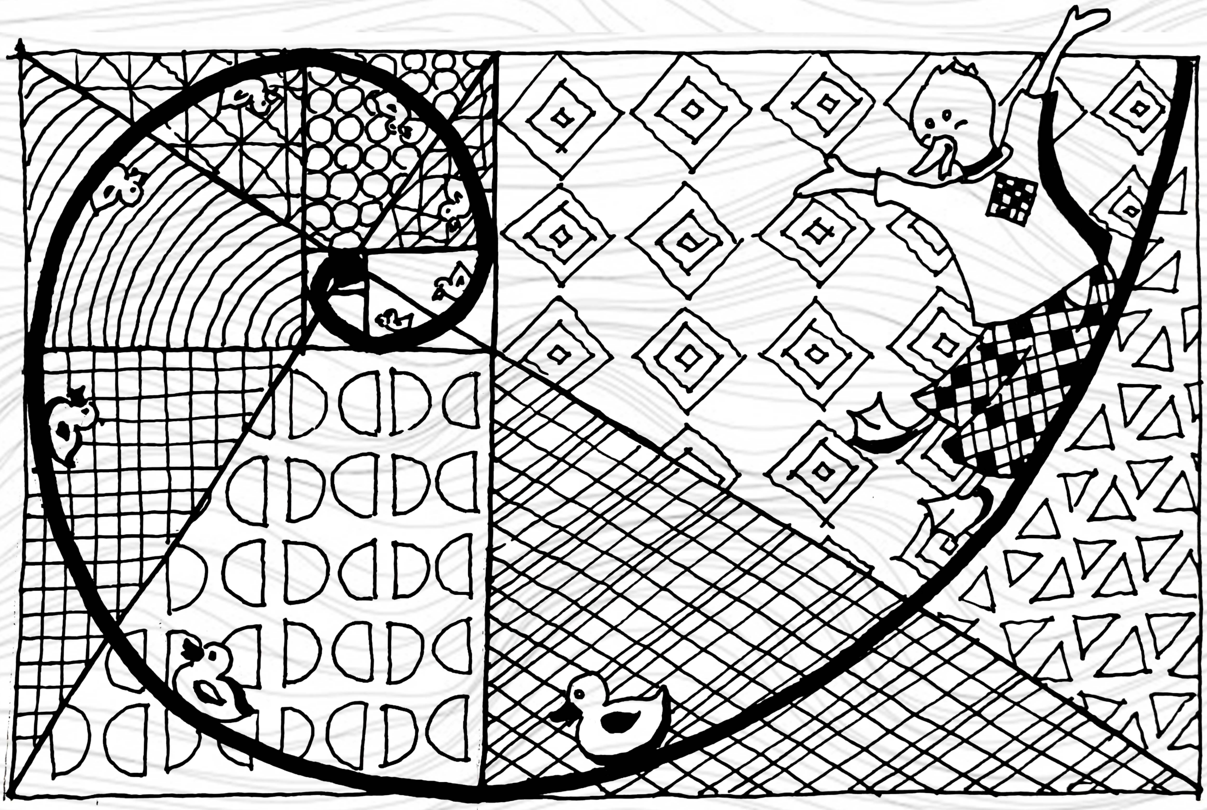
15th Dürer Competition

PROBLEMS, SOLUTIONS



A C E

B D E+

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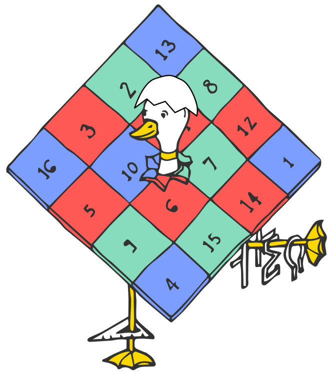
Introduction – About the Dürer Competition

Plenty of mathematics contests are traditionally held in Hungary. From primary schoolers to university students, everybody can find a contest that fits their age and qualifications. These are mostly individual contests where the participants sit down in a room for a few hours, working on the problems quietly. However at the Dürer Competition, there are teams of 3 taking part. For the duration of the contest each team works together to solve the problems, so the contestants can experience the benefits of cooperative thinking. Our experience shows that the majority of students are happier and more relaxed than during an individual contest.



Ducks were the theme of the 15th competition.

It is a very important goal for us to set interesting problems to show the beauty of mathematics and the joy of thinking to lots of students. We also wish to include as many original problems as possible. In each year, about 150 problems appear on the contest – of course not all can be original, but we invent most of the harder problems on our own.

At this point we definitely have to mention that the organising team traditionally consists of young people, mostly university students studying maths. This dates back to the early years of the competition, and ever since then, we can regularly welcome former competitors as new organisers. The success of the competition depends on this community, consisting of 30 to 70 people. Some of them have been organisers for 15 years already (and still take part enthusiastically, even alongside a full-time job) and some of them take important responsibilities as first-year undergraduates already. A lot of our organizers were participating both at EGMO 2022 and IMO 2022 as coordinators.

This is the spirit in which we have been organising the contest for 15 years. The competition attracts more and more students and schools with each year. In the 2021–22 academic year, more than 800 Hungarian students competed in the high school maths categories, and more than 650 in the primary school categories.

This was the third year that we opened our two hardest categories for international competitors. Due to the pandemic the contest was held online with more than 150 students taking part in the regional round of the competition.

Primary school students can take part in our competition in the following two categories:

- **Category A** is open to 5th and 6th grade students.
- **Category B** is open to 7th and 8th grade students.

In these two categories the contest is regional: the first round is organised in 6 cities in north-eastern Hungary, but is open to anyone provided that they travel to one of the locations.

Four categories are available for high school students:

- **Category C** is open to 9th and 10th graders who have never previously qualified for the final of any national math contest.

- **Category D** is open to 9^{th} to 12^{th} graders who are a bit more experienced, but do not come from a school that is outstanding in handling mathematical talents.
- **Category E** is open to 9^{th} to 12^{th} graders who already have good results from other contests, or come from a school outstanding in maths.
- **Category E$^+$** is designed for competitors who actively take part in olympiad training. In this category, most teams include some student who has taken part at an international olympiad (IMO, MEMO, EGMO, RMM, CMC), or is about to qualify for one in the same academic year.

We also organise the contest in physics (*category F* and F^+) and chemistry (*categories K* and K^+), but these are omitted from this booklet.

For high schoolers (in categories C, D and E), the first round is an **online relay round** consisting of 9 problems. The answer to each question is an integer between 0 and 9999. Initially each team gets the first question only. They have three attempts to submit an answer – if they get it right, they score a set number of points, and can proceed to the next question. Each wrong attempt to a question reduces the possible score by 1, and after 3 wrong attempts the team must move on to the next question without scoring. There is also an online **game** in this round.

The second round is a traditional **olympiad-style contest**, where detailed proofs have to be given. The teams have 3 hours to solve 5 problems. The contest can be sat in the whole country, at about 20 locations.

The final round takes place in Miskolc. For high schoolers (categories C, D, E, E$^+$, F, F$^+$ K, K$^+$) we organise it on a weekend in early February from Thursday to Sunday. The first competition day is Friday, with the students working on five **olympiad-style problems** and a **game**. If a team thinks that they have found the winning strategy for the game, they can challenge us. If they can defeat us twice in a row, they get the maximal score for the problem. If they lose, they can still challenge us two more times for a partial score. On Saturday we hold a **relay round** consisting of 16 questions. The rules are similar to the online round. Rankings are based on a combined score from the two competition days.

At the weekend of the final, the students and teachers can participate in many educational and recreational activities, such as lectures, games and discussions about universities.

The competition for primary school students is organized seperatly, but in a similar way. For them, the first round is a **relay round** consisting of 15 questions, while the final round has the same structure as for high school students.

The competition is expanded year after year. In 2022–23 we plan to introduce some changes in the marking of the international competition with involving more volunteers from other coutries. In the long run this would contribute to the establishment of the Dürer Competition as a well-known and renowned contest which is one of our main objectives.

A subset of the organizers of the 15th Dürer Competition

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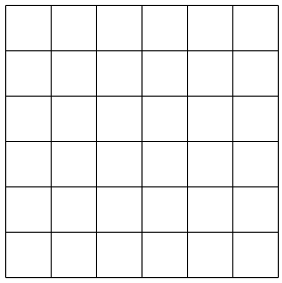
1 Problems – primary school categories

1.1 Regional round

1.1.1 Category A

1. Anna and Panna made up to color some fields of the following table. Anna colored every odd row and Panna every even column. How many fields are there which neither of the girls colored?



(Solution)

2. Bonnie has a fair die on her desk. Bonnie counted that the number of visible dots (the ones not on the bottom face) is 16. How many dots would be visible if we turned over the die? *(When turning over the die the top face from before becomes the bottom one. On a fair die the numbers on opposite faces add up to 7)*

(Solution)

3. At the market of Piripócs, the tradition of trading persists. For three hatchlings, a goose can be gained. For one goose and two hatchlings, a duck can be swapped. Each exchange is valid back and forth. Aunt Kati wishes to raise hatchlings, but only has two ducks and two geese right now. At most how many hatchlings can Aunt Kati gain with these trades?

(Solution)

4. A six member duck family, the mother, the father and the four ducklings are toddling single file to the water. James noticed that the two adults are on the front and back, with the black ducling being ahead of the grey one, and the white duckling walking right after the the yellow one. In how many different orders could the duck family be going to the water?

(Solution)

5. The notation of the date today (2021.11.19.) is interesting in that it contains four identical digits. Writing dates in a similar way, in how many days will there be four identical consecutive digits in the date again? *(The month and day numbers are always written with two digits.)*

(Solution)

6. Sári wrote a message to her friend. Her friend sees that Sári typed for exactly one minute before sending the 76-character message. How many times did Sári press the 'backspace' button if we know that she typed 2 key per second? *With the 'backspace' button, she always deleted exactly one character.*

(Solution)

7. The fifth and sixth grade of the Quackton elementary school go on a joint class excursion to Mallardville. Of the 60 children, 30 prefer swimming and 30 prefer ice cream for the afternoon. The children are divided into five-member rooms, and of the two possibilities, every room chooses the one preferred by a majority of the room. At most how many children will go swimming on the excursion?

(Solution)

8. Andris and Anett went on a 10-kilometer tour. Their friend Ervin Forrest placed signs at a few different locations along the tour path. There are positive integers on the signs, denoting how much further the end of the tour is. Ervin Forrest placed a total of six signs, but he didn't pay good attention, and although every sign was placed an integer number of kilometers from the target, three signs had the wrong number. On the tour, Andris saw the signs in the following order: 8, 6, 2, 7, 3, 4. How many kilometers were they from the target at the third sign?

(Solution)

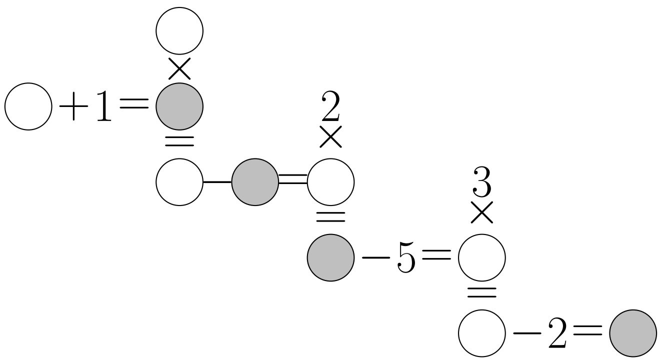
9. The ship Graceful Hippo is occupied by a total of 21 knights and knaves. The knights always tell the truth, the knaves always lie. When the Graceful Hippo completes its three-month journey on the island Oxis, the crew members left the ship one by one such that the captain went last. Apart from the captain, everyone said this when leaving: "There are more knaves left on the ship than knights." How many knaves serve on board the Graceful Hippo?

(Solution)

10. Flea Florence likes to hop on the plane. One afternoon, she began hopping with a one-metre hop to the east, then two metres north. After this, she continue hopping such that each hop was 1 metre longer than the previous one. At least how many hops does she need to complete over the afternoon to return to the starting place, if every hop is made to the west, to the east, to the north, or to the south?

