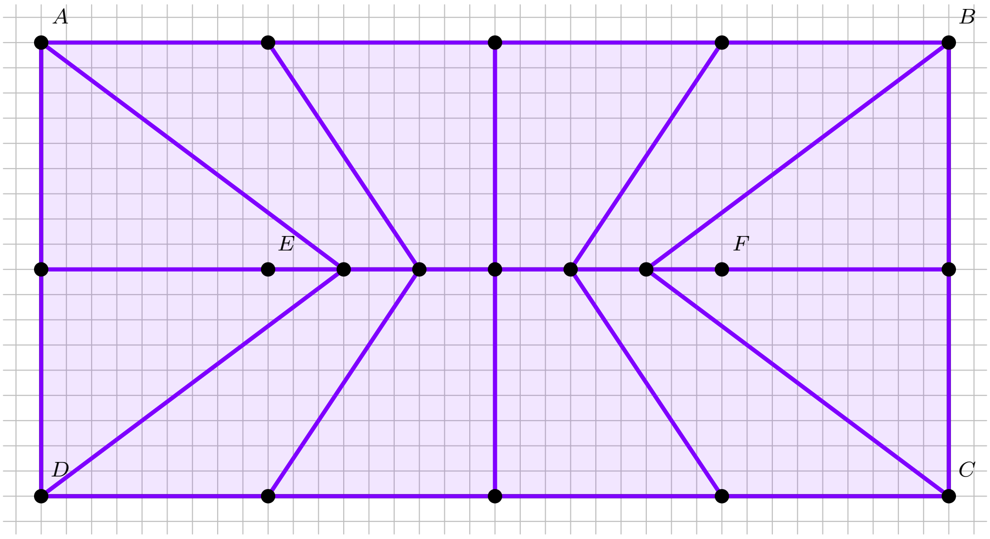
Now let's draw triangles in the following way:



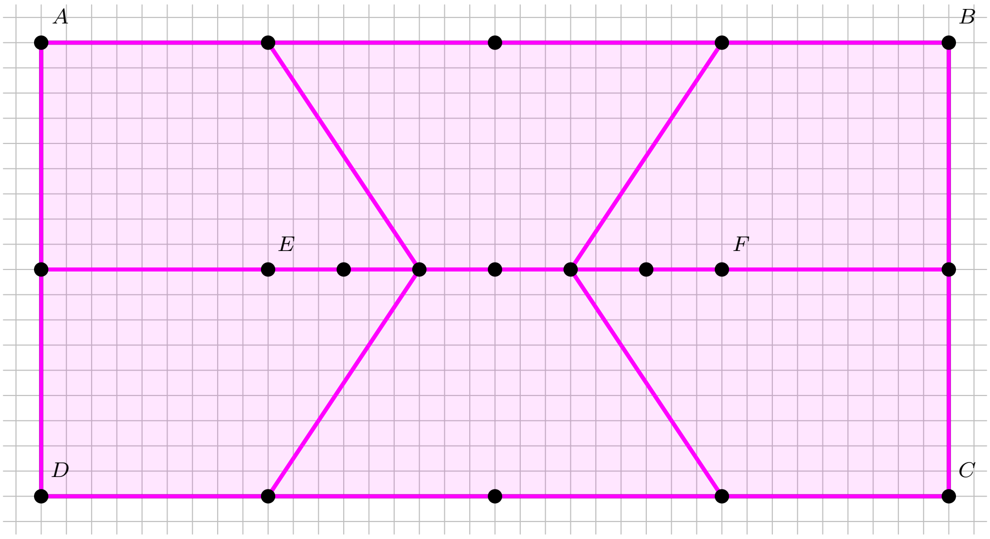
All blue triangles have a side of length 9 cm on the perimeter of the cake. And their corresponding altitude is 9 cm too, since the segment EF was defined in this way. So all blue triangles have equal area.

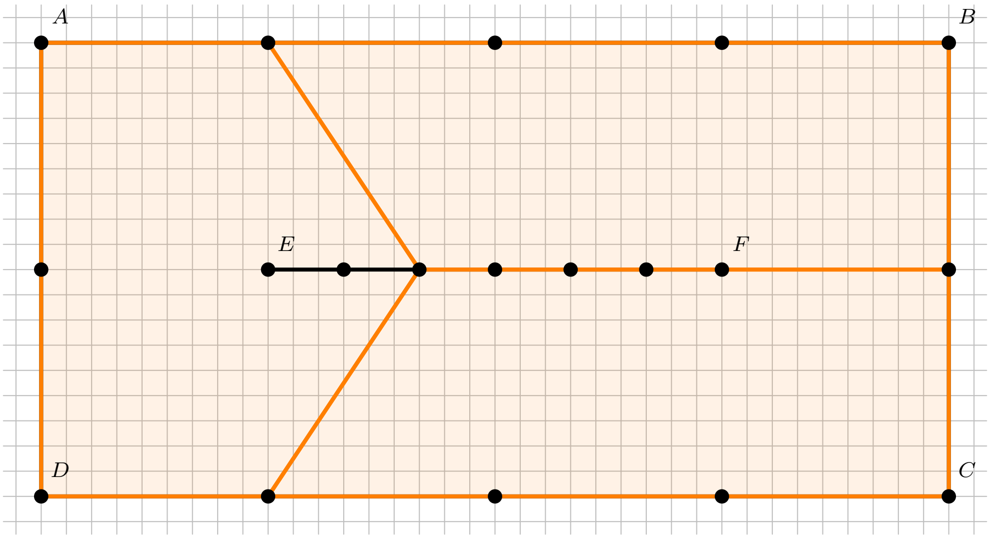
As we have divided segment EF into equal parts, there is a side length that appears in each of the red triangles. Also each red triangle has an altitude of 9 cm corresponding to it, so all red triangles have equal area too.

So if everybody gets a blue slice and a red one, then all will get equal shares of both the area and the chocolatey perimeter:

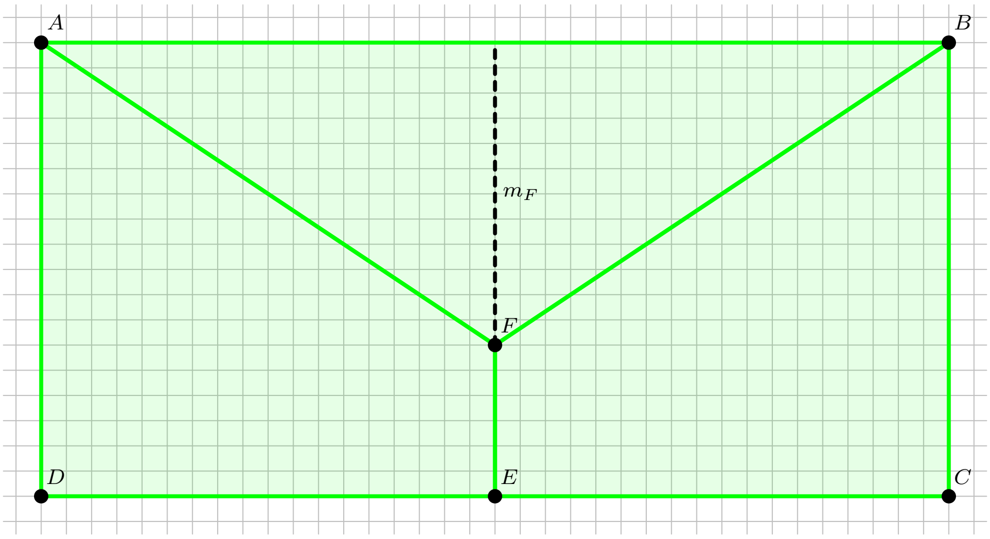


If everyone takes 2 blue and 2 red parts, then this gives a solution for part **b)** , and if everyone takes 4 blue and 4 red, we get a solution for **a)** :



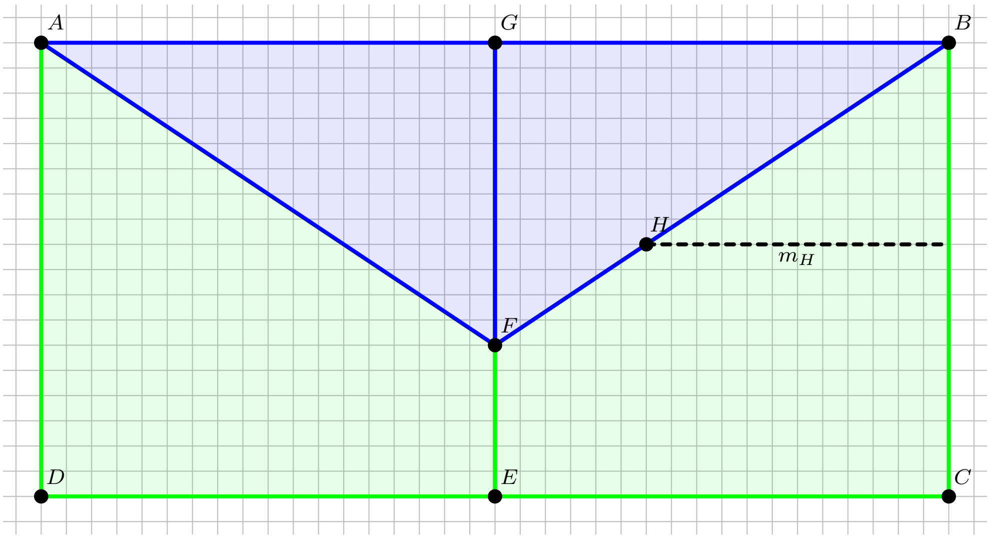


Second solution: **a)** The perimeter is of length $2(18 + 36) = 108$ cm, so everybody has to get a third of this, which is 36 cm. This is exactly the length of the longer side. Take the two endpoints of one of the longer sides (A, B) and the midpoint of the opposite side (E). These divide the perimeter into 3 equal parts. Let's find a point F inside the rectangle, which when connected to these three points, divides the rectangle's area into three equal parts as well:



So that the areas on the left and right are equal, we look for such a point F on the perpendicular bisector of side AB. As the area of triangle ABF has to be equal to the third of the cake's area, it should be $18 \cdot 36/3 = 216$ cm^2, and we also know the length of AB, so we can determine the corresponding altitude: $m_F = \frac{2T_{ABF}}{AB} = \frac{2 \cdot 216}{36} = 12$ cm, which means $EF = 18 - 12 = 6$ cm.

b) From here, we can halve all three parts properly. For triangle ABF this is easy to do: segment FG will divide both the area and the chocolate part of triangle ABF:

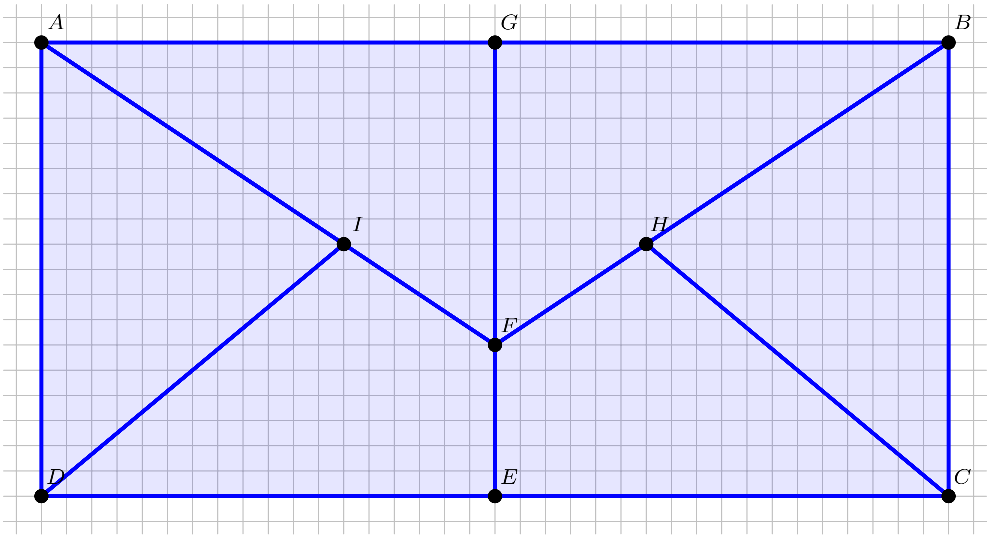


To halve quadrilateral $BCEF$, we need to cut it using a line from point C (so that the chocolate part is cut into two equal halves of length 18 cm each), and the remaining question is where we should take the point H so that CH cuts the area of $BCEF$ into half.