(Solution)

11. Fill the circles with the numbers from 1 to 10 such that the equalities are made true. What will be the sum of numbers written in gray circles? *Every number can be used exactly once, and every circle must contain exactly one number.*



(Solution)

12. Mickey Mouse, Minnie Mouse and Donald Duck live on the same side of a road. Mickey lives in number 35 while Minnie lives in number 59. We know that there are exactly half as many houses between Mickey and Donald as there are between Donald and Minnie. What's the number of Donald Duck's house? *(One side of the road has even house numbers, while the other has odd ones.)*

(Solution)

13. In Duck School, a duckling can get ranks in three different categories: quacking, laying eggs and flying. With these they can get distinctions. Someone gets a distinction if they reach (or exceed) the required rank in at least two of the three categories. These ranks are shown in the table below. Donald Duck managed to get a total of 16 ranks across all the categories until now, with which he already got three distinctions from these four. What's the product of Donald Duck's ranks in the different categories? *The ranks can only increase, and start from 0.*

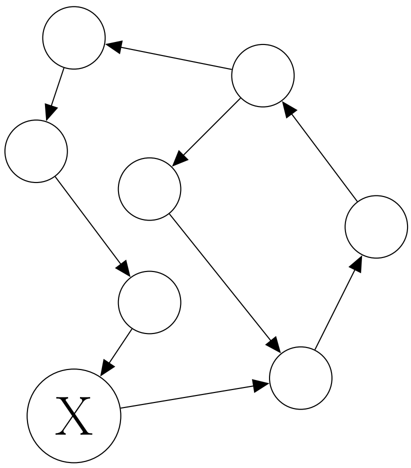
Categories	Quacking	Laying Eggs	Flying
Blue distinction	6th rank	6th rank	7th rank
Red distinction	11th rank	1st rank	8th rank
Yellow distinction	3rd rank	3rd rank	11th rank
Green distinction	7th rank	10th rank	4th rank

(Solution)

14. At a chess contest where everyone played everyone exactly once, there were 6 participants. 2 points were awarded for winning, 1 for a draw, and 0 for losing. At the contest, Niki placed second, Tiki third, and Viki fourth. How many points did they collect at most in total? *(The places are in decreasing order according to score, and for equal scores, the places are decided by drawing lots.)*

(Solution)

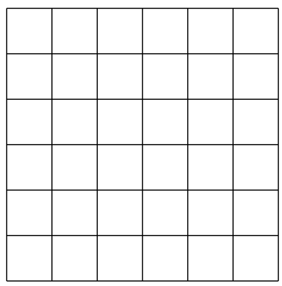
15. The diagram shows the 8 islands of the country Oxis, with ship services along the arrows every day of the year. Lord Adventurer lives on island X, and his plan is to board a ship on each of 100 upcoming days, from the island he is at to another island where he stays the night. Upon hearing the plan, the wife of Lord Adventurer immediately wrote down which of the following 100 days her husband will surely not spend on their own island. How many days did she record? *(Lord Adventurer begins from the island marked with an X, but we do not know where he will arrive on the hundredth day.)*



(Solution)

1.1.2 Category B

1. Anna, Hanna, and Panna made up to color some fields of the following table. Anna colored every odd row, Hanna every even column, and Panna the fields on the two diagonals. How many fields are there which neither of the girls colored?



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4. The seventh and eighth grade of the Quackton elementary school go on a joint class excursion to Mallardville. Of the 60 children, 30 prefer swimming and 30 prefer ice cream for the afternoon. The children are divided into five-member rooms, and of the two possibilities, every room chooses the one preferred by a majority of the room. At most how many children will go swimming on the excursion?

(Solution)

5. Princess Atonce would like to build a castle in the air. To build this palace, enough stone needs to be transported there first. On the first day, they bring one third of the required stones, but during the night the evil gnomes carry half the stones away. Every following day the workers bring stones equal to double the amount already on site, but under cover of darkness the gnomes come, and take half the stones there. How many days does Princess Atonce have to wait for enough stone to be gathered? *Once they brought the sufficient amount, the evil gnomes won't carry more stones away.*

(Solution)

6. Andris and Anett went on a 10-kilometer tour. Their friend Ervin Forrest placed signs at a few different locations along the tour path. There are positive integers on the signs, denoting how much further the end of the tour is. Ervin Forrest placed a total of six signs, but he didn't pay good attention, and although every sign was placed an integer number of kilometers from the target, three signs had the wrong number. On the tour, Andris saw the signs in the following order: 8, 6, 2, 7, 3, 4. How many kilometers were they from the target at the third sign?

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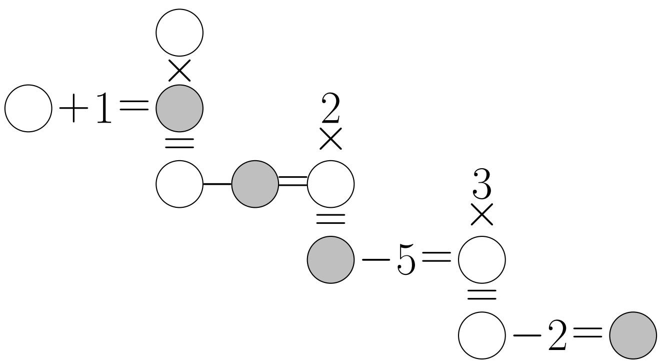

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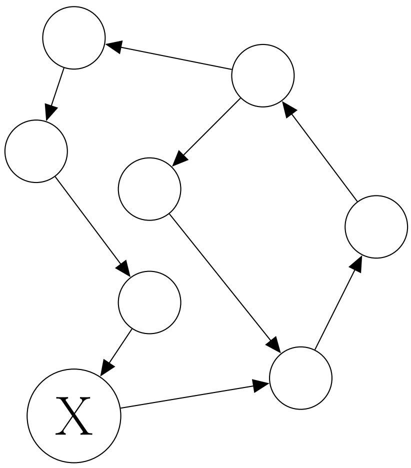


(Solution)

14. In triangle ABC, the internal angle bisector from B meets side AC at D. In triangle ABD, the internal angle bisector from D meets side AB at E. What is the degree measure of angle CAB, if segment DE is parallel to segment BC, and the length of segment AD equals the length of segment BE?

(Solution)

15. Sophie glued three identical standard dice together. He glued full faces with each other, and only if the had the same number of dots. In how many ways could Sophie have glued the dice together? *Two ways to glue are the same, if after rotation their visible faces are identical.*

(Solution)

1.2 Final round – day 1

1.2.1 Category A

1. In the Ducko Doble Dance School, three drakes (Duckevin, Richarduck, Duckmond) and four ducks (Aduck, Rhoduck, Duckatie, Duckota) learn to dance in pairs. Over the course of the evening, there are three dances, and with every dance, each pair consists of a duck and a drake. (This way, with each dance, a duck is left without a pair.)

In the first dance, the pairs were Duckevin-Aduck, Richarduck-Rhoduck, Duckmond-Duckatie. In the second dance, Duckevin-Duckatie, Richarduck-Aduck, Duckmond-Duckota were the pairs. Find a possible pairing for the third dance, if we know that Duckmond does not wish to dance with Aduck, nor would two ducks be in the same pair twice. *It is not necessary to find all cases, but merely to find one possible pairing.*

(Solution)

2. How many parts can two triangles divide the plane into? Give examples for all the cases! *You needn't show that there isn't any other possibility.*

(Solution)

3. Kriszti read her favorite Donald duck book so many times that some consecutive pages fell out of the bind. Thus, opening the book at one place, the sum of the page numbers is 49, but one page later, this sum is already 71.
a) How many pages are missing?
b) Can the smallest missing page number be uniquely determined? *Opening the book, even page numbers are on the left, while odd page numbers are on the right. There are two pages to a sheet.*

(Solution)

4. Put operation symbols ($+, -, \cdot, /$) and brackets on the left side of the following equalities such that they become true. *An operation symbol must be placed between every pair of digits.*

$$\begin{array}{lccccccccl} \textbf{a)} & & & 7 & 6 & 5 & 4 & 3 & 2 & 1 = 20 \\ \textbf{b)} & & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 = 202 \\ \textbf{c)} & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 = 2022 \end{array}$$

(Solution)

5. For the 90th birthday of Grandma, all her grandchildren have gathered. At the dinner, the grandchildren counted how many of their cousins are present, and shared this with the others. Every answer was 6 or 7. How many grandchildren can Grandma have?

Give an example for as many values as possible. When you think no other value is possible, explain why. It is possible that among the answers, 6 and 7 occurred, but it is also possible that every answer was the same.

(Solution)

6. (Game) There are 3 of the 1 pengő, 5 of the 2 pengő, and 7 of the 3 pengő coin in a heap. In a move, the player takes a coin from the heap and replaces a coin of smaller value, or does not replace anything. The winner is the one who takes the last coin from the heap.
Win against the organisers two times consecutively! At the beginning of the game, you may decide whether you wish to be the first or the second player.

(Solution)

1.2.2 Category B

1. How many parts can two triangles divide the plane into? Give examples for all the cases!
You needn't show that there isn't any other possibility.

(Solution)

2. You are participating in a tour, proceeding along a stream and a railway throughout. Initially, the stream is to your left, and the railway is left of the stream. By the end of the tour, you cross the stream eight times and the railway five times. Can you determine what order the path, the stream, and the railway are in at the end of the tour? *The path, the stream, and the railway are side by side throughout the tour, continuing in the same direction. The railway could cross the stream.*

(Solution)

3. Put operation symbols $(+, -, \cdot, /)$ and brackets on the left side of the following equalities such that they become true. *An operation symbol must be placed between every pair of digits.*

a) $7\quad 6\quad 5\quad 4\quad 3\quad 2\quad 1 = 20$

b) $8\quad 7\quad 6\quad 5\quad 4\quad 3\quad 2\quad 1 = 202$

c) $9\quad 8\quad 7\quad 6\quad 5\quad 4\quad 3\quad 2\quad 1 = 2022$

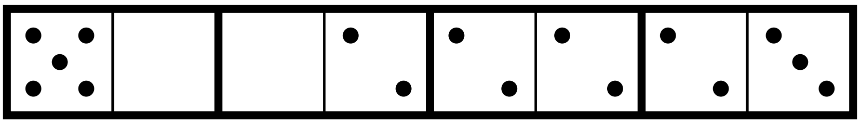
(Solution)

4. For the 90th birthday of Grandma, all her grandchildren have gathered. At the dinner, the grandchildren counted how many of their cousins are present, and shared this with the others. Every answer was 6 or 7. How many grandchildren can Grandma have?

Give an example for as many values as possible. When you think no other value is possible, explain why. It is possible that among the answers, 6 and 7 occurred, but it is also possible that every answer was the same.

(Solution)

5. In a domino set, the dominos are such that each half contains 0, 1, 2, 3, 4, 5, or 6 dots. There is one of each possible combination, so a total of 28 pieces. Dominika chose all dominos where there are at most 4 dots on the two parts in total.
a) How many dominos did Dominika select?
An arrangement of dominos is regular if the number of dots in neighboring dominos is equal. *For example, on the diagram below, some dominos are arranged in a regular way.*



b) How many ways can Dominika order the dominos she chose in a regular way? *Two orderings are different if from left to right, there is a domino part with a different number of dots in the two orderings.*

(Solution)

6. (Game) There are some coins in the heap, of value 1, 2, or 3 pengő. In a move, the player may take a coin from the heap and replace a coin of smaller value, or not replace anything. The winner is the one who takes the last coin from the heap.
Win against the organisers two times consecutively! At the beginning of the game, the organisers decide how many of each coin there is in the heap. Then, you may decide whether you wish to be the first or the second player.

(Solution)

1.3 Final round – day 2

1.3.1 Category A

1. Csongi duck dined on a dozen dozen lawn larvae and yet another dozen. How many dozen lawn larvae did Csongi duck devour?

(Solution)

2. Three ducks, Huey, Dewey, and Louie, have feathers decorating their hats. There are as many feathers on Huey's hat as there are found on Dewey's and Louie's altogether. On Dewey's hat, there are 4 more feathers than on Louie's hat. How many feathers are there on Dewey's hat, if 10 feathers decorate Huey's hat?

(Solution)

3. Three girls and three boys participate in a lottery game. A few of the children win a present, a few of them do not. We know that of the six children, an equal number of girls and boys win presents. At the end of the game, they list the names of those who have won presents. How many different lists of winners could there be, if we know that at least one of the children won a present? *It does not matter what order the children appear in the list.*

(Solution)

4. The sheriff's son collected sticks to duel a catus. During the duel whenever he hit the cactus with a stick, the stick broke into four pieces. How many possibilities are there for the number of sticks the boy collected originally if after the duel he had 17? *The boy hit the cactus at least once during the duel. The broken pieces of sticks are still considered sticks*

(Solution)

5. Waddles, the clever duck, is asked by his parents to improve his swimming practice over the following two weeks. Compared to this, Waddles swims a total of 1 metre on the first day, Monday. As his parents are angry at him, he makes the following promise: although he will not swim a single metre on weekends, he will always swim twice the previous day's distance each following day. How many metres does Waddles swim over the two weeks in all if he keeps his promise?

(Solution)

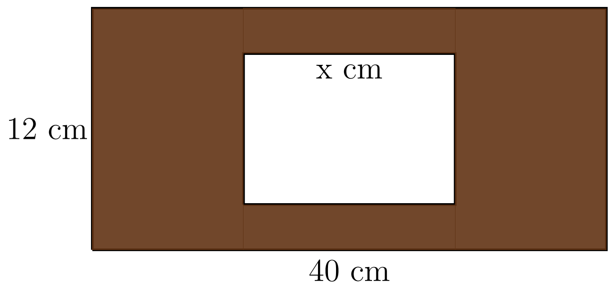
6. Five friends, Andris, Anett, Kartal, Kristóf and Zsófi collected some flat pebbles in a bucket, then they started to skip stones with them at a lake. First, it was Andris who threw 7 pebbles into the water, then Anett threw the third of the remaining ones in the lake. Then Kartal came with 4 throws and afterwards Kristóf threw as many pebbles in the water that only fourth of the pebbles remained of what had been in the bucket before his throw. Thus, there were only 2 pebbles left for Zsófi. How many pebbles did the friends have altogether?

(Solution)

7. Gábor collected all four-digit numbers with 15 appearing as two consecutive digits. How many numbers did Gábor collect?

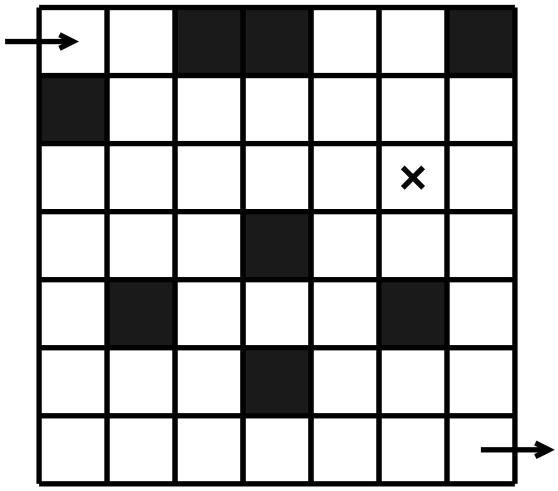
(Solution)

8. Mister Hamster's favorite chocolate is the one with holes in it. The chocolate is made by cutting a smaller rectangle from a 12 cm × 40 cm rectangular chocolate. We know that the smaller rectangle has integer side lengths, that its perimeter is half that of the larger one, and that its area is a third of it. How many centimetres is the side x of the smaller rectangle? *The removed rectangle has side lengths parallel to those of the larger one. Note that the diagram is not proportionate.*



(Solution)

9. Adam mows the lawn in his garden. The garden is drawn as the 7×7 chart on the diagram. The grass needs trimming on the white fields, while the black fields have rose bushes, and those fields are to be avoided. The arrows signal where Adam begins and finishes the lawn mowing. In order to finish as soon as possible, he wants to moves through the garden such that each field is visited exactly once. When he trims the grass on the field marked with an X, how many fields has he mowed? *Adam mows the lawn on every field he visits. He moves from field to field by crossing an edge.*



(Solution)

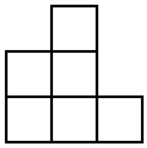
10. Adam, Benedict, Calvin, Daniel, and Hannah take part in an athletic contest. They each present a series of exercises, to be awarded points by the jury. At the end of the contest, it was observed that every contestant reached either first or final place following their exercises. How many ways could the five contestants be placed if they presented in the given order? *They were all given different amounts of points.*

(Solution)

11. Bugs Bunny thought of a positive integer less than 20, which Dodo duck wishes to guess in as few questions as possible. In one question, Dodo duck may ask whether Bugs Bunny thought of a certain number, for example: "Did you think of 13?". Bugs Bunny answers every question with yes or no. At least how many questions does Dodo duck need for finding out the number with certainty, if Bugs Bunny may lie at most once?

(Solution)

12. Lucy made the tower of cubes on the diagram from 6 congruent little cubes. In how many ways could she assemble the cubes? *The diagram depicts the tower viewed from the side. Lucy placed the cubes one after the other.*



(Solution)

13. On the railway from Duckton to Duckaster, trains depart from either city at every hour. Tony travelled from Duckton but realised that he left something there, so he stepped off the train at Duckchurch and boarded the train back. Toby fared similarly, who started from Duckaster but turned back at Duckchurch, having forgotten something. One travel took 100 minutes, and the other took 140 minutes. How many minutes is the train ride from Duckton to Duckaster?

Both ducks boarded the first train back at Duckchurch, where the trains do not stop at the same time. The trains move at a constant speed.

(Solution)

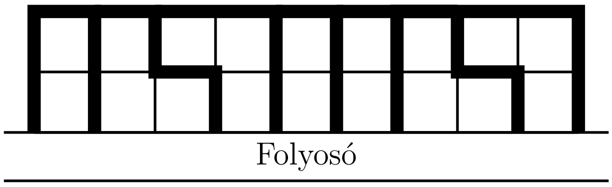
14. A 500-calory packet of mixed nuts is made up of three ingredients: 51 peanuts, 33 raisins, and 16 cashews. How many calories is one cashew if it contains as much calories as a peanut and a raisin in total? *For each ingredient, every piece has the same calory content, given by a positive integer.*

(Solution)

15. 100 chameleons sit in a row on a long tree branch, each looking left or right, only seeing the chameleon before them. Initially, every chameleon has a color, but each time a bird flies over the tree, the chameleon takes on the color of the chameleon it saw before the bird arrived. After 100 birds flew over the branch, the 100 chameleons sported a total of 37 colors. At most how many chameleons are looking right? *If a chameleon at the end of the branch does not see anyone, then naturally, its color does not change.*

(Solution)

16. The basic design of a house is a 2×9 rectangle where the squares each mean one room. The corridor is on one side of the rectangle. We would like to partition the house into 2-room and 3-room apartments. These apartments must be connected, and must have a doorway to the corridor. How many different partitions are possible? *The following diagram shows a possible partition. Two partitions are distinct if there are two neighboring rooms belonging to the same house in one partition but not in the other.*



(Solution)

1.3.2 Category B

1. Csongi duck dined on a dozen dozen lawn larvae and yet another dozen. How many dozen lawn larvae did Csongi duck devour?

(Solution)

2. Three girls and three boys participate in a lottery game. A few of the children win a present, a few of them do not. We know that of the six children, an equal number of girls and boys win presents. At the end of the game, they list the names of those who have won presents. How many different lists of winners could there be, if we know that at least one of the children won a present? *It does not matter what order the children appear in the list.*

(Solution)

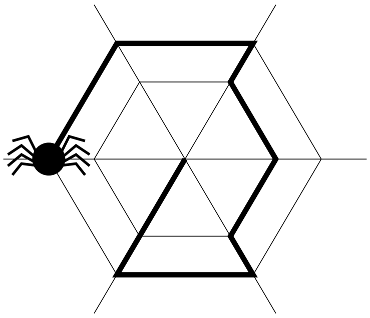
3. The sheriff's son collected sticks to duel a catus. During the duel whenever he hit the cactus with a stick, the stick broke into four pieces. How many possibilities are there for the number of sticks the boy collected originally if after the duel he had 17? *The boy hit the cactus at least once during the duel. The broken pieces of sticks are still considered sticks*

(Solution)

4. Waddles, the clever duck, is asked by his parents to improve his swimming practice over the following two weeks. Compared to this, Waddles swims a total of 1 metre on the first day, Monday. As his parents are angry at him, he makes the following promise: although he will not swim a single metre on weekends, he will always swim twice the previous day's distance each following day. How many metres does Waddles swim over the two weeks in all if he keeps his promise?

(Solution)

5. On the diagram, the spider moves 12 centimetres from the centre of the web. How many centimetres is the thickened path? *Every triangle on the diagram is regular.*

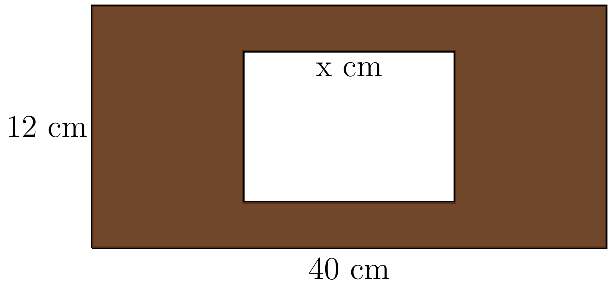


(Solution)

6. Rainville, Cloudford, and Stormland are three villages along a straight line, in this order. Cloudford is 10 kilometers from Rainville and Stormland. Ann lives in Rainville, but she is now in Cloudford, visiting a girlfriend. They were playing a board game when Ann checked the weather forecast. She saw that a raincloud just arrived in Stormland, nearing Cloudford and Rainville at a rate of 20 km per hour. How many minutes does Ann have to head home if she can bike at a speed of 15 km per hour, intending to make it home before the downpour?

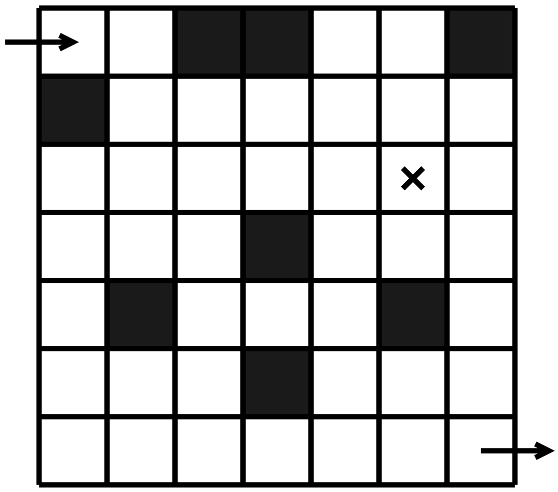
(Solution)

7. Mister Hamster's favorite chocolate is the one with holes in it. The chocolate is made by cutting a smaller rectangle from a 12 cm × 40 cm rectangular chocolate. We know that the smaller rectangle has integer side lengths, that its perimeter is half that of the larger one, and that its area is a third of it. How many centimetres is the side x of the smaller rectangle? *The removed rectangle has side lengths parallel to those of the larger one. Note that the diagram is not proportionate.*



(Solution)

8. Adam mows the lawn in his garden. The garden is drawn as the 7×7 chart on the diagram. The grass needs trimming on the white fields, while the black fields have rose bushes, and those fields are to be avoided. The arrows signal where Adam begins and finishes the lawn mowing. In order to finish as soon as possible, he wants to moves through the garden such that each field is visited exactly once. When he trims the grass on the field marked with an X, how many fields has he mowed? *Adam mows the lawn on every field he visits. He moves from field to field by crossing an edge.*



(Solution)

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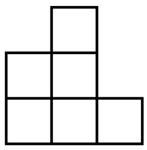
(Solution)

10. Lucy made the tower of cubes on the diagram from 6 congruent little cubes. In how many ways could she assemble the cubes? *The diagram depicts the tower viewed from the side. Lucy placed the cubes one after the other.*



(Solution)

11. On the railway from Duckton to Duckaster, trains depart from either city at every hour. Tony travelled from Duckton but realised that he left something there, so he stepped off the train at Duckchurch and boarded the train back. Toby fared similarly, who started from Duckaster but turned back at Duckchurch, having forgotten something. One travel took 100 minutes, and the other took 140 minutes. How many minutes is the train ride from Duckton to Duckaster?