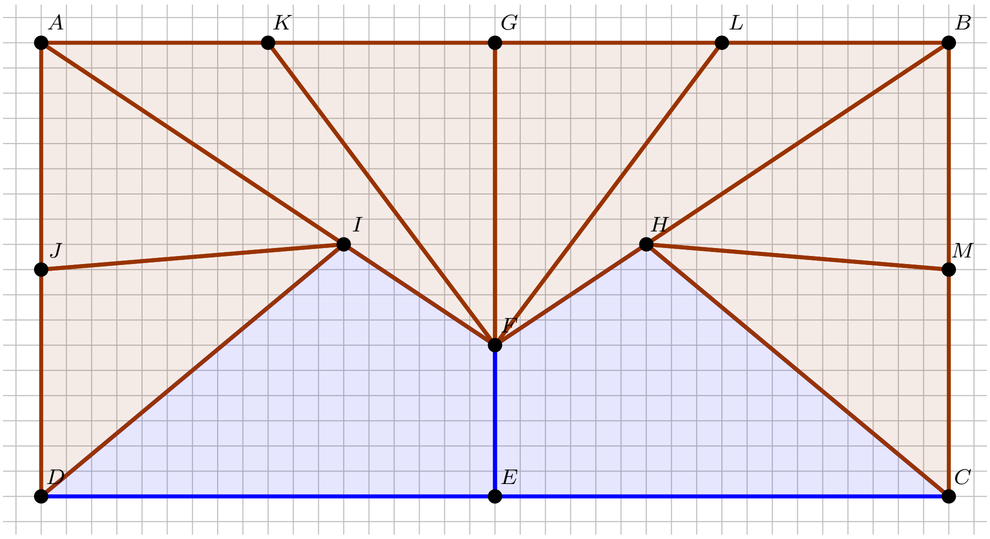
This is again easy to determine, since the area of HBC needs to be one-sixth of the rectangle (which is 108 cm^2), and we know the base BC (18 cm), so the corresponding altitude should be $m_H = \frac{2T_{HBC}}{BC} = \frac{2\cdot 108}{18} = 12$ cm.

Actually this means that H will be the trisection point of segment BF lying closer to F, and also it is at a distance of $\frac{2}{3} \cdot GF = 8$ cm.

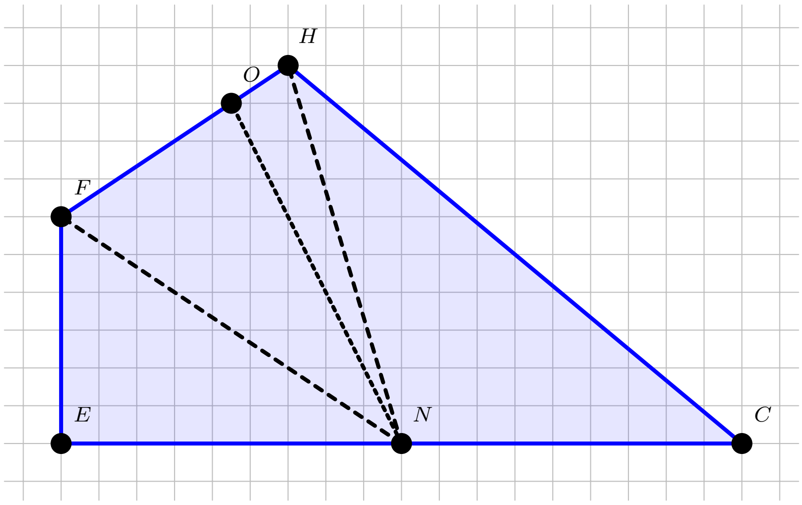
Symmetrically we can halve the quadrilateral $ADEF$ correctly using segment DI:



c) Again we will halve the parts obtained in part **b)**. It is easy to do this correctly for triangles DIA, AFG, GFB and BHC:. we just take their medians IJ, FK, FL and HM, which will halve the area of each triangle and also the opposite side (which is the chocolate part):



It remains to care about quadrilateral $CEFH$. If we can halve this correctly, then we can also do so for the symmetric $DEFI$. Let N be the midpoint of EC, and let's find the point O for which NO halves the area:



The areas of triangles NEF and NHC can be calculated, and we will see that both are less than 54 cm^2, which means that point O has to lie somewhere on segment FH - as we can see in the figure.

$T_{NEF} = \frac{NE \cdot EF}{2} = \frac{9 \cdot 6}{2} = 27$ cm^2.

In order to calculate the area of NHC, let's remind ourselves that H was at a distance of 8 cm from AB, which means that it is at a distance of 10 cm from EC. So $T_{NHC} = \frac{NC \cdot 10}{2} = \frac{9 \cdot 10}{2} = 45$ cm^2.

So we have to choose point O in such a way that $T_{OHN} = 54 - 45 = 9$ cm^2 and $T_{OHF} = 54 - 27 = 27$ cm^2. The ratio of the two areas is $1 : 3$, so if we take O to be the point on HF for which $HO : OF = 1 : 3$, then the two triangles will have equal altitudes from N and the bases are in a ratio of $1 : 3$ so the areas will have the right ratio.

Note: In fact we don't need to exactly determine where point O lies, it is just enough to prove that *there exists* an appropriate point.

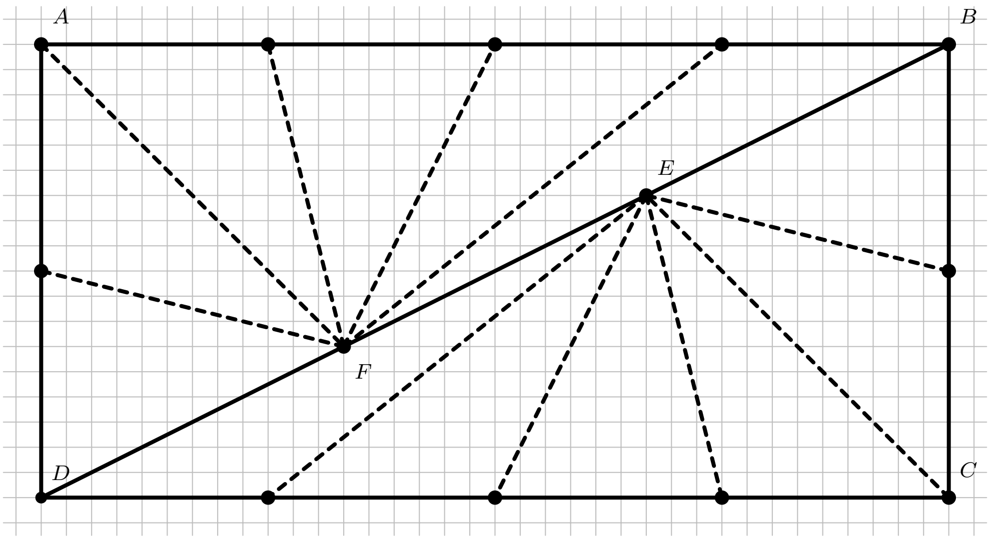
Let's imagine that we place our knife on ray NE, and we start rotating it in a clockwise direction so that one of its ends stays at N. (So our knife would traverse the segments in the figure in the order NE, NF, NO, NH, NC.)

So initially the left area is 0 and the right area is the whole 108 cm^2, and as we rotate the knife, the left area grows and the right area decreases gradually. At the end, the left area will be 108 cm^2, and the right area will be 0. As the area changed continuously, there had to be a position of the knife that cut the area exactly into half.

This solution uses the idea of continuity, but as we have seen, the problem can be solved without it.

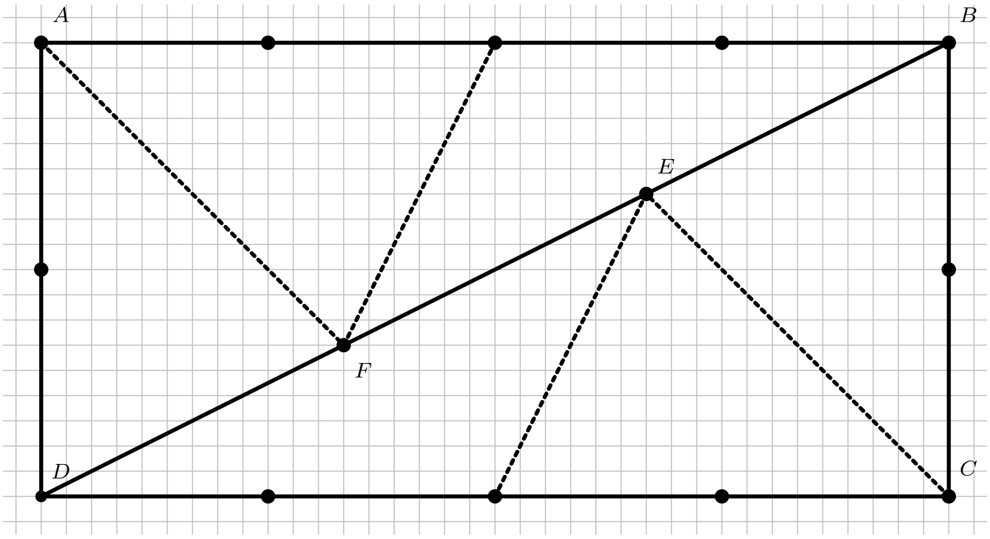
Third solution: Again we will give a correct slicing for part **c)** . Let's take points E and F on the diagonal so that they have an equal distance from sides BC and CD, and sides DA and AB respectively. In other words, E is on the bisector of angle BCD and F is on the bisector of angle DAB.

Again divide the perimeter into 12 equal parts, and connect the dividing points above diagonal BD with F, and those below it with E:



The sides of the triangles lying on the perimeter of the rectangle are all equal (so everybody gets equal chocolate), and the altitudes belonging to them will also be equal, since this is how we have chosen points E and F. So this slicing is correct.

If we only connect every other dividing point with F and E, then we get a new solution for part **b)**:



Unfortunately we can't get a new solution for part **a)** by combining parts in this construction because one of the shapes will always be concave.

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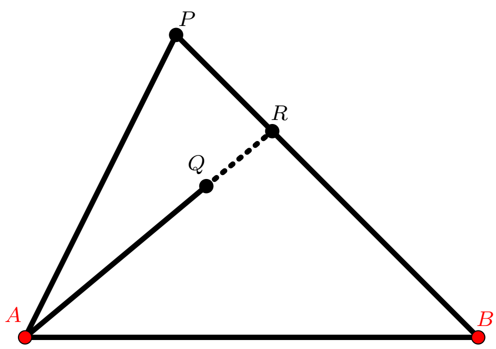
3. Take an arbitrary point P of the museum. We are going to prove that every point of the interior and boundary of the triangle ABP is in the museum.

If X, a point of the museum, can be seen from point A (or B) then the whole section AX (or BX) is in the museum since if there was a point Y on AX that is not in the museum then X could not be seen from point X.

By this argument the sections AB, AP and BP (so the whole boundary of the ABP triangle) is in the museum.

Now we prove that every point Q inside of ABP is in the museum. Let R be the intersection of the lines AQ and BP. R is in the museum (since it is on the boundary of ABP) so R can be seen from A thus the whole AR section is in the museum thus Q is in the museum.



Therefore every point in the interior or on the boundary of ABP is a point of the museum. Hence P can be seen from every point C on the section AB since CP is in the museum.

This is true for every P in the museum so the whole museum can be seen from every point of the AB section.

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4. a) We will show that there are no solutions. Observe that if p^k $(p, k \in Z^+)$ is a prime power that divides all three numbers, then if we divide the numbers by p^k, the equality still remains true, as all terms get divided by p^k. If we do this with all prime powers appearing in the prime factorization of the GCD $d = (a, b, c)$, then in the end we will have $(a, b, c) = 1$ and the equation will still hold. So in the remaining part of the solution we will assume $(a, b, c) = 1$.

Take a prime p that divides at least one of a, b and c. The equation is symmetric in the three variables so we can assume $p \mid a$. Then p divides $[a, b]$, $[a, c]$ and $[a, b, c]$ too, so by the equation it also divides $[b, c]$. So it divides either b or c (but not both, as otherwise $(a, b, c) = 1$ would not hold). So every prime divisor of a, b or c divides exactly two of the numbers.

Moreover, it is also true that if p divides both a and b, then it actually appears in the prime factorizations of a and b to the same power. We will prove this by contradiction. Suppose that p appears to the k-th power in a and to the l-th power in b, and WLOG suppose $k > l$. Then let us compute the power of p appearing in each of the terms of the equation: $[a, b]$, $[a, c]$ and $[a, b, c]$ all have p^k and $[b, c]$ has p^l (using the fact that p does not divide c). Thus p^k divides both sides of the equation, so $l \geq k$ which is a contradiction. Therefore any prime that appears in the factorizations of any of the numbers appears in the factorizations of exactly two of the numbers, and it appears in both to the same power.

If we group all primes appearing in a, b or c depending on which two of the numbers they divide, we can write the numbers in the following form: $a = de$, $b = ef$ and $c = fd$, where d, e and f are pairwise coprime. (For example e is the product of prime powers dividing both a and b.) Then $[a, b] + [a, c] + [b, c] = def + def + def = 3def$, and $[a, b, c] = def$. But this is impossible since d, e and f are positive so there are no solutions.

b) Again observe that if a prime divides all three numbers then we can again divide all three numbers by it, and the equation remains true. So for the moment, let us only consider the cases where we have $d = (a, b, c) = 1$.

Take a prime p that divides at least two numbers out of the three. By symmetry we can assume that p divides a and b. Then it will divide all of $[a, b]$, $[a, c]$, $[b, c]$ and $[a, b, c]$, so by the equation it must also divide (a, b, c). However $(a, b, c) = 1$ so this is impossible. So any prime can only divide at most one of the three numbers, so the three numbers are pairwise coprime. So the equation will be the following: $ab + ac + bc = abc + 1$.