Both ducks boarded the first train back at Duckchurch, where the trains do not stop at the same time. The trains move at a constant speed.

(Solution)

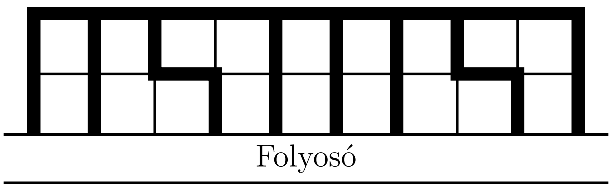
12. A 500-calory packet of mixed nuts is made up of three ingredients: 51 peanuts, 33 raisins, and 16 cashews. How many calories is one cashew if it contains as much calories as a peanut and a raisin in total? *For each ingredient, every piece has the same calory content, given by a positive integer.*

(Solution)

13. 100 chameleons sit in a row on a long tree branch, each looking left or right, only seeing the chameleon before them. Initially, every chameleon has a color, but each time a bird flies over the tree, the chameleon takes on the color of the chameleon it saw before the bird arrived. After 100 birds flew over the branch, the 100 chameleons sported a total of 37 colors. At most how many chameleons are looking right? *If a chameleon at the end of the branch does not see anyone, then naturally, its color does not change.*

(Solution)

14. The basic design of a house is a 2×9 rectangle where the squares each mean one room. The corridor is on one side of the rectangle. We would like to partition the house into 2-room and 3-room apartments. These apartments must be connected, and must have a doorway to the corridor. How many different partitions are possible? *The following diagram shows a possible partition. Two partitions are distinct if there are two neighboring rooms belonging to the same house in one partition but not in the other.*



(Solution)

15. Anett has a 1000-page novel to read over the summer vacation. She comes up with the following pastime: after reading a page, she circles the page number

- in red if it is divisible by 7,
- in yellow if it contains a digit 7,
- in green if it is divisible by 11,
- in blue if it contains a digit 1.

If more of the four conditions hold, then she circles the number with all the suitable colors. After Anett finished reading the book, Andris opened it somewhere and found that both page numbers are circled with three colors. What is the sum of these page numbers? *Opening the book, the even page numbers are always on the left, and the odd page numbers are always on the right.*

(Solution)

16. We have three large boxes of cookies: one chocolate, one honey, and one strawberry flavored. Each box contains 20 cookies. We take out a total of 8 pieces from them and arrange them in the shape of a flower: one in the middle, the rest around. How many different flowers can we obtain this way, if rotations are seen as indistinct?

(Solution)

2 Problems – high school categories

2.1 Online round

2.1.1 Category C

1. Imagine a building with a ground floor and five further floors. (Now the first floor is the one *above* the ground floor.)
The ground floor, the second floor and the fourth floor have 19 rooms each, the rest of the floors have 21 rooms each. Each room has 3 windows. How many windows does the building have?

(Solution)

2. Benedek went to the meadow looking for 4-leaf clovers. He found three heaps but he could not see the stems of the clovers. Because of this, he could only count the number of clover leaves in each of the heaps, these counts were 42, 37, and 32. At least how many 4-leaf clovers were in the three heaps in total, if every clover has either 3 or 4 leaves.

(Solution)

3. Anett has a diary, and she sometimes writes down what she did on that day. Unfortunately, she only writes down the days of the week (and even those not always), but not the date.

Marvin stole the diary, and the days with something written next to them were the following: Wednesday ... Thursday ... Tuesday ... Friday ... Saturday ... Wednesday ... Sunday ... Saturday ... Thursday (Dürer online round!)

If we know that the day of the Dürer online round is today, at least how many days ago did Anett start writing her diary?

(Solution)

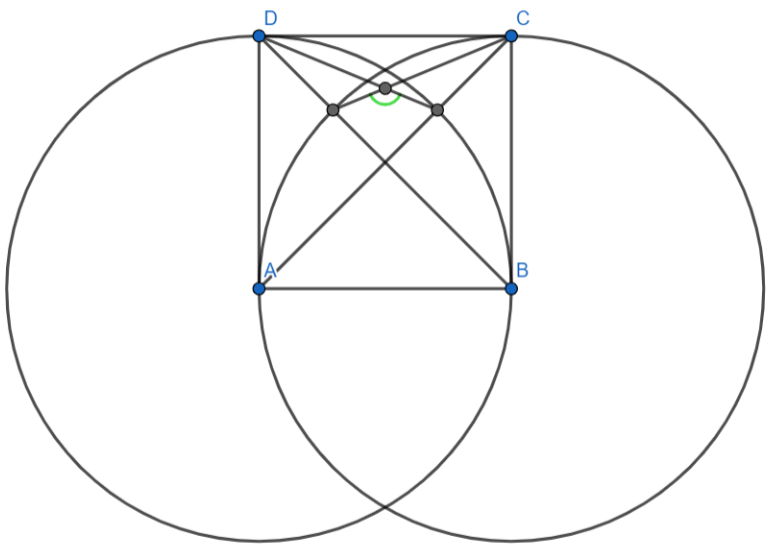
4. Csenge draw four lines on a paper. Then Csongi picked 4 different points on the paper, and for each of them he wrote down the number of lines going through this point. What is the maximal possible sum of these four numbers?

(Solution)

5. Anett, Andris, and Orsi went for a bike ride. They went for a 45 km long trip, and they have three bicycles: a yellow, a blue, and a red one. [with one bicycle pedal turn– na ez hogy van angolul?? – esetleg one revolution of the bicycle pedal?] the blue bike can go 1 meter, the yellow one goes 2.5 meters and the red bike can go 1.5 meters. Anett can go 3 pedal turns in a second, Orsi can make 5, and Andris can make 2 pedal turns. How many minutes are needed until the slowest member of the team arrives at the destination, if they are free to choose bikes?

(Solution)

6. How many degrees is the angle marked on the picture? For square $ABCD$ we drew two circles around A and B with radius AB. Then we connected the intersection points of the diagonals and the circles to nodes C and D.



(Solution)

7. The sum of a single-digit number divisible by one, a two-digit number divisible by two, a 3-digit number divisible by 3, and a 4-digit number divisible by 4 is a 5-digit number divisible by 5. What is the smallest possible value of the 4-digit number?

(Solution)

8. Máté is rolling a die on a checkered paper, he always rolls it to a neigbouring side and each number appears on the top exactly once. He always writes the number on top of the die on a piece of paper. How many ways can the six-digit number on Máté's paper look like, if we know that in the beginning the number on top was 1?

(Solution)

9. Scrooge McDuck keeps his wealth in a big safe, but he doesn't trust anyone, so only he knows the opening code. To make sure he won't forget it, he makes a riddle, where he needs to write the numbers from 1 to 9 in the empty white fields in a way that he can use a number only once. From this table, read the three 3-digit numbers in the rows, and add them together. This way we get the secret code. What is Scrooge McDuck's code?

You need to fill out the table in a way that the equations are true in the rows and columns . The / sign means division, × means multiplication. Pay attention to the order of operations!

	/		×		=	3
+		×		+		
	/		×		=	28
+		/		−		
	×		+		=	29
=		=		=		
20		1		4		

(Solution)

10. Game: A lot of people know how a knight moves on a chessboard, but the move of a duck in much less well-known: a duck can move to any of the four neighboring squares. Two players alternatingly place ducks on a 4×6 table in a way that new duck should not attack any of the other pieces. The player who cannot place a duck loses.

Beat the computer twice in this game! You can decide whether you want to be the first or second player.

(Solution)

2.1.2 Category D

1. Benedek went to the meadow looking for 4-leaf clovers. He found three heaps but he could not see the stems of the clovers. Because of this, he could only count the number of clover leaves in each of the heaps, these counts were 42, 37, and 32. At least how many 4-leaf clovers were in the three heaps in total, if every clover has either 3 or 4 leaves.

(Solution)

2. Anett has a diary, and she sometimes writes down what she did on that day. Unfortunately, she only writes down the days of the week (and even those not always), but not the date.

Marvin stole the diary, and the days with something written next to them were the following: Wednesday ... Thursday ... Tuesday ... Friday ... Saturday ... Wednesday ... Sunday ... Saturday ... Thursday (Dürer online round!)

If we know that the day of the Dürer online round is today, at least how many days ago did Anett start writing her diary?

(Solution)

3. Anett, Andris, and Orsi went for a bike ride. They went for a 45 km long trip, and they have three bicycles: a yellow, a blue, and a red one. [with one bicycle pedal turn– na ez hogy van angolul?? – esetleg one revolution of the bicycle pedal?] the blue bike can go 1 meter, the yellow one goes 2.5 meters and the red bike can go 1.5 meters. Anett can go 3 pedal turns in a second, Orsi can make 5, and Andris can make 2 pedal turns. How many minutes are needed until the slowest member of the team arrives at the destination, if they are free to choose bikes?

(Solution)

4. On Béluska's keyboard, the number keys 1, 2, 3, 4, 5, 6, 7, 8, 9 are in this order next to each other. We call a number *easy-to-type* if all of its digits are different and it can be written with consecutive keys, going from left to right or right to left. The single-digit numbers are all easy-to-type. Béluska wants to send an easy-to-type, at most 3 digit prime number to Lizuska. How many possible numbers can he send?

(Solution)

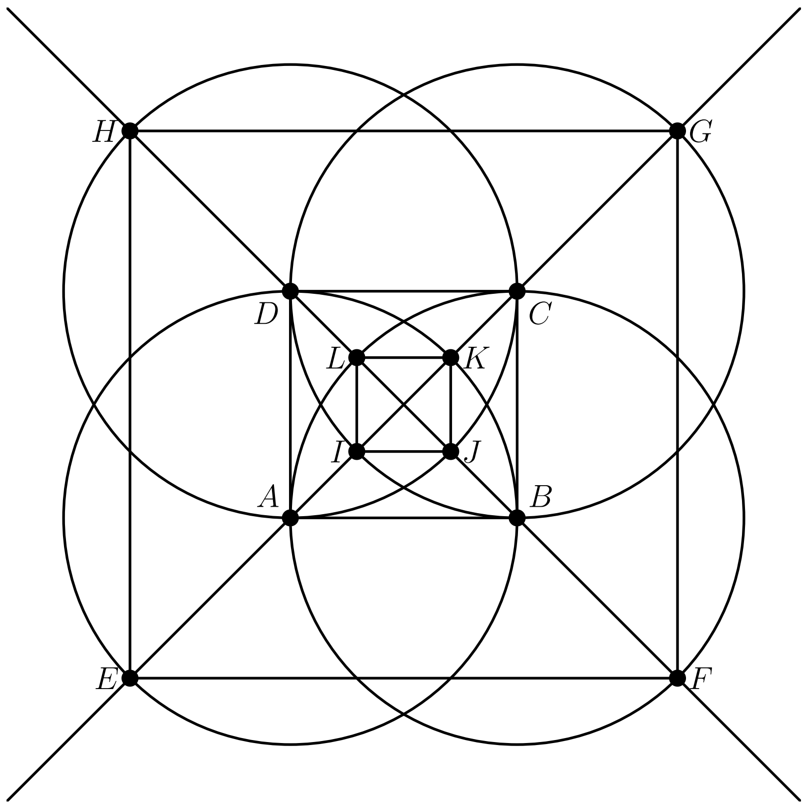
5. The sum of a single-digit number divisible by one, a two-digit number divisible by two, a 3-digit number divisible by 3, and a 4-digit number divisible by 4 is a 5-digit number divisible by 5. What is the smallest possible value of the 4-digit number?

(Solution)

6. The 10 members of a duck family are standing in a circle, where the places are marked with $0, 1, \ldots 9$. To confuse their swan friends, they start swapping places the following way: First the ducks on place 0 and 1 switch places, then the ducks standing on places 1 and 2, and so on, finally the ducks standing on places 9 and 0. Then they repeat the same process again and again. At least how many swaps do we need until each family member is standing on their own original place again?

(Solution)

7. The side lenght of square $ABCD$ is 3 units. In the picture, we drew the circles with radii 3 around centres A, B, C and D, and the two diagonals of square $ABCD$. The intersection points of these diagonals and circles are E, F, G, H, I, J, K and L. What is the product of the area of square $EFGH$ and the area of square $IJKL$?