By symmetry, we can assume that a is the smallest of the three numbers (it is not necessarily distinct from the other two, as there can be multiple smallest numbers). Consider three cases according to the possible values of a:

- If $a = 1$ then $b + c + bc = bc + 1$, so $b + c = 1$ which is impossible.
- If $a \geq 3$ then $ab + ac + bc \leq \frac{abc}{3} + \frac{abc}{3} + \frac{abc}{3} < abc + 1$, which is also impossible.
- Finally if $a = 2$ then $2b + 2c + bc = 2bc + 1$, which after reordering gives $bc - 2b - 2c + 1 = 0$, so $(b - 2)(c - 2) = 3$. As b and c are positive integers, the only possible solution is $a = 2$, $b = 3$ and $c = 5$.

So up to reordering of the variables, the only solution with $(a, b, c) = 1$ is $a = 2$, $b = 3$ and $c = 5$, and we can check that this is indeed a solution.

Now let us turn to the cases where $d = (a, b, c) > 1$. As dividing everything by d gives a good solution where the GCD of the three numbers is 1, the original numbers must be (up to reordering) of the form $a = 2d$, $b = 3d$ and $c = 5d$ for some positive integer d. These are all correct solutions, as $[a, b] + [a, c] + [b, c] = 6d + 10d + 15d = 31d$ and $[a, b, c] + (a, b, c) = 30d + d = 31d$ too.

Second solution: a) We know that $[a, b] \mid [a, b, c]$ and similarly $[b, c]$ and $[c, a]$ also divide $[a, b, c]$. So there exist positive integers p, q and r such that $[a, b, c] = p[a, b] = q[b, c] = r[c, a]$. Divide the equation in the problem by $[a, b, c]$ to get

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

The positive integer solutions of this equation (up to reordering) are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

Here is a possible way to derive this: suppose WLOG that $p \le q \le r$. If $p = 1$, the LHS would be greater than 1 so this is impossible. If $p > 3$ then the LHS is at most $\frac{3}{4}$ so this is also impossible. If $p = 3$ then the LHS can only be 1 if $p = q = r = 3$, which is indeed a solution. Finally if $p = 2$ then we have $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. Reordering this, we get $2r + 2q = qr$ so $qr - 2r - 2q + 4 = 4$, and this factorizes as $(q - 2)(r - 2) = 4$. This gives two possible solutions for q and r: $(q, r) = (3, 6)$ and $(q, r) = (4, 4)$.

Now note that if p, q and r are not pairwise coprime, then we can reach a contradiction: assume that a prime d divides both p and q. Then $d \mid \frac{[a,b,c]}{[a,b]}$ means that d appears in c to a higher power than in a, and $d \mid \frac{[a,b,c]}{[b,c]}$ means that d appears in a to a higher power than in c. And this is impossible.

So p, q and r must be pairwise coprime, but none of the solutions $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$ satisfy this. So there is no solution to the original equation.

b) Let us start similarly to part **a)**: as $(a, b, c) \mid [a, b, c]$, let s be the positive integer such that $[a, b, c] = s(a, b, c)$. Also observe that (a, b, c) divides the pairwise LCM's of the three numbers (e.g. $(a, b, c) \mid a \mid [a, b]$), so if we define p, q and r in the same way as above, then p, q and r wil'l all be divisors of s. Rearranging the equation and dividing by $[a, b, c]$ now gives

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{s} = 1$$

As the equation is still symmetric in a, b and c, we can assume that $p \le q \le r$.

If $p = 1$ then as $r \mid s$, we have $r \le s$ so $\frac{1}{r} - \frac{1}{s} \ge 0$, so since $\frac{1}{q} > 0$, the LHS is greater than 1, a contradiction.

If $p \ge 3$, then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{s} < 3 \cdot \frac{1}{3} + 0$, so the LHS is smaller than 1.

So we must have $p = 2$. This yields

$$\frac{1}{q} + \frac{1}{r} - \frac{1}{s} = \frac{1}{2}$$

Suppose $q = 2$. Then we get $\frac{1}{r} - \frac{1}{s} = 0$, so $r = s$. This means $(a, b, c) = [c, a]$, so as $(a, b, c) \le a \le [c, a]$, we have $(a, b, c) = a = [c, a]$. This gives $a = c$. Then it is easy to see that the original equation reduces to $[a, b] + a = (a, b)$, which is impossible as $[a, b] + a > a \ge (a, b)$.

Suppose $q \ge 4$. Then $\frac{1}{q} + \frac{1}{r} - \frac{1}{s} < 2 \cdot \frac{1}{4} + 0 = \frac{1}{2}$, again a contradiction.

So we must have $q = 3$. Then we get

$$\frac{1}{r} - \frac{1}{s} = \frac{1}{6}$$

Here $r \geq q = 3$, and also $r \leq 5$, since $r \geq 6$ would mean that $\frac{1}{r} - \frac{1}{s} < \frac{1}{6}$. So $r \in \{3, 4, 5\}$, and in each case s can be determined. We get the following three possibilities for the quadruple p, q, r, s:

$$(p; q; r; s) \in (2; 3; 3; 6), (2; 3; 4; 12), (2; 3; 5; 30)$$

The observation in part **a)** that p, q and r are pairwise coprime is true in this case as well, so this only leaves the case $(p; q; r; s) = (2; 3; 5; 30)$. This means

$$[a, b, c] = 2[a, b] = 3[b, c] = 5[c, a] = 30(a, b, c)$$

So $[a, b, c] = 2[a, b]$ means that the exponent of the prime 2 in c is higher than in both a and b. So in $[a, b, c]$, 2 appears to a higher power than in (a, b, c). Similarly 3 and 5 also appear to a higher power in $[a, b, c]$ than in (a, b, c). So $\frac{[a,b,c]}{(a,b,c)} \geq 2 \cdot 3 \cdot 5 = 30$.

Since we have equality here, the only possible case is that the powers of 2 in a and b are the same, and the power of 2 in c is one larger than that. The analogue is true for powers of 3 and 5 too (but of course they have to be the largest in b and a respectively). Also every prime appearing in $[a, b, c]$ other than 2, 3 or 5 has to appear in (a, b, c) to an equal power as in $[a, b, c]$. So these primes appear in all of a, b and c to the same power.

To summarize, we have $a = 2^x 3^y 5^{z+1} k$, $b = 2^x 3^{y+1} 5^z k$ and $c = 2^{x+1} 3^y 5^z k$ for some natural numbers $x, y, z \geq 0$ and a positive integer k that is coprime to 2, 3 and 5.

Taking out the common term $d = 2^x 3^y 5^z k$, this means that $a = 5d$, $b = 3d$ and $c = 2d$ where d can be any positive integer.

So the solution is this up to reordering: $(a; b; c) = (5d; 3d; 2d)$ where d is an arbitrary positive integer. In the original equation we can check that these are indeed solutions: $15d + 10d + 6d = 30d + d$.

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5. If there is one main villain, who is the enemy of everyone else, and the others are all friends with each other, then there are 20 duels in total, and to each of them the winner has taken all of their bullets with them, so in this case the sheriff has $20 \cdot 240 = 4800$ bullets by the end.

We claim that it is not possible that the sheriff has a larger number of bullets by himself after the duels. Consider the bandit who brings the least amout of bullets to his duels, if there are more bandits with this property then we consider the bandit who has the narrowest hat among them. This bandit looses each of his duels, so none of his 240 bullets end up by the sheriff. The other bandits have $20 \cdot 240 = 4800$ bullets in total, so the sheriff cannot have more than 4800 bullets by the end.

Note: *You can read two more solutions for this problem in the E+ answer booklet (at problem 2).*

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2.2.4 Category E$^+$

1. For the solution, see Category E Problem 4.

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2. If there is one main villain, who is the enemy of everyone else, and the others are all friends with each other, then there are 20 duels in total, and to each of them the winner has taken all of their bullets with them, so in this case the sheriff has $20 \cdot 240 = 4800$ bullets by the end.

We claim that it is not possible that the sheriff has a larger number of bullets by himself after the duels. Consider the bandit who brings the least amout of bullets to his duels, if there are more bandits with this property then we consider the bandit who has the narrowest hat among them. This bandit looses each of his duels, so none of his 240 bullets end up by the sheriff. The other bandits have $20 \cdot 240 = 4800$ bullets in total, so the sheriff cannot have more than 4800 bullets by the end.

Second solution: We get the lower bound $20 \cdot 240$ in the same way as in the first solution.

Let us rewrite this exercise as a graph theory problem. We define a graph G, in which the vertices correspond to the bandits, and there is an edge between two bandits if and only if they are enemies. The bandit V brings at most $\frac{240}{d_V}$ bullets to a duel, where d_V denotes the number of the enemies of V. Translating this to a graph theoretic point of view, d_V is the degree of the vertex corresponding to bandit V in G. On the uv edge the sheriff can take at most $\max\left(\frac{240}{d_v}, \frac{240}{d_u}\right)$ bullets, therefore we will prove the following:

$$\sum \max\left(\frac{240}{d_v}, \frac{240}{d_u}\right) \leq 20 \cdot 240$$

where we take the sum over all edges in G, so the LHS is the number of bullets taken.

Let us divide the above statement by 240 and generalise the statement we want to prove for an arbitrary graph $G = (V, E)$:

$$\sum_{uv \in E} \max\left(\frac{1}{d_v}, \frac{1}{d_u}\right) \leq |V| - 1$$

We are going to prove this generalised statement. It is clear that it is enough to prove the statement for connected graphs.

Let Δ denote the maximum degree in G. It is clear, that $\Delta \leq |E|$, and $\max\left(\frac{1}{d_v}, \frac{1}{d_u}\right) \leq \frac{1}{d_u} + \frac{1}{d_v} - \frac{1}{\Delta}$, thus

$$\sum_{uv \in E} \max\left(\frac{1}{d_v}, \frac{1}{d_u}\right) \leq \sum_{uv \in E} \left(\frac{1}{d_u} + \frac{1}{d_v} - \frac{1}{\Delta}\right) = |V| - \frac{|E|}{\Delta} \leq |V| - 1$$

The equality in the middle is satisfied, because every vertex u is by definition the endpoint of exactly d_u edges, hence it will appear d_u times in the sum.

Our estimate also shows that equality is only possible in the case when $|E| = \Delta$ holds, i.e. only in the case of a star graph (one of the vertices is connected with every other vertex by an edge, and the graph has no more edges), and it is clear that in this case the equality holds.

Third solution, sketch: Here we provide a different proof for the more general graph theoretic inequality from the previous solution.

Let u be an arbitrary edge of the graph G, d_u its degree and e an edge which is incident to u. Let us consider a random spanning tree of G, choosing each of the spanning trees with equal probability. The probability that e is contained in this randomly chosen spanning tree is at least $\frac{1}{d_u}$. This is a nice statement, the proof of which we leave as an exercise to the reader. The statement implies that every uv edge is in the randomly chosen spanning tree with a probability of at least $\max\left(\frac{1}{d_v}, \frac{1}{d_u}\right)$. Let $\mathbb{I}_{uv}$ denote the indicator function of the event that uv is contained in the spanning tree, i.e. it takes the value 1 if uv is contained in the spanning tree, and 0 otherwise. The expected value of $\mathbb{I}_{uv}$ is equal to the probability that the edge is contained in the spanning tree. Because of linearity of expectation:

$$\sum_{uv\in E} \max\left(\frac{1}{d_v}, \frac{1}{d_u}\right) \leq \sum_{uv\in E} \mathbb{E}(\mathbb{I}_{uv}) = \mathbb{E}\left(\sum_{uv\in E} \mathbb{I}_{uv}\right) = |V| - 1,$$

where the last equality is satisfied because the expected value of the sum of the indicator functions is the expected number of edges in the spanning tree.