(Solution)

8. Samantha Swan, Susie Swan, Giselle Goose, Gabriel Goose, George Goose, Danny Duck, Daisy Duck, and Donald Duck are making a family photo. They will stand in two rows, four in each row, in a way that the taller ones should not cover the short ones. Every swan is 1.5 meter tall, every goose is 1 m, and every duck is 0,5 m. How many ways can they stand?

Animals of equal height cover each other too.

(Solution)

9. Scrooge McDuck keeps his wealth in a big safe, but he doesn't trust anyone, so only he knows the opening code. To make sure he won't forget it, he makes a riddle, where he needs to write the numbers from 1 to 9 in the empty white fields in a way that he can use a number only once. From this table, read the three 3-digit numbers in the rows, and add them together. This way we get the secret code. What is Scrooge McDuck's code?

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+		×		+		
	/		×		=	28
+		/		−		
	×		+		=	29
=		=		=		
20		1		4		

(Solution)

10. Game: There are n stones in a heap. Two players alternatingly remove some stones. The number of stones removed in one step has to be a power of two. The player who takes the last stone wins.

Beat the computer twice in this game! Knowing the initial state, you can decide whether you want to be the first or second player.

(Solution)

2.1.3 Category E

1. Anett has a diary, and she sometimes writes down what she did on that day. Unfortunately, she only writes down the days of the week (and even those not always), but not the date.

Marvin stole the diary, and the days with something written next to them were the following: Wednesday ... Thursday ... Tuesday ... Friday ... Saturday ... Wednesday ... Sunday ... Saturday ... Thursday (Dürer online round!)

If we know that the day of the Dürer online round is today, at least how many days ago did Anett start writing her diary?

(Solution)

2. Anett, Andris, and Orsi went for a bike ride. They went for a 45 km long trip, and they have three bicycles: a yellow, a blue, and a red one. [with one bicycle pedal turn– na ez hogy van angolul?? – esetleg one revolution of the bicycle pedal?] the blue bike can go 1 meter, the yellow one goes 2.5 meters and the red bike can go 1.5 meters. Anett can go 3 pedal turns in a second, Orsi can make 5, and Andris can make 2 pedal turns. How many minutes are needed until the slowest member of the team arrives at the destination, if they are free to choose bikes?

(Solution)

3. On Béluska's keyboard, the number keys 1, 2, 3, 4, 5, 6, 7, 8, 9 are in this order next to each other. We call a number *easy-to-type* if all of its digits are different and it can be written with consecutive keys, going from left to right or right to left. The single-digit numbers are all easy-to-type. Béluska wants to send an easy-to-type, at most 3 digit prime number to Lizuska. How many possible numbers can he send?

(Solution)

4. The sum of a single-digit number divisible by one, a two-digit number divisible by two, a 3-digit number divisible by 3, and a 4-digit number divisible by 4 is a 5-digit number divisible by 5. What is the smallest possible value of the 4-digit number?

(Solution)

5. The 10 members of a duck family are standing in a circle, where the places are marked with $0, 1, \ldots 9$. To confuse their swan friends, they start swapping places the following way: First the ducks on place 0 and 1 switch places, then the ducks standing on places 1 and 2, and so on, finally the ducks standing on places 9 and 0. Then they repeat the same process again and again. At least how many swaps do we need until each family member is standing on their own original place again?

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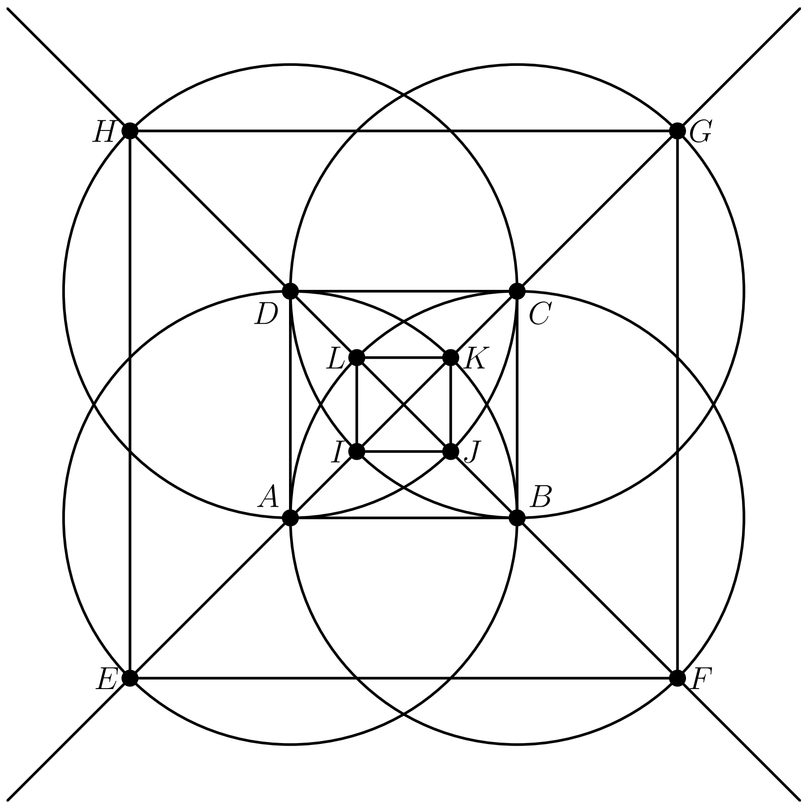
6. The side lenght of square $ABCD$ is 3 units. In the picture, we drew the circles with radii 3 around centres A, B, C and D, and the two diagonals of square $ABCD$. The intersection points of these diagonals and circles are E, F, G, H, I, J, K and L. What is the product of the area of square $EFGH$ and the area of square $IJKL$?



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Animals of equal height cover each other too.

(Solution)

8. Csongi really likes watching the ducks at the lakeside. He noticed that each duck goes under water for 20 seconds, then spends 30 seconds looking around above the water, then goes under water for 20 seconds again, and so on. Today, when he went duck-watching, he noted down how many duck heads he is seeing, then again $10, 20, 30$ and 40 seconds later. He got the following numbers: $24, 22, 20, 17, 25$. How many ducks are in the lake?

(Solution)

9. Scrooge McDuck keeps his wealth in a big safe, but he doesn't trust anyone, so only he knows the opening code. To make sure he won't forget it, he makes a riddle, where he needs to write the numbers from 1 to 9 in the empty white fields in a way that he can use a number only once. From this table, read the three 3-digit numbers in the rows, and add them together. This way we get the secret code. What is Scrooge McDuck's code?

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(Solution)

2.2 Regional round

2.2.1 Category C

1. The following five cities are situated along a straight road in some order: Bácsfeketehegy, Kishegyes, Petrőc, Sóvé, and Topolya. Each city has an army, and any two cities have a different number of soldiers. A city can send a message to those cities which its army can reach along the road such that it does not pass through a city with a larger army. Watch out! The ability to send messages is not necessarily mutual. We know the following:

- Bácsfeketehegy has only one neighbour, and its army is larger than that of Sóvé;
- The army of Kishegyes numbers 4830, and it cannot send a message to Bácsfeketehegy;
- The city of Petrőc has the smallest army, but despite this, it can send a message to Topolya;
- Sóvé neighbors the fourth city along the road, and can mutually send messages with Bácsfeketehegy;
- Topolya is the second city along the road, and its army numbers 16171.

a) In what order are the cities along the path?
b) List the cities in decreasing order according to their army sizes.
Show your reasoning.

(Solution)

2. Dorothy organized a party for the birthday of Duck Mom and she also prepared a cylinder-shaped cake. Since she was originally expecting to have 15 guests, she divided the top of the cake into this many equal circular sectors, marking where the cuts need to be made. Just for fun Dorothy's brother Donald split the top of the cake into 10 equal circular sectors in such a way that some of the radii that he marked coincided with Dorothy's original markings. Just before the arrival of the guests Douglas cut the cake according to all markings, and then he placed the cake into the fridge.
This way they forgot about the cake and only got to eating it when only 6 of them remained. Is it possible for them to divide the cake into 6 equal parts without making any further cuts?

(Solution)

3. We know that a, b, c are distinct positive integers, such that $a \mid b \cdot c$, $b \mid a \cdot c$, $c \mid a \cdot b$, and $a + b + c$ is prime.
a) Give an example of such a triple a, b, c.
b) Prove that $a \cdot b \cdot c$ is a square number for all such triples.
The notation $k \mid n$ means that n is divisible by k.

(Solution)

4. We inscribed in triangle ABC the rectangle $DEFG$ such that D and E fall on side AB, F on side BC, and G on side AC. We know that AF bisects angle BAC, and that $\frac{AD}{DE} = \frac{1}{2}$. What is the measure of angle CAB?

(Solution)

5. Benedek filled a 3×3 table with the numbers $1, 1, 2, 3, 4, 5, 6, 7, 7$. After this, he wrote on a sheet of paper the sums of all pairs of numbers which are found in fields neighboring by an edge. This way, 12 numbers appeared on the paper in total. He noticed that the numbers he recorded were pairwise distinct.
a) Provide such a way to fill the table.
b) Is there a way to fill the table where not only the 12 sums so far, but also the column and row sums, in total 18 sums, are all distinct?

(Solution)

2.2.2 Category D

1. The following five cities are situated along a straight road in some order: Bácsfeketehegy, Kishegyes, Petrőc, Sóvé, and Topolya. Each city has an army, and any two cities have a different number of soldiers. A city can send a message to those cities which its army can reach along the road such that it does not pass through a city with a larger army. Watch out! The ability to send messages is not necessarily mutual. We know the following:

- The city of Topolya has only one neighbour, and its army is larger than that of Petrőc;
- Sóvé cannot send a message to Topolya or Kishegyes;
- The army of Bácsfeketehegy numbers 3945, and it can send a message to Kishegyes;
- Petrőc neighbors the fourth city, and its army is larger than that of Bácsfeketehegy;
- Topolya and Petrőc can mutually send messages;
- The army of Kishegyes numbers 4830.

a) In what order are the cities along the path?
b) List the cities in decreasing order according to their army sizes.
Show your reasoning.

(Solution)

2. We know that a, b, c are distinct positive integers, such that $a \mid b \cdot c$, $b \mid a \cdot c$, $c \mid a \cdot b$, and $a + b + c$ is prime.
a) Give an example of such a triple a, b, c.
b) Prove that $a \cdot b \cdot c$ is a square number for all such triples.
The notation $k \mid n$ means that n is divisible by k.

(Solution)

3. We inscribed in triangle ABC the rectangle $DEFG$ such that D and E fall on side AB, F on side BC, and G on side AC. We know that AF bisects angle BAC, and that $\frac{AD}{DE} = \frac{1}{2}$. What is the measure of angle CAB?

(Solution)

4. Benedek filled a 3×3 table with the numbers $1, 1, 2, 3, 4, 5, 6, 7, 7$. After this, he wrote on a sheet of paper the sums of all pairs of numbers which are found in fields neighboring by an edge. This way, 12 numbers appeared on the paper in total. He noticed that the numbers he recorded were pairwise distinct.
a) Provide such a way to fill the table.
b) Is there a way to fill the table where not only the 12 sums so far, but also the column and row sums, in total 18 sums, are all distinct?

(Solution)

5. What is the least number of groups the numbers $1, 2, 3, \ldots, 100$ can be divided into such that within every group, the numbers are pairwise coprime or pairwise non-coprime?
Two numbers are coprime if their greatest common divisor is 1*. Also show your reasoning.*

(Solution)

2.2.3 Category E

1. Dorothy organized a party for the birthday of Duck Mom and she also prepared a cylinder-shaped cake. Since she was originally expecting to have 15 guests, she divided the top of the cake into this many equal circular sectors, marking where the cuts need to be made. Just for fun Dorothy's brother Donald split the top of the cake into 10 equal circular sectors in such a way that some of the radii that he marked coincided with Dorothy's original markings. Just before the arrival of the guests Douglas cut the cake according to all markings, and then he placed the cake into the fridge.
This way they forgot about the cake and only got to eating it when only 6 of them remained. Is it possible for them to divide the cake into 6 equal parts without making any further cuts?

(Solution)

2. In the acute triangle ABC the circle through B touching the line AC at A has centre P, the circle through A touching the line BC at B has centre Q. Let R and O be the circumradius and circumcentre of triangle ABC, respectively. Show that $R^2 = OP \cdot OQ$.

(Solution)

3. Paraflea makes jumps on the plane, starting from the origin $(0,0)$. From point (x,y) it may jump to another point of the form $(x+p,\ y+p^2)$, where p is any positive real number. (The value of p may differ for each jump.)
a) Is there any point in quadrant I which cannot be reached by the flea? (Quadrant I contains points (x,y) for which x and y are positive real numbers.)
b) What is the minimum number of jumps that the flea must make from the origin so that it gets to the point $(100,1)$?

(Solution)

4. We want to partition the integers $1,2,3,\ldots,100$ into several groups such that within each group either any two numbers are coprime or any two are not coprime. At least how many groups are needed for such a partition?
We call two integers coprime if they have no common divisor greater than 1*.*

(Solution)

5. **a)** A game master divides a group of 12 players into two teams of six. The players do not know what the teams are, however the master gives each player a card containing the names of two other players: one of them is a teammate and the other is not, but the master does not tell the player which is which. Can the master write the names on the cards in such a way that the players can determine the teams? (All of the players can work together to do so.)

b) On the next occasion, the game master writes the names of 3 teammates and 1 opposing player on each card (possibly in a mixed up order). Now he wants to write the names in such a way that the players together cannot determine the two teams. Is it possible for him to achieve this?

c) Can he write the names in such a way that the players together cannot determine the two teams, if now each card contains the names of 4 teammates and 1 opposing player (possibly in a mixed up order)?

(Solution)

2.2.4 Category E$^+$

1. We want to partition the integers $1,2,3,\ldots,100$ into several groups such that within each group either any two numbers are coprime or any two are not coprime. At least how many groups are needed for such a partition?
We call two integers coprime if they have no common divisor greater than 1*.*

(Solution)

2. Determine all triangles that can be split into two congruent pieces by one cut. A cut consists of segments P_1P_2, P_2P_3, ..., $P_{n-1}P_n$ where points $P_1, P_2, \ldots, P_n$ are distinct, points P_1 and P_n lie on the perimeter of the triangle and the rest of the points lie in the interior of the triangle such that the segments are disjoint except for the endpoints.

(Solution)

3. a) A game master divides a group of 40 players into four teams of ten. The players do not know what the teams are, however the master gives each player a card containing the names of two other players: one of them is a teammate and the other is not, but the master does not tell the player which is which. Can the master write the names on the cards in such a way that the players can determine the teams? (All of the players can work together to do so.)

b) On the next occasion, the game master writes the names of 7 teammates and 2 opposing players on each card (possibly in a mixed up order). Now he wants to write the names in such a way that the players together cannot determine the four teams. Is it possible for him to achieve this?

c) Can he write the names in such a way that the players together cannot determine the four teams, if now each card contains the names of 6 teammates and 2 opposing players (possibly in a mixed up order)?

(Solution)

4. Let ABC be an acute triangle, and let F_A and F_B be the midpoints of sides BC and CA, respectively. Let E and F be the intersection points of the circle centered at F_A and passing through A and the circle centered at F_B and passing through B. Prove that if segments CE and CF have midpoints N and M, respectively, then the intersection points of the circle centered at M and passing through E and the circle centered at N and passing through F lie on the line AB.

(Solution)

5. Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be real numbers for which

$$\sum_{i=1}^{n} a_i^{2k+1} = 0$$

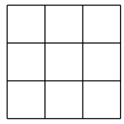
holds for all integers $0 \leq k < n$. Show that in this case, $a_i = -a_{n+1-i}$ holds for all $1 \leq i \leq n$.

(Solution)

2.3 Final round – day 1

2.3.1 Category C

1. Csongor is drawing squares on a piece of paper (any two squares may overlap). After drawing some, the following grid has been made. At least how many squares did he need to draw in order to get this?



(Solution)

2. Pisti was quite bored during quarantine and decided to write numbers on a piece of paper. He started with the number 11 and then he followed the rules below. Once he has written down the number x then he could write

- the sum of the digits of x,
- $2(x+1)+1$,
- or $x+4$.

a) Show that with using only the steps above Pisti could write down any positive integer.
b) When Pisti only used the first and the second steps, no matter how hard he tried, he could not write down 2022. Show that it was not Pisti's fauilt and that it was indeed impossible to get 2022 this way.

(Solution)

3. To the exterior of side AB of square $ABCD$, we have drawn the regular triangle ABE. Point A reflected on line BE is F, and point E reflected on line BF is G. Let the perpendicular bisector of segment FG meet segment AD at X. Show that the circle centered at X with radius XA touches line FB.

(Solution)

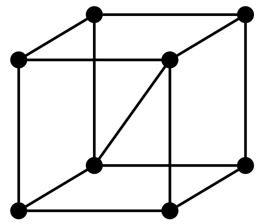
4. a) Are there 12 consecutive positive integers whose sum is a square number?
b) Are there 11 consecutive positive integers whose sum is a square number?
c) Are there 10 consecutive positive integers whose sum is a square number?
Show an example, or prove that they do not exist.

(Solution)

5. Donna Duck layed her very first egg on her first birthday, since then she lays exactly one egg each day, always either brown, grey or orange. Every time she lays an egg she writes in her notebook how many days ago she last layed an egg of the same color, even using a pen of the same color as the current egg (when she lays her first egg of either color she doesn't write anything.) Donna realised that so far she hadn't written the same number twice in the same color. Maximum how many eggs could she have layed in her life until now?

(Solution)

6. Game: Given is the skeleton of a cuboid in which one of the solid diagonals is drawn. Within a move the upcoming player colours one of the yet uncoloured vertices with one of three colours (meaning that he puts a red, yellow or blue disc on top of the vertex) in a way that no adjacent vertices can have the same colour. The game ends when the next player cannot make a move any longer. The player who started wins if all the vertices have been coloured, whereas the second player wins if there remain uncoloured vertices.



Beat the organizers in this game twice in a row! It is up to you if you would rather be the first or the second player.

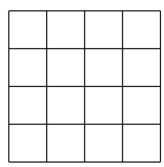
(Solution)

2.3.2 Category D

1. The Duckcongress annually elects a Duckdelegate. There is however an ancient rule that for a consecutive years the delegate has to be a wild duck. Then, for b years a domestic duck is elected, followed by a wild duck for a years and a domestic duck for b years, and so on. (a and b denote positive integers). We know that in 1999, 2004, 2005 and 2011 a wild duck was elected, whereas in 2010, 2015 and this year in 2022 a domestic duck was appointed. For how many years does the wild duck need to wait to get back his respectable position as the duckdelegate? *Find all the possible solutions and show that there are no other.*

(Solution)

2. Orsi is drawing squares on a piece of paper (any two squares may overlap). After drawing some, the following grid has been made. At least how many squares did she need to draw in order to get this?



(Solution)

3. a) Are there 20 consecutive positive integers whose sum is a square number?
b) Are there 21 consecutive positive integers whose sum is a square number?
c) Are there 2022 consecutive positive integers whose sum is a square number?
Show an example, or prove that they do not exist.

(Solution)

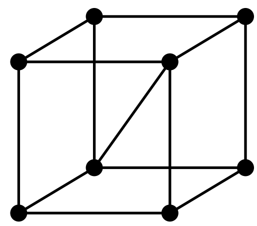
4. The longer base of trapezoid $ABCD$ is AB, while the shorter base is CD. Diagonal AC bisects the interior angle at A. The interior bisector at B meets diagonal AC at E. Line DE meets segment AB at F. Suppose that $AD = FB$ and $BC = AF$. Find the interior angles of quadrilateral $ABCD$, if we know that $BEC\sphericalangle = 54°$.

(Solution)

5. Donna Duck layed her very first egg on her first birthday, since then she lays exactly one egg each day, always either brown, grey or orange. Every time she lays an egg she writes in her notebook how many days ago she last layed an egg of the same color, even using a pen of the same color as the current egg (when she lays her first egg of either color she doesn't write anything.) Donna realised that so far she hadn't written the same number twice in the same color. Maximum how many eggs could she have layed in her life until now?

(Solution)

6. Game: Given is the skeleton of a cuboid in which one of the solid diagonals is drawn. Within a move the upcoming player colours one of the yet uncoloured vertices with one of three colours (meaning that he puts a red, yellow or blue disc on top of the vertex) in a way that no adjacent vertices can have the same colour. The game ends when the next player cannot make a move any longer. The player who started wins if all the vertices have been coloured, whereas the second player wins if there remain uncoloured vertices.



Beat the organizers in this game twice in a row! It is up to you if you would rather be the first or the second player.

(Solution)

2.3.3 Category E

1. To the exterior of side AB of square $ABCD$, we have drawn the regular triangle ABE. Point A reflected on line BE is F, and point E reflected on line BF is G. Let the perpendicular bisector of segment FG meet segment AD at X. Show that the circle centered at X with radius XA touches line FB.

(Solution)

2. Anett is drawing X-es on a 5×5 grid. For each newly drawn X she gets points in the following way: She checks how many X-es there are in the same row (including the new one) that can be reached from the newly drawn X with horizontal steps, moving only on fields that were previously marked with X-es. For the vertical X-es, she gets points the same way.
a) What is the maximum number of points that she can get with drawing 25 X-es?
b) What is the minimum number of points that she can get with drawing 25 X-es?
For example, if Anett put the X on the field that is marked with the circle, she would get 3 points for the horizontal fields and 1 point for the vertical ones. Thus, she would get 4 points in total.

			X	
		X	X	
		X		
X	○	X		X

(Solution)

3. n students, numbered from 1 to n are sitting next to each other in a class. In the beginning the 1st student has n pieces of paper in one pile. The goal of the students is to distribute the n pieces in a way that everyone gets exactly one. The teacher claps once in a mintue and for each clap the students can choose one of the following moves (or do nothing):

- They divide one of their piles of paper into two smaller piles.
- They give one of their piles of paper to the student with the next number.

At least how many times does the teacher need to clap in order to make it possible for the students to distribute all the pieces of paper amongst themselves?

(Solution)

4. Show that the divisors of a number $n \geq 2$ can only be divided into two groups in which the product of the numbers is the same if the product of the divisors of n is a square number.

(Solution)

5. Annie drew a rectangle and partitioned it into n rows and k columns with horizontal and vertical lines. Annie knows the area of the resulting $n \cdot k$ little rectangles while Benny does not. Annie reveals the area of some of these small recatngles to Benny. Given n and k at least how many of the small rectangle's areas did Annie have to reveal, if from the given information Benny can determine the areas of all the $n \cdot k$ little rectangles?
For example in the case $n = 3$ and $k = 4$ revealing the areas of the 10 small rectangles if enough information to find the areas of the remaining two little rectangles.

30	36	25	
48	56	40	72
42		35	63

(Solution)

6. Game: There are $n \leq 25$ cells in a row. Before the game gets started, a red disc is placed on the far left cell and a blue dics is placed on the far right cell. The player who makes the first move will be with the red discs, wheareas the second player will be with the blue ones. When making a move a player has three options:

- The player moves one of his own discs one or two cells away, on an empty cell (they may as well jump over another disc).
- The player places one of his own discs on an empty cell that is adjacent to a cell containing his own disc.
- The player passes and decides not to do anything.

After each move when a disc is placed on an empty cell, the adcjacent discs that are of opposite colour change and take up the colour of the new disc.

If at any point of the game there are more than $\frac{n}{2}$ red discs on the board, the player with the red discs will immediately win, whereas if there are at least $\frac{n}{2}$ blue discs, the blue player will immediately win. If this does not happen up until 200 moves, the blue player wins.

Beat the organisers in this game twice in a row! Knowing what number n is, you can decide if you would rather be the first or the second player.

(Solution)

2.3.4 Category E$^+$

1. Let $c \geq 2$ be a fixed integer. Let $a_1 = c$ and for all $n \geq 2$ let $a_n = c \cdot \varphi(a_{n-1})$. What are the c numbers for which sequence (a_n) will be bounded?
φ denotes Euler's Phi Function, meaning that $\varphi(n)$ gives the number of integers within the set $\{1, 2, \ldots, n\}$ that are relative primes to n.