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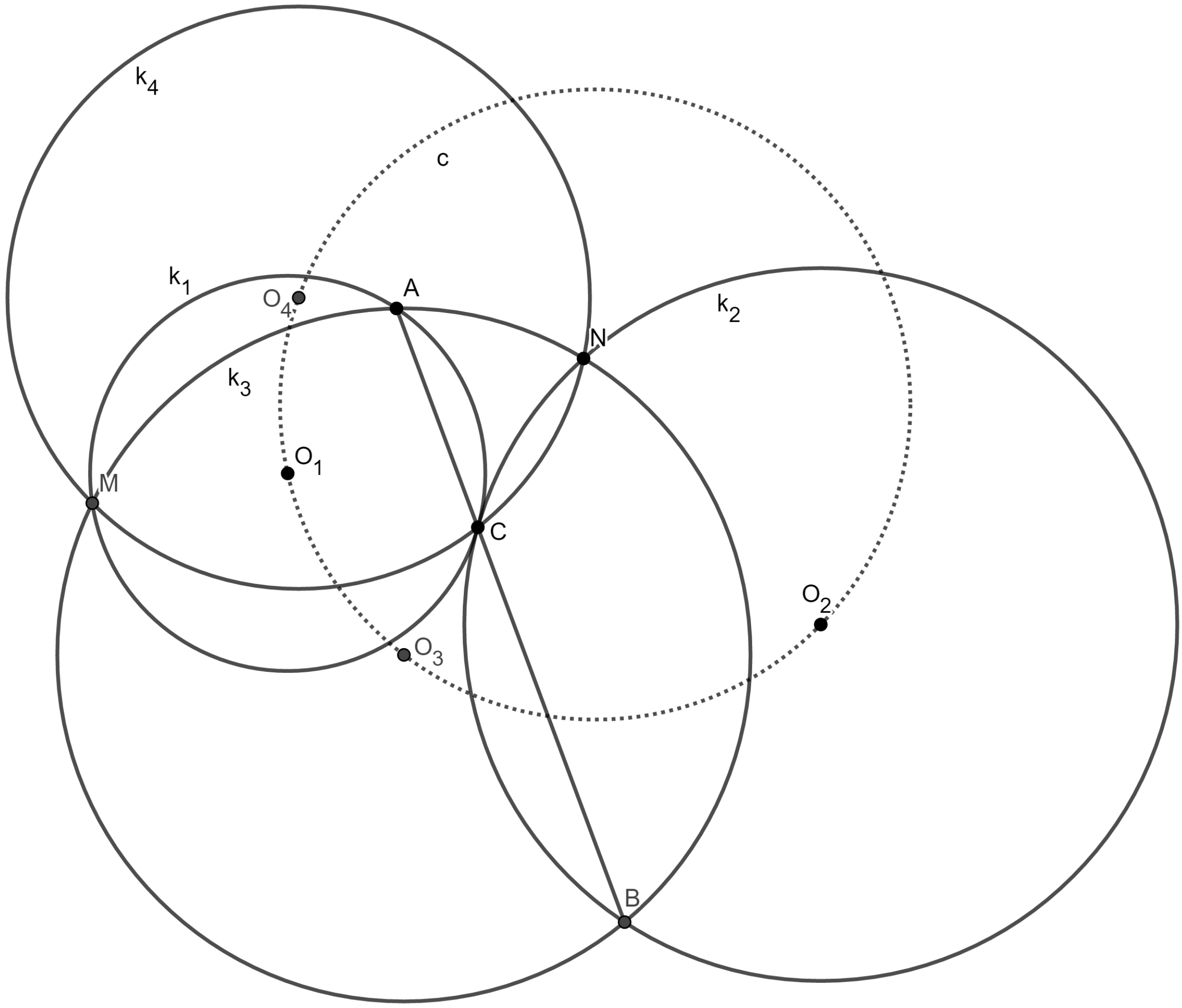
3. The main observation is that the line connecting the intersection points of two circles is perpendicular to the line through their centers. We are going to use angle chasing (working with directed angles modulo π) to solve the problem.
Let O_1, O_2, O_3, O_4 be the centers of the circles k_1, k_2, k_3, k_4, respectively. $O_1O_4 \perp MC$, since the circles k_1 and k_4 intersect at points M and C. Similarly, $O_1O_3 \perp AM$, hence $\sphericalangle O_3O_1O_4 = \sphericalangle AMC$. Analogously, $O_2O_3 \perp BN$ and $O_4O_2 \perp CN$, so $\sphericalangle O_3O_2O_4 = \sphericalangle BNC$. k_1 is tangent to k_2, hence $\sphericalangle AMC = \sphericalangle BNC$. (This is because if we let X be a point on the common internal tangent of k_1 and k_2, then $\sphericalangle AMC = \sphericalangle ACX = \sphericalangle BCX = \sphericalangle BNC$ using the inscribed angle theorem.) Comparing this against our previous results, have $\sphericalangle O_3O_1O_4 = \sphericalangle O_3O_2O_4$, whence the points O_1, O_2, O_3, O_4 are concyclic, as required.



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4. Let k be the degree of q. Then $p(x)q(x)$ has degree $n + k$.

If $k > n$, then $p(x^2) + q(x^2)$ has degree $2k \neq n + k$. If $k < n$, then $p(x^2) + q(x^2)$ has degree $2n \neq n + k$.

The only possibility is therefore that the degree of q is also n.

Let a be the leading coefficient of p and b the leading coefficient of q. Then $p(x)q(x) = p(x^2) + q(x^2)$ has leading coefficient $ab = a + b$.

So $ab - a - b = 0$, i.e. $(a - 1)(b - 1) = 1$. Since a and b are non-zero integers, this is only possible if $a = b = 2$.

If 0 is a root of multiplicity r in $p(x)$, and a root of multiplicity s in $q(x)$, then it is a root of multiplicity $r + s$ in $p(x)q(x)$.

If $r < s$, then $p(x^2) + q(x^2)$ is divisible by x^{2r}, but not divisible by any greater power of x. $2r \neq r + s$.

Similarly, if $s < r$, then $p(x^2) + q(x^2)$ is divisible by x^{2s}, but not divisible by any higher power of x. $2s \neq r + s$.

Therefore equality is only possible if $r = s$.

Let $p(x) = x^r P(x)$ and $q(x) = x^r Q(x)$, where 0 is a root of neither P nor Q.

Then $x^{2r}P(x)Q(x) = x^{2r}(P(x^2) + Q(x^2))$, and so $P(x)Q(x) = P(x^2) + Q(x^2)$.

It is still true that P and Q have both degree m, and both have leading coefficient 2.

$p(x)$ has only non-negative real roots, hence the roots of $P(x)$ are positive real numbers.

Since $P(x^2)+Q(x^2)$ is an even function, $P(x)Q(x)$ is also even, thus if x is a root, then the same is true for $-x$.

This implies that if we multiply the positive roots of $P(x)$ by -1 we get roots of $P(x)Q(x)$, and because they are not positive, they cannot be roots of $P(x)$, only of $Q(x)$.

So if $P(x) = 2(x - r_1)(x - r_2)\dots(x - r_m)$, then $Q(x) = 2(x + r_1)(x + r_2)\dots(x + r_m)$.

Therefore $P(x)Q(x) = 4(x^2 - r_1^2)(x^2 - r_2^2)\dots(x^2 - r_m^2) = 2(x^2 - r_1)(x^2 - r_2)\dots(x^2 - r_m) + 2(x^2 + r_1)(x^2 + r_2)\dots(x^2 + r_m)$

Denoting x^2 by y we get that

$4(y - r_1^2)(y - r_2^2)\dots(y - r_m^2) = 2(y - r_1)(y - r_2)\dots(y - r_m) + 2(y - r_1)(y - r_2)\dots(y - r_m)$.

Let us assume that P is not a constant polynomial.

The constant term of P is a non-zero integer because 0 is by definition not a root of P. So the absolute value of the constant term is at least 1.

The constant term is the product of the roots with some sign, so either every root r_i has absolute value 1, or there is a root among them with absolute value greater than 1.

We know that the roots r_i are all positive, so they are either all equal to 1, or the biggest root among them (which we will denote by r_m) is greater than 1.

If every $r_i = 1$, then $4(y - 1)^m = 2(y - 1)^m + 2(y + 1)^m$, so $(y - 1)^m = (y + 1)^m$, which is impossible.

If r_m is the biggest root and $r_m > 1$, then in case of $y = r_m^2$, $4(y-r_1^2)(y-r_2^2)\dots(y-r_m^2) = 0$.

However $r_m^2 > r_m \geq r_i$ for every i, hence $2(y-r_1)(y-r_2)\dots(y-r_m)$ is a product of positive terms for $y = r_m^2$, and naturally $2(y + r_1)(y + r_2)\dots(y + r_m)$ is also a product of positive numbers for $y = r_m^2$.

Hence $2(y - r_1)(y - r_2)\dots(y - r_m) + 2(y + r_1)(y + r_2)\dots(y + r_m)$ is positive for $y = r_m^2$.

So $4(y-r_1^2)(y-r_2^2)\dots(y-r_m^2) \neq 2(y-r_1)(y-r_2)\dots(y-r_m)+2(y+r_1)(y+r_2)\dots(y+r_m)$.

The only remaining possibility is that P and Q are constant functions. We have already shown that the leading coefficient is 2, thus $P(x)$ and $Q(x)$ are both the constant polynomial 2.

Therefore the solution can only be of form $p(x) = 2x^n, q(x) = 2x^n$. These solutions all satisfy the contidions because $p(x)$ has n non-negative real roots and $p(x)q(x) = 4x^{2n} = p(x^2)+q(x^2)$.

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5. For even n: $\frac{n^2}{4}$, for odd n: $\frac{(n-1)(n+1)}{4}$, so in general $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil$.

Upper bound: Consider n given lines in the three-dimensional space, such that no 2 of them are parallel and no 3 of them are concurrent. Consider the graph G, whose vertices are the lines, and for every plane determined by two of the n lines we create exactly one edge between two of the lines, which determine this plane. This way the graph will have exactly as many edges as there are determined planes. We claim that G is triangle-free.

Let's use proof by contradiction. Assume G has a triangle in it, that is, there are 3 distinct lines e_1, e_2 and e_3 and 3 distinct planes s_1, s_2 and s_3 such that e_1 and e_2 determine s_3, e_1 and

e_3 determine s_2, e_2 and e_3 determine s_1. As there are no parallel lines, two lines determine a plane if and only if they intersect, thus e_1, e_2 and e_3 are pairwise intersecting. Since they are not concurrent, this is only possible if they form a triangle, but then they lie on one plane, so s_1, s_2 and s_3 can't be distinct planes which is a contradiction.

Thus G really is triangle-free, so according to Turán's theorem it can have at most $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil$ edges, that is, this is the maximal number of planes n lines can determine, and this is what we wanted to prove.

Lower bound: We have to give a construction with n lines, which determine $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil$ planes. We know that equality in Turán's theorem occurs when the n lines are divided into two almost equal groups, so we have to look for such a construction, where two lines intersect if and only if they are in a different group. The idea is to look for lines of the following form:
Let one of the groups have lines whose orthogonal projection onto the xy plane is parallel to the x axis, namely let e_k's projection's equation be $y = k$ where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Likewise, let the other group have lines f_k whose orthogonal projection to the xy plane has equations $x = k$ for all $1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$. After this, we want to set the slope of these lines with respect to the xy plane such that the e lines have pairwise different slopes, the f lines also have pairwise different slopes, and any two e_i and f_j should intersect. Luckily, we can achieve this:

For all $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ let e_k's equations be $y = k$ and $z = kx$. Likewise, for all $1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil$ let f_k's equations be $x = k$ and $z = ky$. Then for arbitrary $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil$ both e_i and f_j passes through the point (j, i, ij), as this point satisfies both lines' equations. It's clear that these intersection points are different for different pairs of lines, so there are no 3 lines which are concurrent. As we chose the slopes of the lines to be different there can't be any parallel lines either. Hence this really is a correct construction for n lines which determine $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil$ planes.

Remark: A construction can also be given using a hyperboloid of revolution.

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2.3 Final round – day 1

2.3.1 Category C

1. a) Eagleeye would like to make exchanges starting from 50 heads of cattle such that he ends up with at least 20 of each kind of animal. We will show a way to do this.

He can do the following changes.

1. $5C \to 3P + 5G$
2. $1C + 1G \to 2P$
3. $3P + 3G \to 2C$

First he executes the first exchange option 5 times. Then he will have 25 heads of cattle, 15 pigs and 25 goats. Then he uses the second option three times exchanging 3 heads of cattle and 3 goats to 6 pigs. Then he has $15 + 6 = 21$ pigs, $25 - 3 = 22$ goats and $50 - 25 - 3 = 22$ heads of cattle. Then he has at least 20 animals from each kind.

b) We show a sequence of changes that results in less than 15 animals. Here is an execution.

First he exchanges 50 heads of cattle to 30 pigs and 50 goats using the first option 10 times. Then he uses the third option 10 times to get 20 heads of cattle for 30 goats and 30 pigs. So now he has 20 heads of cattle and 20 goats. Then he uses the second option 10 times to change 10 goats and 10 heads of cattle to 20 pigs. So now he has 10 heads of cattle, 10 goats and 20 pigs.

Now he can use the third option three times resulting in 16 heads of cattle, 11 pigs and 1 goat. Using the first option 3 times it will result 1 head of cattle, 20 pigs and 16 goats.

Now using the third option 5 times he will have 11 heads of cattle, 5 pigs and 1 goats. Using the first option twice he will have 1 head of cattle, 11 pigs and 11 goats. Finally using the third option 3 times he will have 7 heads of cattle, 2 pigs and 2 goats. Thus he can leave the fair with $7 + 2 + 2 = 11 < 15$ animals.

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2. Let us note that if Ludmilla didn't erase the last digit of A, then the last digit of A and B are equal, therefore their sum is even. However 20210521 is odd, so Ludmilla erased the last digit of A.
Let x be the last digit of A. Then $A = 10 \cdot B + x$, and $A + B = 11 \cdot B + x = 20210521$.
$20210521 = 11 \cdot 1837320 + 1$ and since x is a digit, the only possibility is that $x = 1$ and $B = 1837320$. Then the first number ludmilla wrote is $A = 18373201$.
18373201 has 8 digits and by erasing the last digit we get 1837320, for which $18373201 + 1837320 = 20210521$ also holds, so the solution is correct.

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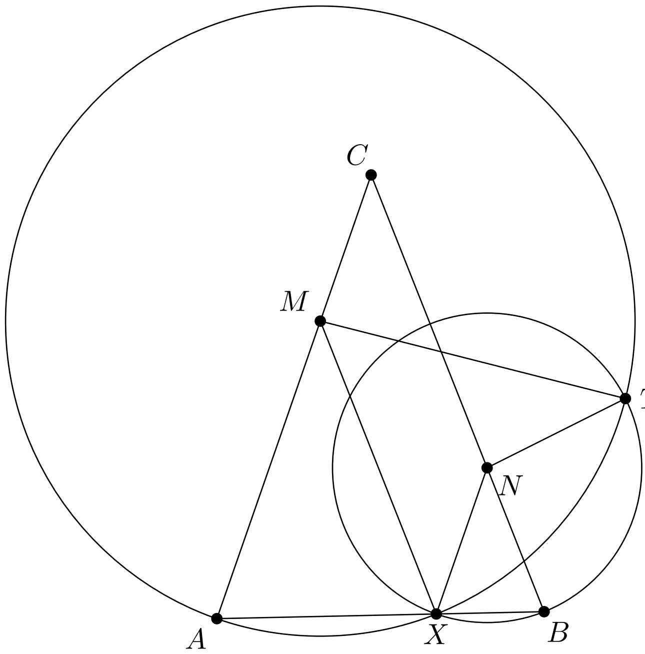
3. The other intersection point of the two circles beside M is X. The segments Mt and Mx are of the same length, since they are radiuses of the same circle. In the same way $|NX| = |NT|$. Since three of their sides are equal the triangles MNT and MNX are equal, so $MTN\angle = NXM\angle$. (Therefore the quadrilateral $MXNT$ is a deltoid, which is symmetric with respect to the line spanned by MN.)

We know that NX and AC are parallel and so are MX and BC so the quadrilateral $CMXN$ is a paralelogramm, implying $MCN\angle = NXM\angle$, which in turn equals the angle $MTN\angle$. Thus the size of angles $MCN\angle$ and $MTN\angle$ truly is equal.



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4. a) We show a sultion with 16 glasses. Let Grandma pour $\frac{1}{3}$ litres of apple juice in 8 of the glasses each, and $\frac{3-8/3}{8} = \frac{1}{24}$ litres in the other glasses each. If 9 children come, the first 8 wil each get one of the first 8 glasses and the 9th child drink the contents of the remaining 8 glasses. This way the 9th grandchild also drinks $8 \cdot \frac{1}{24} = \frac{1}{3}$ litres of apple juice like the rest.
If only 8 grandchildren come to visit let each of them drink a larger ($\frac{1}{3}$ litres) and a smaller ($\frac{1}{24}$ litres) glass of juice. This way all 8 grandchildren drink the same amount ($\frac{3}{8}$ litres) of apple juice and no two of them drank from the same cup.

b) 1st solution: Let us assume that Grandma can solve the problem with 15 glasses. We know that all of the apple juice has to be consumed and every glass can be used by only one grandchild. If 9 grandchildren arrive each grandchild has to drink $\frac{3}{9} = \frac{1}{3}$ litres of apple juice - not a drop more. If Grandma were to pour more than $\frac{1}{3}$ litres in one of the glasses, the child who would drink from this put pf the 9 wpuld drink more than the others. Therefore every glass can contain $\frac{3}{9} = \frac{1}{3}$ at most. However, this way if only 8 children arrive each one would have to drink from two glasses, since $\frac{1}{3} < \frac{3}{8}$ (each glass contains less than a child would drink in total). So 15 glasses aren't enough and 16 are needed.

b) 2nd solution: We can also formulate the main idea of the solution backwards. This way - there being 15 glasses - it is enough to have 8 grandchildren for there to be a child who candrink from 1 glass at most (otherwise there would have to be 16 glasses). If there are 8 of them, they all drink $\frac{3}{8}$ litres each, so a child drinking from only one glass would have to get a glass containing $\frac{3}{8}$ litres. If 9 grandchildren were to arrive, no child could get the 'bigger' glass, since the child drinking from it would drink more than $\frac{1}{3}$ litres. However, this way they can't drink all the apple juice. Therefore the division cannot be realized with 15 glasses.

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5. a) The first natural number Csongor has written could have been 999981 for example. The last, 38th number is 1000018. Between 999981 and 999989 the sum of digits ranges from 45 to 54, so their remainders after division by 11 range from 1 to 9. From 999990 to 999999 the remainders range from 1 to 10, as they do from 100000 to 100009. Finally, between 1000010 to 1000018 the remainders range from 2 to 10. This means none of the 38 listed number have a sum of digits divisible by 11, so the construction is valid.

The table below shows the numbers in compressed form and the remainders of division by 11 of the sum of their digits:

999981	...	999989	999990	...	999999	1000000	...	1000009	1000010	...	1000018
1	...	9	1	...	10	1	...	10	2	...	10

b) Let us first note the following: if Csongor wrote 10 consecutive natural numbers, the smallest of which ends with digit 0, then these numbers only differ in the number of their ones. Let's call a group of 10 such numbers a *decimal group*. The sums of digits and the remainders of these after division by 11 in a decimal group are consecutive numbers (where we consider 0 to follow 10 as a remainder modulo 11). In order for a decimal group not to contain a number divisile by 11 the remainders have to range from 1 to 10 (e.g. this holds from 100 to 109 or 480 to 489).

Our second observation: Let's consider it generally, how large the difference of the sums of decimal digits of two consecutive numbers can be. If the smaller number ends with exactly k 9s the next number will end in k 0s and the digit of the $(k+1)$th smallest place value will increase by exactly 1 ($\overline{\ldots n99\ldots 9} \rightarrow \overline{\ldots (n+1)00\ldots 0}$). The sum of digits increases/decreases by $1-9k$. This also means that if by stepping over to the next number crosses 10 the sum digits increases by 1. If 10 is crossed, but 100 isn't then the sum decreases by 8.

Now let's divide the analysis into two parts:

Case I: The numbers Csongor wrote can only differ by their last two digits (in short they are part of the same *centesimal group*). If we write at least 19 consecutive numbers let us call the first one ending with 0 A, which comes before the 11th number. Starting from A there are 10 consecutive numbers which only differ by their last digit (A, $A+1$, ..., $A+9$). Based on our first observation the remainders after division by 11 of the sums of digits have to range from 1 to 10.

Let's check the numbers in the decimal groups before and after A. $A-1$ ends in exactly one 9, so the sum of its digits (according to our second observation) is greater than the sum of A's digits by 8, so the modulo 11 remainder of the sum of digits of $A-1$ is $1+8=9$. If we go down stepwise by one, we see that the sum of digits of $A-9$ modulo 11 is 1 and the number itself also ends with 1, but it is 0 for $A-10$ - here we stop writing numbers. Similarly the smallest number of the next decimal group is $A+10$ whose sum of digits is 8 less than for $A+9$, so the sum of digits modulo 11 is 2 for $A+10$. Moving further this remainder is 10 for $A+18$ but the sum of digits of $A+19$ isalready divisible by 11.

Let's check this also in the table below with the sum of digits modulo 11 in the bottom row:

number	$A-10$	$A-9$	...	$A-1$	A	...	$A+9$	$A+10$	...	$A+18$	$A+19$
remainder	0	1	...	9	1	...	10	2	...	10	0

We can see that in this case at most $9+10+9=28$ numbers can be written from consecutive decimal groups (e.g. from 471 to 498 the remainders according to the table).

Case II: 100 is crossed in the numbers written by Csongor. Let us now denote a number that Csongor wrote and is divisible by 100 C (for now we don't know if he could've written more than one). Let us note that at most 19 numbers can be chosen from both the *centesimal group* ending with $C-1$ and the one beginning with C.

The numbers from $C-10$ to $C-1$ constitute a decimal group. Based on the analysis of **case I.** if the decimal group $(C-10)-(C-1)$ can be written appropriately then the sum of digits of $C-10$ is 1 modulo 11. In this case we can only choose 9 other numbers from the previous decimal group at most, since the remainders of the sums of digits in the decimal group $(C-19)-(C-10)$ are 'one below' the one in $(C-9)-(C-1)$. Similarly in the decimal group from C to $C+9$ the remainders modulo 11 range from 1 to 10 while in $(C+10)-(C+19)$ $C+19$ is divisible by 11 so it cannot be written.

Since a maximum of 19 numbers can be chosen from both groups, we can write 38 numbers in total in this case at most. Since in **case I.** we couldn't write more than 28 numbers, we can see that more than 38 numbers can't be written, thus solving part **b)** . However, it is worth continuing the argument since we could produce an example for **a)** based precisely on these observations.

In order to write 38 numbers that meet the conditions it is necessary for the sum of digits modulo 11 to be 1 for C and 10 for $C-1$, which ends with 99. If this condition is met then all remainders will be suitable (but only in this case can 38 numbers be written). Let's look at the table; the bottom row still shows the modulo 11 remainder of the sum of digits of the number above:

number	$C-19$	...	$C-11$	$C-10$	...	$C-1$	C	...	$C+9$	$C+10$	...	$C+18$
remainder	1	...	9	1	...	10	1	...	10	2	...	10

Let k denote the number of 9s at the end of $C-1$. At the beginning of part **b)** we observed that the sum of Cs digits is $9k-1$ smaller than for $C-1$. We know that on the one hand the modulo 11 remainder of the sum of digits of $C-1$ is 10 and on the other hand it is equal to the modulo 11 remainder of the sum of digits of $(9k-1)+1=9k$. Let's find a positive integer k for which the remainder after division of $9k$ with 11 is 10. If we examine the multiples of 9 we will find that $k=6$ is suitable ($9k=54=4\cdot 11+10$). Since $6\cdot 9$ modulo 11 is 10, the number 999999, which is constituted of 6 9s will be a suitable choice for $C-1$. In this case C is 1000000, whose sum of digits is 1 which is also correct. This is how we actually arrived at the solution in **a)** : the smallest number corresponds to 999981 and the las number CSongor wrote is 1000018, which we can check is correct.

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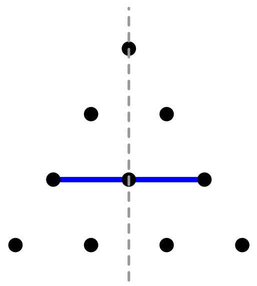
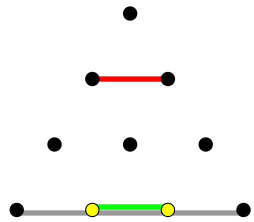
6. *We will refer Indians as players and to ropes as segments or lines as well.* In this game the first player has a winning strategy. What's more they can win after an arbitrary first move, no matter what moves the second player makes. We will examine the winning strategy of the first player, where they span a rope between two poles at distance two, e.g. the blue one in the picture on the left. (If they span any other rope of length two, the same position can be obtained by rotating the picture).





After this the first player will mirror the moves of its opponent if possible with the gray dashed line as an axis of symmetry. This is always possible when their opponent doesn't draw a line from the ones in the picture on the right. These segments are their own mirror images with respect to the axis, so if one player draws them, the other can't draw their mirror image. If therefore their opponent's move is one of these lines (one of the colored lines in the picture on the right), then this is how they should anwser:

- If the other player spans the green or the gray rope, the first player should choose the red one.
- If the other player draws the red line in their move, then the anwser should be to draw whichever is possible from either the green or gray lines. Since the picture was symmetric after every move thus far it is true that of the two columns in the middle of the bottom row marked with yellow dots either both are reached by a rope or neither of them is. Th epoints is to get a rope between th two columns (green line), but if the rope can be extended (gray line) then they will choose the gray line of 3 units length obviuosly.

When the second player draws one of these colored lines, the first player - according to their strategy - will not yet have drawn such a line, so they will always be able to choose the colored line which will be an appropriate anwser to the second player's move. This way the picture will remain symmetric.

In summary: the picture will remain symmetric with respect to the gray dashed line after the first player's every move. While there are still empty places where the second player can span a rope the first player will also have an empty space because of axial symmetry, so the first player wil never run out of possibilities to move. Thus with this strategy they will surely win the game.

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2.3.2 Category D

1. For the solution, see Category C Problem 1.

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2. If $k = 1$, then rearranging the equation gives $n^3 = 3$, which doesn't have an integer solution.

If $k > 1$, then $k!$ is even and so is the left-hand side of the equation. From this it follows that n^3 is also even, meaning $n = 2m$ for some positive integer m. After substitution: $8m^3 - 2 = k!$. We can factor out 2 from the left-hand side bringing it into the form $2(4m^3 - 1)$. Since $4m^3 - 1$ is odd the left-hand side isn't divisible by 4 so the right-hand side isn't either. Since $k!$ is the product of positive integers up until k, $k < 4$ has to hold. If $k = 2$ the equation is of the form $n^3 = 4$, which doesn't have an integer solution, while if $k = 3$, $n^3 = 8$. This has exactly one solution among positive integers: $n = 2$.

The equation in question has one solution: $k = 3$, $n = 2$.