We call a sequence (x_n) bounded if there exist a constant D such that $|x_n| < D$ for all positive integer n-s.

(Solution)

2. Annie drew a rectangle and partitioned it into n rows and k columns with horizontal and vertical lines. Annie knows the area of the resulting $n \cdot k$ little rectangles while Benny does not. Annie reveals the area of some of these small recatngles to Benny. Given n and k at least how many of the small rectangle's areas did Annie have to reveal, if from the given information Benny can determine the areas of all the $n \cdot k$ little rectangles?
For example in the case $n = 3$ and $k = 4$ revealing the areas of the 10 small rectangles if enough information to find the areas of the remaining two little rectangles.

30	36	25	
48	56	40	72
42		35	63

(Solution)

3. Let x, y, z denote positive real numbers for which $x+y+z = 1$ and $x > yz$, $y > zx$, $z > xy$. Prove that

$$\left(\frac{x-yz}{x+yz}\right)^2 + \left(\frac{y-zx}{y+zx}\right)^2 + \left(\frac{z-xy}{z+xy}\right)^2 < 1.$$

.

$$\left(\frac{x-yz}{x+yz}\right)^2 + \left(\frac{y-zx}{y+zx}\right)^2 + \left(\frac{z-xy}{z+xy}\right)^2 < 1.$$

(Solution)

4. $ABCD$ is a cyclic quadrilateral whose diagonals are perpendicular to each other. Let O denote the centre of its circumcircle and E the intersection of the diagonals. J and K denote the perpendicular projections of E on the AB and BC sides. Let F, G and H be the midpoint of the OE, AD and DC line segments. Show that lines GJ, FB and HK either pass through the same point or are parallel to each other.

(Solution)

5. There are n people sitting at a round table. In the beginning, everyone writes down a positive integer on piece of paper in front of them. From now on, in every minute, they write down the number that they get if they subtract the number of their right-hand neighbour from their own number. They write down the new number and erase the original. Give those numbers n for which there exists an integer k in a way that regardless of the starting numbers, after k minutes, everyone will have a number that is divisible by n.

(Solution)

6. Game: There are $n \leq 25$ cells in a row. Before the game gets started, a red disc is placed on the far left cell and a blue dics is placed on the far right cell. The player who makes the first move will be with the red discs, wheareas the second player will be with the blue ones. When making a move a player has three options:

- The player moves one of his own discs one or two cells away, on an empty cell (they may as well jump over another disc).
- The player places one of his own discs on an empty cell that is adjacent to a cell containing his own disc.
- The player passes and decides not to do anything.

After each move when a disc is placed on an empty cell, the adcjacent discs that are of opposite colour change and take up the colour of the new disc.

If at any point of the game there are more than $\frac{n}{2}$ red discs on the board, the player with the red discs will immediately win, whereas if there are at least $\frac{n}{2}$ blue discs, the blue player will immediately win. If this does not happen up until 200 moves, the blue player wins.

Beat the organisers in this game twice in a row! Knowing what number n is, you can decide if you would rather be the first or the second player.

(Solution)

2.4 Final round – day 2

2.4.1 Category C

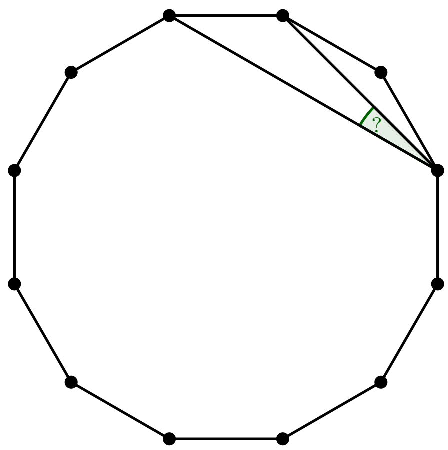
1. Thirty-three minutes, thirty-three seconds, and three times three seconds: how many seconds in total?

(Solution)

2. A prime number is called *middle aged* if the numbers 4 smaller and 4 larger than it are also primes. What is the sum of middle aged primes less than 60?

(Solution)

3. A regular 12-gon is visible on the diagram, with two diagonals drawn. How many degrees is the angle marked with a question mark?



(Solution)

4. Csaba stands in the middle of a 15 m × 15 m room at a workplace where everyone strictly adheres to 1,5 m social distancing. At least how many people are there other than Csaba in the room if Csaba cannot reach any wall without the others moving?

The people are viewed as points.

(Solution)

5. In duck language, only letters **q**, **a**, and **k** are used. There is no word with two consonants after each other, because the ducks cannot pronounce them. However, all other four-letter words are meaningful in duck language. How many such words are there?

In duck language, too, the letter ***a*** *is a vowel, while* ***q*** *and* ***k*** *are consonants.*

(Solution)

6. Momma duck laid 50 eggs this years, and Pappa duck guessed how many will be boy ducklings and how many will be girls. After the 50 ducklings hatched, it turned out that there were 10% fewer girl ducklings and 15% more boy ducklings than Pappa duck guessed initially. How many boy ducklings hatched?

The sum of Pappa duck's guesses was 50.

(Solution)

7. At the beginning of each year, Kartal writes a sentence about how many times each digit occurs in the number of the year. Last year (in 2021), he wrote the sentence

In this year number, there are 2 numbers 2, 1 number 1, and 1 number 0.

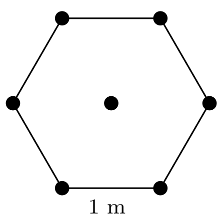
This year (in 2022), Kartal's sentence sounds like this:

In this year number, there are 3 numbers 2 and 1 number 0.

Kartal noticed that the four digits occurring in this sentence are all distinct. In how many years will there once more be a sentence where the digits involved are all distinct?

(Solution)

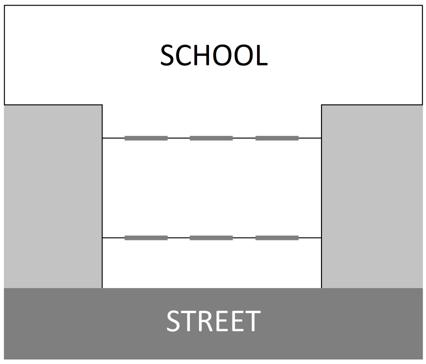
8. At the water bird olimpics, 7 ducks perform synchronised swimming. Six of them are located in the six vertices of a regular hexagon with side length 1 m, and the seventh is in the center (see diagram). The swan judges view the swim from 3 m height. For all of them to judge the harmony of the swimming properly, they wish to position themselves such that to every swan, there are at least three ducks with equal distance from it. At most how many swan judges can there be?



(Solution)

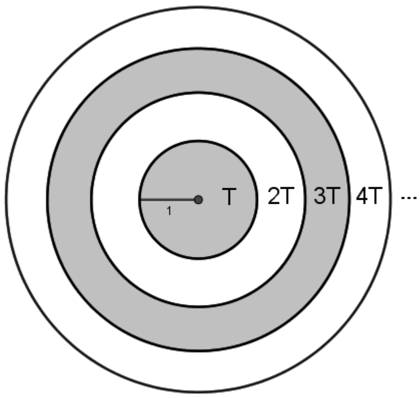
9. In Dürer's duck school, there are two rows of doors, as seen on the diagram; both rows are made up of three doors. Dodo duck wishes to enter the school from the street in a way that she uses all six doors exactly once. (On her path, she may go to the street again, or leave the school, so long as she finishes her path in the school.) How many ways can she perform this?

Two paths are considered different if Dodo takes the doors in a different order.



(Solution)

10. Benedek draws circles with the same center in the following way. The first circle he draws has radius 1. Next, he draws a second circle such that the ring between the first and second circles has twice the area of the first circle. Next, he draws a third circle such that the ring between the second and third circles is three times the area of the first circle, and so on (see the diagram).
What is the smallest n fow which the radius of the n-th circle is an integer greater than 1?



(Solution)

11. Three palaces, each rotating on a duck leg, make a full round in 30, 50, and 70 days, respectively. Today, at noon, all three palaces face northwards. In how many days will they all face southwards?

(Solution)

12. One angle of a triangle equals the sum of the other two, and the square of the longest side is 168 cm^2 less than the sum of the squares of the two smaller sides. How many cm^2 is the area of the triangle?

(Solution)

13. 6 teams took part in a soccer contest, and every team played every other team exactly once. Every win scored 3 points, every draw 1 point, and every loss 0 points. If five of the teams had final scores 13, 11, 8, 5, and 0, then what was the score of the sixth team?

(Solution)

14. Write some positive integers in the following table such that

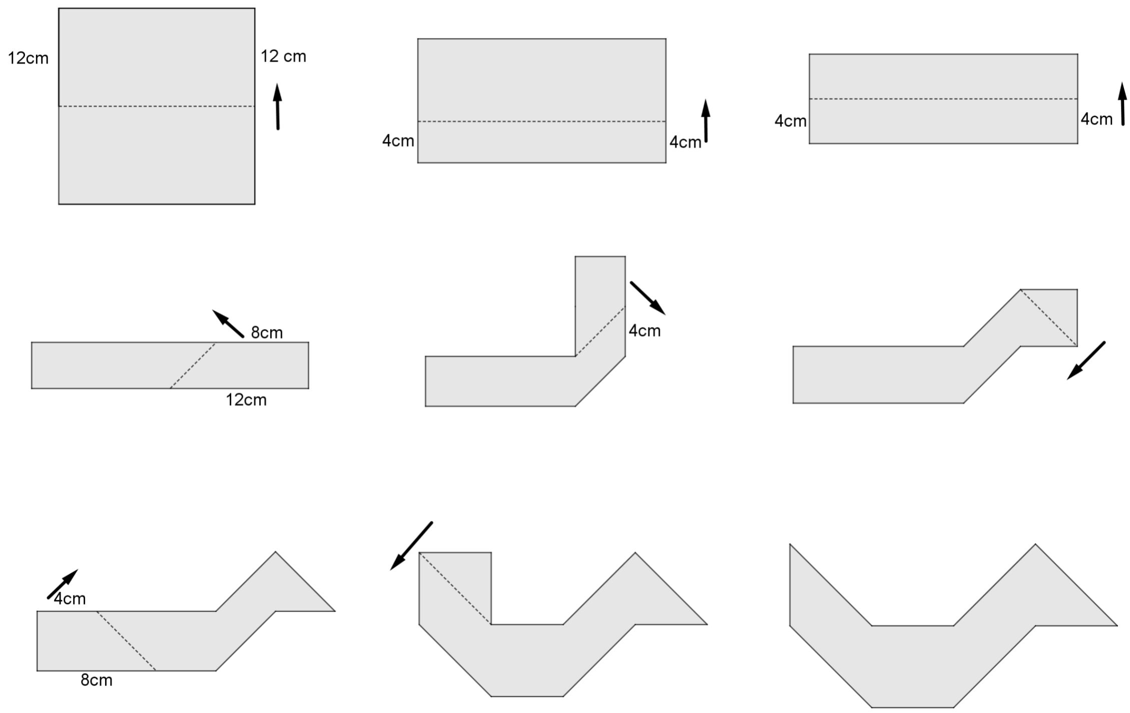
- there is at most one number in each field
- each number is equal to how many numbers there are in edge-adjacent fields,
- edge-adjacent fields cannot have equal numbers.

What is the sum of numbers in the resulting table?

					1
1			3		
		4			2
	2				
					2
1					

(Solution)

15. Csongi taught Benedek how to fold a duck in 8 steps from a 24 cm × 24 cm piece of paper. The paper is meant to be folded along the dashed line in the direction of the arrow. Once Benedek folded the duck, he undid all the steps, finding crease lines on the square sheet of paper. On one side of the paper, he drew in blue the folds which opened towards Benedek, and in red the folds which opened toward the table. How many cm is the difference between the total length of the blue lines and the red lines?