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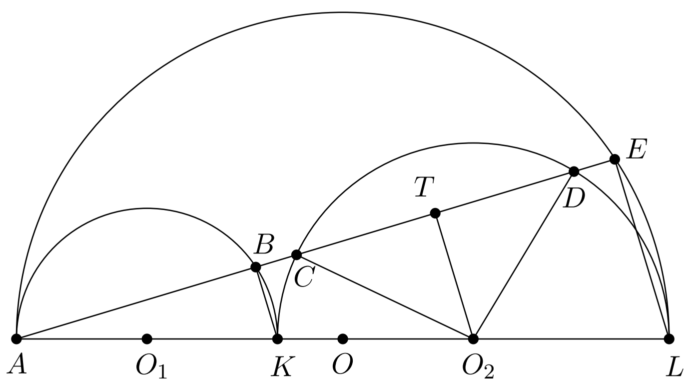
3.



Let K and L be the intersection points of circles with centers O_1 and O_2, and O_2, O respectively.

Since AK is a diameter in the circle with center O_1 it follows form Thales's theorem that $ABK\angle = 90°$. Similarly we get that $AEL\angle = 90°$ in the O centered circle, meaning that the $BKLE$ quadrilateral is a trapezoid. Let us draw the midline of trapezoid, which intersects side BE in point T. We know the the midline of the trapezoid halves the lateral sides of and is parallel to the bases, which means this will coincide with the segment TO_2 - since O_2 is the midpoint of KL - which we will find to be perpendicular to AE. Then we can see that CO_2T and DO_2T are congruent, since they both have a right angle at T and their hypoteneuses (radiuses CO_2 and DO_2) and legs are the same. Therefore $CT = TD$, which, combined with $BT = TE$ (T is the midpoint of side BE) gives $BC = DE$.

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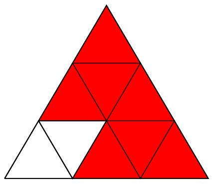
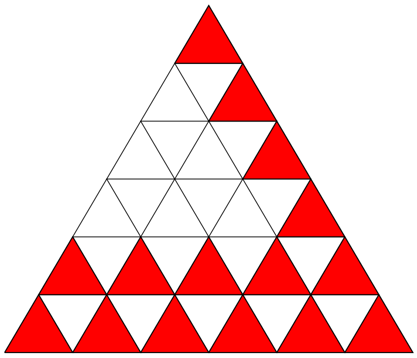
4. a) Let us show that $N \geq 5$, i.e. no matter how Charlie colors five fields, Piroska won't be able to color the whole triangle. We have 3 triangles of baselength 2, therefore Piroska could have colored this many at most, so thjer had to at least 6 fields in the beginning. However, we can solve the problem with this many fields, e.g. by coloring the six in the middle.

b) $M = 8$, since for the case of 7 we still can give a construction where Piroska can't color the whole triangle as shown in the picture. With 8 colored fields only one small triangle will miss out, which she certainly will be able to color.



c) Similarly to the argument in section **b)** M is at least $n^2 - 1$, but is also sufficient for coloring this many fields.

Now we will determine the value of N generally. In every step the number of colored triangles increases by one, and Piroska can have at most as many triangles as there are traingles with baselength 2. There are $1 + 2 + \ldots + n - 1 = \frac{n(n-1)}{2}$ triangles pointing upwards and $1 + \ldots + n - 3 = \frac{(n-3)(n-2)}{2}$ triangles pointing downwards. Thus Piroska can color at most $n^2 - 3n + 3$ small triangles, meaning that in the beginning at least $n^2 - (n^2 - 3n + 3) = 3n - 3$ were colored if she colored the whole triangle by the end. For this reason fewer than $3n - 3$ is certainly not enough. But Charlie can lay this many triangles in such a way that Piroska is able to color the whole triangle as can be seen in the example in the picture. Therefore $N = 3n - 4$.



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5. Let us call a sequence n-mystical, if $x_0 < n$, $x_{i+1} = \lfloor x_i \rfloor \{x_i\}$, and the sequence contains a nonzero integer. The problem asks for the number of 2021-mystical sequences, the following solution works for all n. We propose that there are $2^{n-1} - 1$ n-mystical sequences.

Let $S(n)$ be the set of n-mystical sequences and $A(n)$ denote their number. It is clear that $A(1) = 0$, and we will show that $A(n+1) = 2A(n) + 1$.

If $x_0 < n + 1$ then $x_1 < n$ since $x_1 = \lfloor x_0 \rfloor \{x_0\}$ and $\lfloor x_0 \rfloor \leq n$, $\{x_0\} < 1$. Let us consider the sequences in $S(n+1)$. Obviously $S(n) \subset S(n+1)$, so it suffices to write down sequences for which $x_0 \geq n$. There is a nonzero element in these sequences. If this is the zeroth one, than it can only be $x_0 = n$. In all the others a later element will be nonzero, which means that starting the sequence from x_1 we get a sequence in $S(n)$. Since here $\lfloor x_0 \rfloor = n$, and $x_1 = \lfloor x_0 \rfloor \{x_0\}$, $\{x_0\} = x_1/n$ also has to hold, which is possible in exactly one way for all cases where $x_1 < n$. Thus the sequences in $S(n+1)$ for which $x_0 > n$ are in bijection with the elements where $x_1 < n$, which are precisely the elements constituting $S(n)$. Thus, we finally get $A(n+1) = 2A(n) + 1$. From this and $A(1) = 0$, $A(n) = 2^{n-1} - 1$ follows by induction.

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6. For the solution, see Category C Problem 6.

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2.3.3 Category E

1. Let a and b be the two positive integers for which we study the difference of cubes. Without loss of generality we can assume that $a \geq b$. Since a and b are positive integers just like the difference of their cubes, the difference must be $a^3 - b^3$. By the well-known identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ the difference can only be a prime if $a - b = 1$ since $a, b \geq 1$ implies that $a^2 + ab + b^2 \geq 3$. Hence $a = b + 1$ and the difference is

$$a^2 + ab + b^2 = (b+1)^2 + (b+1)b + b^2 = 3(b^2+b) + 1.$$

Note that b^2 and b have the same parity implying that their sum is divisible by 2. This means that $3(b^2 + b)$ is divisible by 6. This shows that $3(b^2 + b) + 1$ gives 1 as remainder after division by 6.

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2. First we make some general remarks which hold for both maps. Let's call cities vertices and have an edge connect two cities if they are neighbours. For any vertex v elt $f(v)$ denote the cities wealth measured in denari. Firstly, let us notice that if $f(v) > 1000$ for some city v, then the conditions put forth in the problem would still hold and the overall wealth would decrease if the city's wealth were $f(v) - 1000$. Thus we can assume $0 < f(v) \leq 1000$ for all vertices v.

The key observation is that with these conditions for any vertices connected by edge (u, v), $f(v)$ is unambigiuously defined $f(u)$, since $1000 \mid f(u) + f(v)$, so if $f(u) < 1000$ then $f(u) + f(v) < 2000$, meaning that only $f(v) = 1000$ is possible. Therefore if we have determined the value of a vertex, then we knows its neighbours' and the neighbours those and so on, one vertex determines the value of all vertices. This leaves two options:
First possibility: If $f(v) = 1000$ for any vertex v, then the value is 1000 for all vertices so the total wealth is $1000n$ deanri, where n denotes the number of cities.
Second possiblity: If $f(v) < 1000$ for some vertex v then it is smaller than 1000 for all vertices, so the total wealth is less than $1000n$ denari. It can be seen that the first possibility always satisfies the condition stated in the problem, but if there is a valid realization of the second possibility ,then the total wealth will be less.

a) Here we propose that it is possible to realize the second possibility described above and the minimal total wealth is 9000 denari.
Let us observe that the vertices can be divided into two groups (let's call them A and B) in such a way that if any two vertices are in the same group, then they don't share an edge. This is clear - simply just put the edges into groups A and B in an alternating fashion and we will succeed.
We can assume that group A is larger, in which case there are 15 vertices in A, while there are (in B. Let $a \in A$ and suppose $x = f(a)$ which we already know. Furthermore let us suppose that $x < 1000$. Then all neighbours b of a are in B and $f(b) = 1000 - x$. Their neighbours on the other hand are in A and their values are x. Continuing this, we can see that we have assigned x to all vertices in A and $1000 - x$ to all vertices in B. Thus the total wealth is $15x + 8(1000 - x) = 8000 + 7x$. $X > 0$, so the total wealth is more than 8000, that is it has

to bne at least 9000. This is achievable by choosing $x = \frac{1000}{7}$. In this case the total wealth is indeed 9000 denari and the conditions set by the problem are met, so this is the solution.
Remark: The argument described above can be said about any arbitrary connected bipartite graph.

b) Here the key observation is that there exists a cycle of length 5, i.e. vertices v_1, v_2, v_3, v_4, v_5 where v_i and v_{i+1} are connected for $i = 1, 2, 3, 4$ and so are v_5 and v_1. Let $f(v_1) = x$ and suppose $x < 1000$. Let's move along this cycle:

$$f(v_2) = 1000 - x, \quad f(v_3) = x, \quad f(v_4) = 1000 - x, \quad f(v_5) = x, \quad f(v_1) = 1000 - x.$$

Thus we got $x = f(v_1) = 1000 - x$ which can only be the case if $x = 500$. Starting from vertex v_1 we can determine everyone: we assign $1000 - 500 = 500$ to the neighbours of v_1 and 500 to their's and so on - 500 to every vertex. there are 15 vertices so the total wealth is $15 \cdot 500 = 7500$ denari, which isn't divisible by 1000 so we arrived at a contradiction.
Thus the second possibility is out of question, and the only option is to assign 1000 to every vertex, totaling 15000 denari as wealth which is minimal.
Remark: It also follows from the above argument that in case of a connected graph which contains a cycle of odd length, if the graph has an even number of verices then we can reach the minimum by assigning 500 to every vertex and 1000 in case of an odd number of vertices.

It is a known theorem that any connected graph contains a cycle of odd length or is bipartite, therefore the solutions to parts **a)** and **b)** together solve the problem for arbitrary connected graphs.

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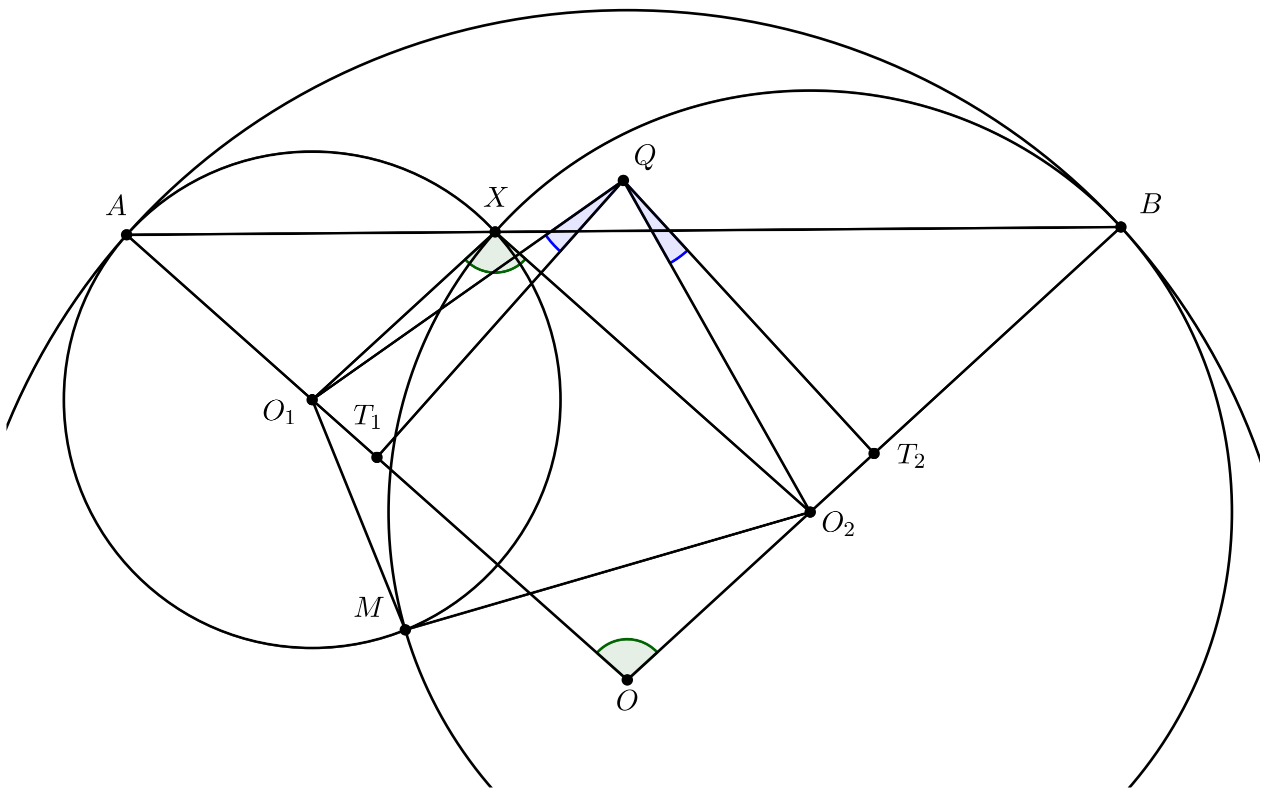
3. We show that Q and M points lie on the circumcircle of O_1O_2O. We know that $BAO\angle = OBA\angle = O_1XA\angle = BXO_2\angle$, therefore OO_1XO_2 is a paralelogram. Thus, $AO_1 = 0_1X_OO_2$. Let T_1 and T_2 be the perpendicular projections of Q unto AO and BO respectively. Q is the circumcenter of AOB, thus QT_1 and QT_2 are perpendicular bisectors of AO and BO, in addition, $QT_1 = QT_2$ also follows from $AO = BO$. From this it is clear triangles QT_1O_1 and QT_2O_2 are congruent, since $O_1T_1 = AT_1 - AO_1 = OT_2 - OO_2 = O_2T_2$, and so two of their sides and the enclosed angle are equal. Therefore $O_1QO_2\angle = T_1QT_2\angle = 180° - O_1OO_2\angle$, which proves the proposition for Q.

$O_1X = O_1M$ and $O_2X = O_2M$, thus O_1XO_2M is a deltoid, moreover we know that OO_1XO_2 is a paralelogram, so $O_1MO_2\angle = O_2XO_1\angle = O_1OO_2\angle$, which proves the statement for M as well.



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4. For the solution, see Category D Problem 5.

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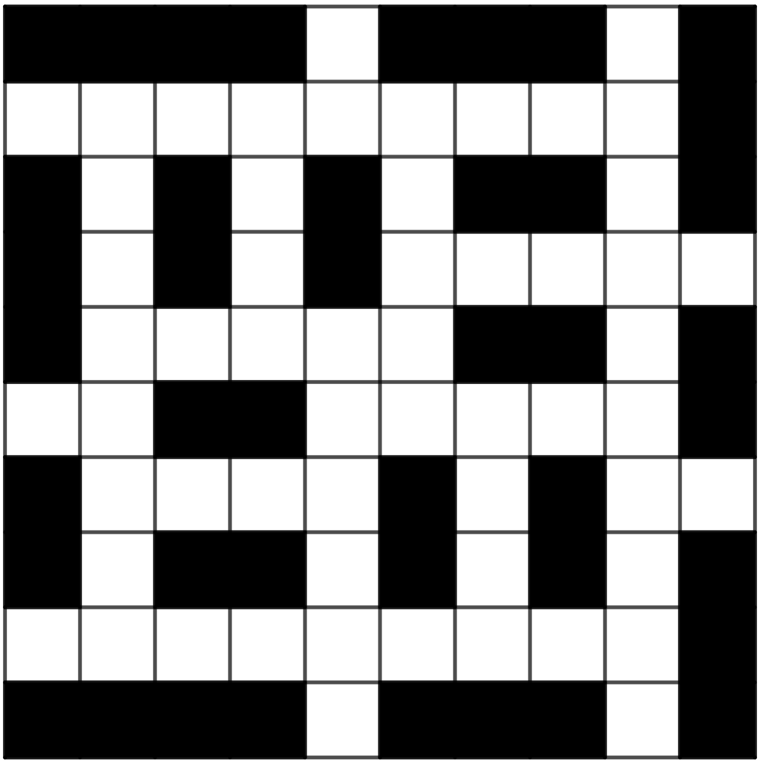
5. a) It can't be done. Let's divide the board into 5×5 squares of size 2×2. In one such small area any two fields touch so each area can contain only one ship. A ship of size 1×1 and 1×2 is contained by at least 1, while ships of size 1×3 and 1×4 are contained by at least 2 small areas, so the ships would have to be contained by at least $2 \cdot 2 + 4 \cdot 2 + 6 \cdot 1 + 8 \cdot 1 = 26$ small areas, so they can't be placed on the board.

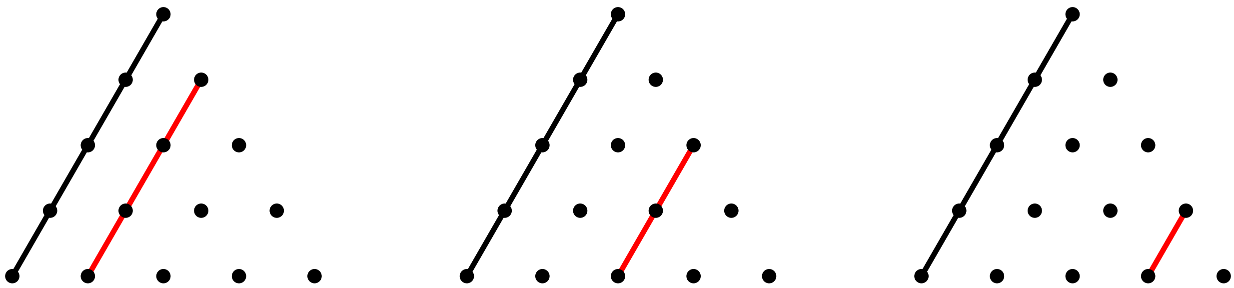
b) Let us assume that we somehow managed to place the ships. Let's increase the size of each ship by half a unit in each direction, i.e. a ship of size $1 \times c$ will increase to $2 \times (c + 1)$. Clearly the ships that we obtain don't touch each other and are contained by the board which we get by increasing the big board by halfa unit in every direction, that is a board of size 11×11. The increased total area of the ships is $2 \cdot 10 + 4 \cdot 8 + 9 \cdot 6 + 4 \cdot 4 = 122$, while the area of the increased board is $11 \cdot 11 = 121$, so we can't place tha ships.

c) The construction in this case:



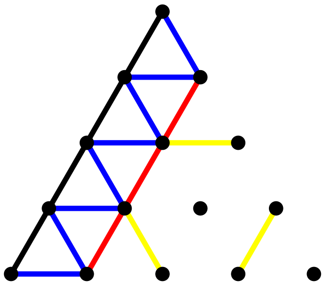
(Back to problems)

6. *We will refer to Indians as players and to ropes as segments or lines as well.* The second player has a winning strategy in the game. Based on the first step we will distinguish three cases which we will analyse:



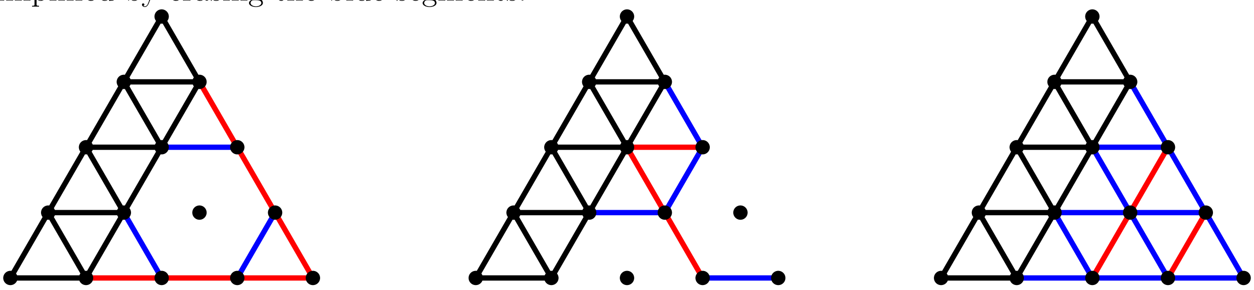
In case the first player draws the red segment - we can assume they will choose one of the red or black segments based on rotational symmetry - the second player will draw the black one. If the first player draws a black segment, the second can draw an arbitrary red one.

We define trivial segments as follows: it has length 1, is not drawn yet and has rope touching the poles whic hare its endpoints. E.g. of the lines which aren't drawn yet - considering the first case - the blue ones are trivial while the yellow ones aren't in the picture below.



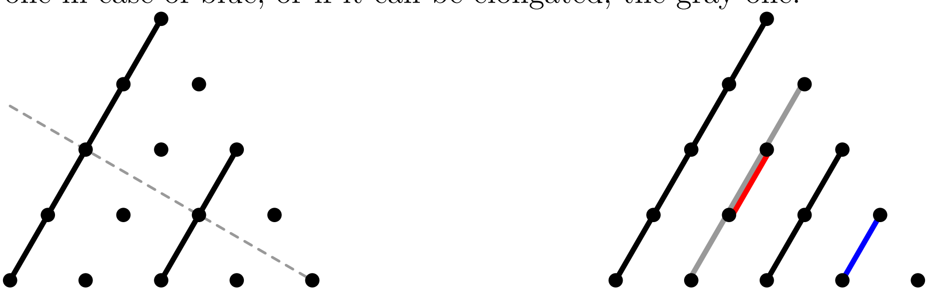
Trivial segments make it much easier to analyse the given cases: In case we have an even number of such segments after one of our moves (e.g. after the first two moves), then we can choose to draw a trivial segment if and only if the other player also drew one. This way we can always simplify the current state of the game: we can 'draw' the trivial segments which have merged thus far, because it is always us that will draw the last one of these, so it suffices to find a strategy on the rest of the board.

First case: The picture on the left in the top row shows segments which are the first two drawn segments. Here the state can be simplified in such a way if only the black lines were drawn in the pictures below. Then (ignoring axial symmetry) if the first player draws one of the red segments, the second player will anwser by drawing another red segment and the picture is simplified by erasing the blue segments.



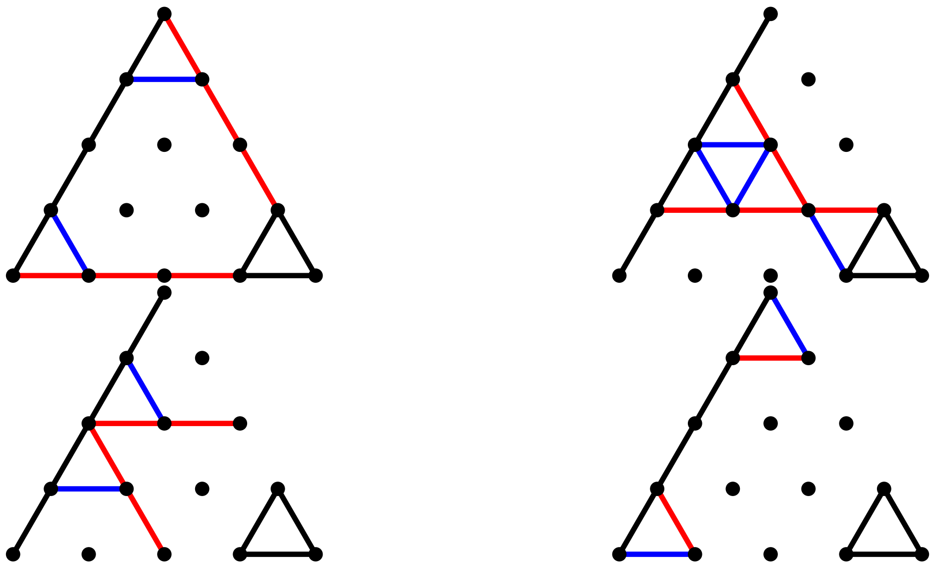
It is easy to see that in these simplified states the second player wins: in the first and third cases the number of turns to play is fixed while in the second they can mirror the moves of the first player.

Second case: Here the second player mirrors the moves of the first player with the gray dashed line as axis of symmetry, as is shown in the picture on the left below, except were such a move is impossible. In a such a case the first player must draw one of the non-black segments and the second player will draw the other such segment, i.e. the blue one in case of red and gray, the red one in case of blue, or if it can be elongated, the gray one.



With this strategy all segments parallel to the black ones will be drawn in an even number of steps (2 or 4), and so will the other, non-parallel ones. Thus there will remain an even number of steps after drawing all the black ones and so the second player can win.

Third case: Here the game state can be simplified to when only the black segments are drawn in the pictures below. Then (ignoring axial symmetry) if the first player draws one of the red segments, the second player will anwser by drawing the other red segment in the appropriate picture, and it will be simplified by erasing the blue segments.



In the state shown in the first picture the number of moves to be made is constant and is even. We can see this by observing that there is one big area inside which will eventually be divided into 13 areas. Since every segment creates a new area, there remain 14 steps and the second player will win.

In the second picture they can mirror all the moves of the first player so the second player can win.

In the first picture let us observe that exactly 8 moves will take place so the second player will have the last move and can mirror in the other two parts.

Finally let's look at the last case. We can either anwser the first player's moves (and win) or the first player will draw one of the previously undrawn segments in the side of the big triangle. However, here the second player will draw the other one and similarly as in the first picture will win.

The strategy presented in our solution isn't the only winning strategy and there can be several winning moves in a given state. Nonetheless, our strategy guarantees victory for the second player so by following it Red Fire will win against Silent Stream.

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2.3.4 Category E$^+$

1. For the solution, see Category E Problem 3.

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2. For the solution, see Category D Problem 5.

(Back to problems)

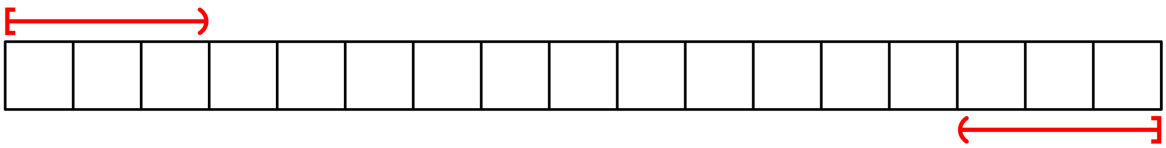
3. We propose that they can divide the bars between themselves if and only if $n \leq k$ or $n > k$ and $n - k \mid n$. First we show a construction, proving that there exists a solution for these cases.

Construction ($n \leq k$). First a contruction for the case where $n \leq k$: everyone has to get $\frac{k}{n}$ bars of chocolate. Let us order the k bars in a line and divide this line into n equal parts. Since $\frac{k}{n} \geq 1$ any bar contains at most one cut, so this division meets the conditions.

Construction ($k < n$ and $n - k \mid n$). Let us cut each piece of chocolate according to the ratio $k : (n - k)$. Then k people get one piece of size $\frac{k}{n}$ each, while $n - k$ people share k pieces of size $\frac{n-k}{n}$ equally. The latter division is possible, since $n - k \mid n - (n - k) = k$. Therefore all members of the second group also get $\frac{k}{n}$ bars of chocolate.

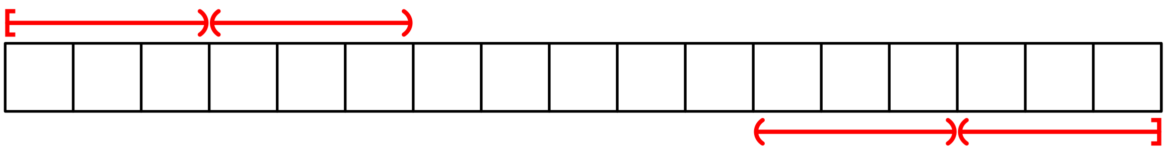
Proof of impossibility. We show that in any other case they can't divide the chocolate according to the conditions. It is clear, that in the case of $n > 2k$ the division is not possible, since we create at most $2k$ pieces of chocolate by breaking the bars but there are more children than this. Thus $k + 1 \leq n \leq 2k - 1$ holds. We define a unit of chocolate as $\frac{1}{n}$. In this case one child gets k units of chocolate. Let $d = n - k$.

It is not possible to break a piece from the chocolate that is greater than k units. (on the following picture the chocolate cannot be broken in the red intervals.)



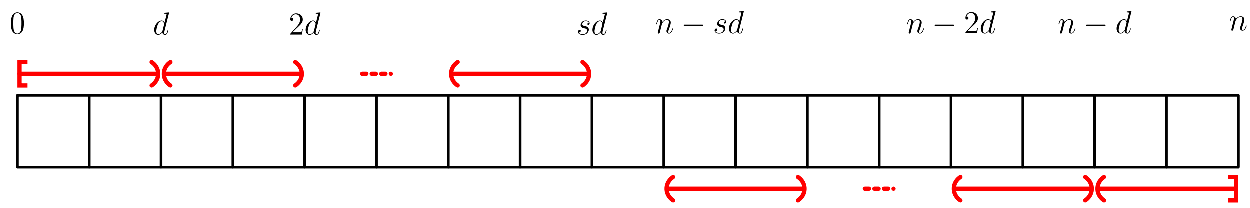
The person who got the large part of the bar would get more than average. From this it immediately follows that there can't be any piece smaller than $d = n - k$ units either (since then the other piece would be greater than k).

Let's go one step further! It is not possible to break a piece from the chocolate with size greater than $n - 2d$ but less than $n - d$ units.



If there were such a piece, whoever received it would need to supplement it to $k = n - d$. However, for this they would need to use a piece smaller than d which we have already seen cannot happen. We can go even further: there cannot be a piece of chocolate with a size greater than $n - sd$ but smaller than $n - (s - 1)d$ units or a size greater than $(s - 1)d$ and less than sd. By induction let us suppose that this is known up to $s - 1$.

If the size of the piece was between $(s-2)d$ and $(s-1)d$ then any piece used to supplement it would have a size of at most $(s-1)d$ as $n > k$ so each child gets at least one unit less than 1 bar worth of chocolate. Furthermore it is clear, that this has to be a piece whose size isn't divisible by d. In conclusion a piece of size in one of the intervals $(0, d)$, $(d, 2d)$, $\ldots$ or $((s-2)d, (s-1)d)$ would have to be used, which isn't possible according to the induction hypothesis.



We can see that only those breaks are allowed, where the sizes of both resulting pieces are divisible by d that is only if $d \mid n$ can there remain breaks that are allowed. Thus we have proven our proposition.

Second solution: We show another solution proving that $n-k \mid n$ is a necessary condition for division when $k+1 \le n \le 2k$. Let the people be the vertices of a graph. Two people share an edge if they got a piece from the same chocolate. So each edge in the graph represents a piece of chocolate.

Le us note that in this graph there can be no cycle. If there were a cycle of length x, the x people contained in the cycle would need to get at least x pieces of chocolate. Thus someone would get more than their share. Therefore all connected components are trees.

Let us view one connected component of the graph, containing g people, who get c amount of chocolate. Since this component is a tree $c = g-1$. It follows that any person in this component gets $\frac{g-1}{g}$ amount of chocolate. But this means all people get this same amount. On the other hand we know that there are n people and k amount of chocolate. This means that

$$\frac{k}{n} = \frac{g-1}{g}$$

for some integer g. Transforming this we get $n = g(n-k)$, that is the division can be solved only if $n-k$ is a divisor of n.

(Back to problems)

4. For the solution, see Category E Problem 5.

(Back to problems)

5. We prove a more general statement: Given two $n, k > 1$ positive integers all positive divisors of the expression $kn^2 - 1$ modulo kn are differen. First we prove a lemma.

Lemma. *Let x, y positive integers be relative primes. Then all positive divisors of x are different modulo y if and only if the conditions $yc + x \mid x^2$ and $yc + x > 0$ are only satisfied by $c = 0$ among integers.*

Proof. **The implication:** If there exists another solution c then it can be written in the form $yc + x = d_1 d_2$ where$d_1, d_2 \mid x$ and $d_1, d_2 > 0$. However,

$$x \equiv yc + x = d_1 d_2 \quad (y)$$

implies $x \equiv d_2 \ (y)$. We can divide by d_1 because $(x, y) = 1$ so $(d_1, y) = 1$. But this according to the assumption can only be if $x/d_1 = d_2$ so $c = 0$.

The converse: If there were two positive divisors giving the same remainder $d_1 \equiv d_2 \ (y)$ then

$$\frac{x}{d_1} d_2 \equiv x \quad (y).$$

Clearly $(d_2 x/d_1) \mid x^2$ so

$$c = \frac{d_2 x/d_1 - x}{y}$$

is a solution and $d_1 \neq d_2$ implies $c \neq 0$. Thus we have proven the lemma.

□

Applying our lemma to the statement of the problem we get that $c = 0$ is the only solution satisfying conditions $knc + kn^2 - 1 \mid (kn^2 - 1)^2$ and $knc + kn^2 - 1 > 0$. Let's rearrange this in a prettier fashion: Let $c' = c - n$ - substituting this the statement to prove is the following: Conditions $knc' - 1 \mid (kn^2 - 1)^2$ and $knc' - 1 > 0$ are only satisfied by $c' = n$. From here on let's denote c' with c.

Let's consider k fixed. A pair (n, c) is suitable if it satisfies th previous conditions. Let us assume that if (n, c) is a suitable pair, then (c, n) is also suitable, since

$$0 \equiv (kn^2 - 1)^2 \equiv (kn^2 - (knc)^2)^2 = (kn^2)^2 (kc^2 - 1)^2 \pmod{knc - 1}.$$

It is clear, that $knc - 1$ is a relative prime to kn^2, so we got $knc - 1 \mid (kc^2 - 1)^2$ which is exactly the condition for (c, n) being suitable.

Finally if (n, c) is suitable then there exists a suitable (n, c') for which $n > c'$. Let us note that

$$\frac{(kn^2 - 1)^2}{knc - 1} = knc' - 1$$

for some c' since the left-hand side is congruent to -1 modulo kn. It is easy to see that $c' < n$ holds and that (n, c') is a suitable pair, so (c', n) is also suitable. This means we have found a suitable pair whose elements are both smaller than in the original pair, and we could continue with this infinitely - descente infinie - which is a contradiction. Similarly we arrive at a contradiction if $n > c$ so $n = c$ has to hold.

(Back to problems)

6. For the solution, see Category E Problem 6.

(Back to problems)

2.4 Final round – day 2

2.4.1 Category C

#	ANS	Problem	P
C-1	1887	Kartal, Balint and Timi are playing with some cards.	3p
C-2	9288	In Sixcountry there are 12 months,	3p
C-3	31	How many integers between 11 and 2021 have the property	3p
C-4	4	Dóra plays with the following domino set:	3p
C-5	11	In the evening some kids are playing a card game called *Moore*	4p
C-6	73	The figure shows a line intersecting a square lattice.	4p
C-7	276	What is the number of 4-digit numbers	4p
C-8	32	Trapezoid $ABCD$ has bases AD and BC.	4p
C-9	4045	The Good Fairy ATM works as follows.	5p
C-10	247	We call a positive integer *absolutely relative*	5p
C-11	288	Billy owns a nice piece of land in the Wild West,	5p
C-12	13	Billy let his herd freely.	5p
C-13	8	A digital clock displays the digits with dashes as follows.	6p
C-14	325	Jimmy's garden has right angled triangle shape	6p
C-15	1764	Two teams of 3 are travelling to the Durer competition by tram.	6p
C-16	2011	A date is called *rearranging*	6p

2.4.2 Category D

#	ANS	Problem	P
D-1	31	How many integers between 11 and 2021 have the property	3p
D-2	1887	Kartal, Balint and Timi are playing with some cards.	3p
D-3	1010	Find the number of integers n between 1 and 2021	3p
D-4	11	In the evening some kids are playing a card game called *Moore*	3p
D-5	4045	The Good Fairy ATM works as follows.	4p
D-6	73	The figure shows a line intersecting a square lattice.	4p
D-7	375	Given a right angled triangle ABC in which $ACB\sphericalangle = 90°$.	4p
D-8	288	On an 8×8 chessboard,	4p
D-9	247	We call a positive integer *absolutely relative*	5p
D-10	29	Greatgranny has 9 greatgrandchildren.	5p
D-11	90	A triangle is given. Its side a is of length 20 cm,	5p
D-12	13	Billy let his herd freely.	5p
D-13	12	Japanese businessman Rui lives in America	6p
D-14	1764	Two teams of 3 are travelling to the Durer competition by tram.	6p
D-15	17	A digital clock displays the digits with dashes as follows.	6p
D-16	137	Benedek wrote the following 300 statements on a piece of paper.	6p

2.4.3 Category E

#	ANS	Problem	P
E-1	9288	In Sixcountry there are 12 months,	3p
E-2	1010	Find the number of integers n between 1 and 2021	3p
E-3	73	The figure shows a line intersecting a square lattice.	3p
E-4	276	What is the number of 4-digit numbers	3p
E-5	807	How many integers $1 \leq x \leq 2021$	4p
E-6	46	Bertalan thought about a 4-digit positive number.	4p
E-7	325	Jimmy's garden has right angled triangle shape	4p
E-8	8	John found all real numbers p	4p
E-9	288	On an 8×8 chessboard,	5p
E-10	90	A triangle is given. Its side a is of length 20 cm,	5p
E-11	12	Japanese businessman Rui lives in America	5p
E-12	13	Billy let his herd freely.	5p
E-13	133	The trapezoid $ABCD$ satisfies	6p
E-14	48	How many functions $f : \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 16\}$	6p
E-15	20	King Albrecht founded a family.	6p
E-16	2911	Consider a table consisting of 2×7 squares.	6p

2.4.4 Category E$^+$

#	ANS	Problem	P
E$^+$-1	375	Given a right angled triangle ABC in which $ACB\sphericalangle = 90°$.	3p
E$^+$-2	2401	In a french village the number of inhabitants is a perfect square.	3p
E$^+$-3	807	How many integers $1 \leq x \leq 2021$	3p
E$^+$-4	46	Bertalan thought about a 4-digit positive number.	3p
E$^+$-5	146	Joe, who is already feared by all bandits in the Wild West,	4p
E$^+$-6	90	A triangle is given. Its side a is of length 20 cm,	4p
E$^+$-7	288	On an 8×8 chessboard,	4p
E$^+$-8	137	Benedek wrote the following 300 statements on a piece of paper.	4p
E$^+$-9	12	Japanese businessman Rui lives in America	5p
E$^+$-10	306	Billy owns some really energetic horses.	5p
E$^+$-11	5105	How many functions $f : \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 16\}$	5p
E$^+$-12	133	The trapezoid $ABCD$ satisfies	5p
E$^+$-13	100	At a table tennis competition,	6p
E$^+$-14	20	King Albrecht founded a family.	6p
E$^+$-15	2911	Consider a table consisting of 2×7 squares.	6p
E$^+$-16	22	The angles of a convex quadrilateral form an arithmetic sequence	6p