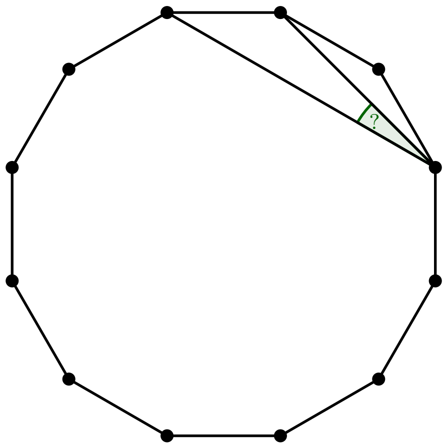
(Solution)

16. The product of Albrecht's three favorite numbers is 2022, and if we add one to each number, their product will be 1514. What is the sum of their squares, if we know their sum is 0?

(Solution)

2.4.2 Category D

1. A regular 12-gon is visible on the diagram, with two diagonals drawn. How many degrees is the angle marked with a question mark?



(Solution)

2. A prime number is called *middle aged* if the numbers 4 smaller and 4 larger than it are also primes. What is the sum of middle aged primes less than 60?

(Solution)

3. Csaba stands in the middle of a 15 m $\times$ 15 m room at a workplace where everyone strictly adheres to 1,5 m social distancing. At least how many people are there other than Csaba in the room if Csaba cannot reach any wall without the others moving?

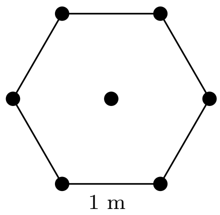
The people are viewed as points.

(Solution)

4. How many 10-digit sequences are there, made up of 1 four, 2 threes, 3 twos, and 4 ones, in which there is a two in between any two ones, a three in between any two twos, and a four in between any two threes?

(Solution)

5. At the water bird olimpics, 7 ducks perform synchronised swimming. Six of them are located in the six vertices of a regular hexagon with side length 1 m, and the seventh is in the center (see diagram). The swan judges view the swim from 3 m height. For all of them to judge the harmony of the swimming properly, they wish to position themselves such that to every swan, there are at least three ducks with equal distance from it. At most how many swan judges can there be?

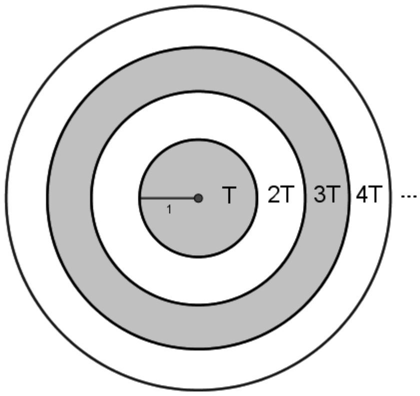


(Solution)

6. Three palaces, each rotating on a duck leg, make a full round in 30, 50, and 70 days, respectively. Today, at noon, all three palaces face northwards. In how many days will they all face southwards?

(Solution)

7. Benedek draws circles with the same center in the following way. The first circle he draws has radius 1. Next, he draws a second circle such that the ring between the first and second circles has twice the area of the first circle. Next, he draws a third circle such that the ring between the second and third circles is three times the area of the first circle, and so on (see the diagram).
What is the smallest n fow which the radius of the n-th circle is an integer greater than 1?



(Solution)

8. 6 teams took part in a soccer contest, and every team played every other team exactly once. Every win scored 3 points, every draw 1 point, and every loss 0 points. If five of the teams had final scores 13, 11, 8, 5, and 0, then what was the score of the sixth team?

(Solution)

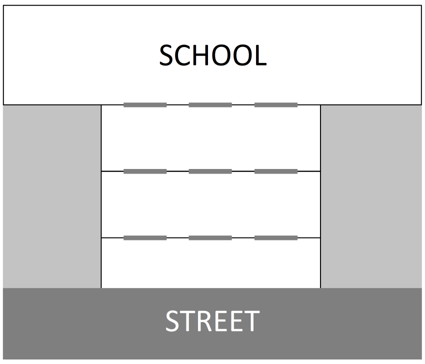
9. The *fragments* of a positive integer are the numbers seen when reading one or more of its digits in order. The fragment sum equals the sum of all the fragments, including the number itself. For example, the fragment sum of 2022 is 2022+202+022+20+02+22+2+0+2+2 = 2296. There is another four-digit number with the same fragment sum. What is it?

As the example shows, if a fragment occurs multiple times, then all its occurrences are added, and the fragments beginning with 0 also count (for instance, 022 *is worth* 22*).*

(Solution)

10. In Dürer's duck school, there are three rows of doors, as seen on the diagram; both rows are made up of three doors. Dodo duck wishes to enter the school from the street in a way that she uses all six doors exactly once. (On her path, she may go to the street again, or leave the school, so long as she finishes her path in the school.) How many ways can she perform this?

Two paths are considered different if Dodo takes the doors in a different order.

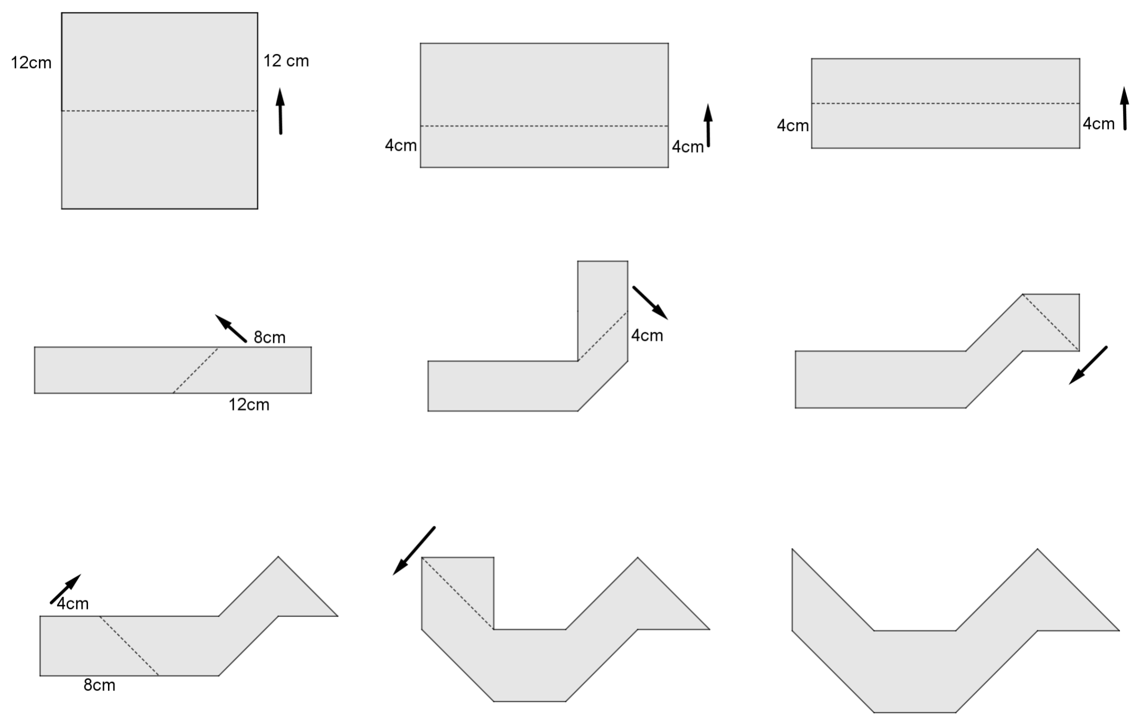


(Solution)

11. In Kacs Aladár street, houses are only found on one side of the road, so that only odd house numbers are found along the street. There are an odd number of allotments, as well. The middle three allotments belong to Scrooge McDuck, so he only put up the smallest of the three house numbers. The numbering of the other houses is standard, and the numbering begins with 1. What is the largest number in the street if the sum of house numbers put up is 3133?

(Solution)

12. Csongi taught Benedek how to fold a duck in 8 steps from a 24 cm × 24 cm piece of paper. The paper is meant to be folded along the dashed line in the direction of the arrow. Once Benedek folded the duck, he undid all the steps, finding crease lines on the square sheet of paper. On one side of the paper, he drew in blue the folds which opened towards Benedek, and in red the folds which opened toward the table. How many cm is the difference between the total length of the blue lines and the red lines?



(Solution)

13. Write some positive integers in the following table such that

- there is at most one number in each field
- each number is equal to how many numbers there are in edge-adjacent fields,
- edge-adjacent fields cannot have equal numbers.

What is the sum of numbers in the resulting table?

					1
1			3		
		4			2
	2				
					2
1					

(Solution)

14. Every side of a right triangle is an integer when measured in cm, and the difference between the hypotenuse and one of the legs is 75 cm. What is the smallest possible value of its perimeter?

(Solution)

15. The pair of positive integers (a, b) is such that a does not divide b, b does not divide a, both numbers are at most 100, and they have the maximal possible number of common divisors. What is the largest possible value of $a \cdot b$?

(Solution)

16. Doofy duck buy tangerines in the store. All tangerines have equal weight and are divided into 9, 10, 11, 12, or 13 equal wedges, although this cannot be seen without peeling them. How many tangerines does Doofy duck need to buy if he wishes to eat exactly one tangerine's worth while eating at most one wedge from every tangerine?

Doofy duck only peels the tangerines at home.

(Solution)

2.4.3 Category E

1. In duck language, only letters **q**, **a**, and **k** are used. There is no word with two consonants after each other, because the ducks cannot pronounce them. However, all other four-letter words are meaningful in duck language. How many such words are there?

*In duck language, too, the letter **a** is a vowel, while **q** and **k** are consonants.*

(Solution)

2. Csaba stands in the middle of a 15 m $\times$ 15 m room at a workplace where everyone strictly adheres to 1,5 m social distancing. At least how many people are there other than Csaba in the room if Csaba cannot reach any wall without the others moving?

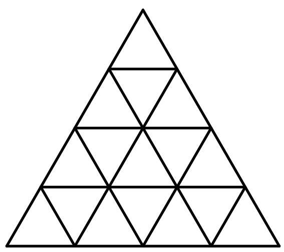
The people are viewed as points.

(Solution)

3. Three palaces, each rotating on a duck leg, make a full round in 30, 50, and 70 days, respectively. Today, at noon, all three palaces face northwards. In how many days will they all face southwards?

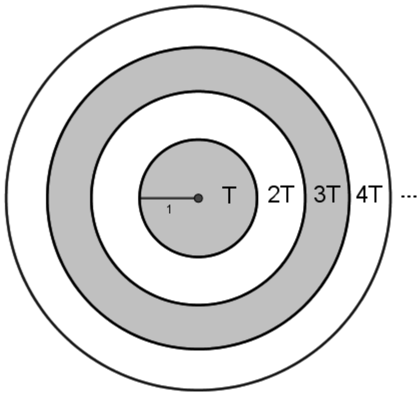
(Solution)

4. At least how many regular triangles are needed to cover the lines of the following diagram? (Only the perimeter of the triangles is involved in the covering, and the entire perimeter need not be incident on the diagram.)



(Solution)

5. Benedek draws circles with the same center in the following way. The first circle he draws has radius 1. Next, he draws a second circle such that the ring between the first and second circles has twice the area of the first circle. Next, he draws a third circle such that the ring between the second and third circles is three times the area of the first circle, and so on (see the diagram).
What is the smallest n fow which the radius of the n-th circle is an integer greater than 1?



(Solution)

6. In Kacs Aladár street, houses are only found on one side of the road, so that only odd house numbers are found along the street. There are an odd number of allotments, as well. The middle three allotments belong to Scrooge McDuck, so he only put up the smallest of the three house numbers. The numbering of the other houses is standard, and the numbering begins with 1. What is the largest number in the street if the sum of house numbers put up is 3133?

(Solution)

7. The *fragments* of a positive integer are the numbers seen when reading one or more of its digits in order. The fragment sum equals the sum of all the fragments, including the number itself. For example, the fragment sum of 2022 is 2022+202+022+20+02+22+2+0+2+2 = 2296. There is another four-digit number with the same fragment sum. What is it?

As the example shows, if a fragment occurs multiple times, then all its occurrences are added, and the fragments beginning with 0 also count (for instance, 022 *is worth* 22*).*

(Solution)

8. The product of Albrecht's three favorite numbers is 2022, and if we add one to each number, their product will be 1514. What is the sum of their squares, if we know their sum is 0?

(Solution)

9. Every side of a right triangle is an integer when measured in cm, and the difference between the hypotenuse and one of the legs is 75 cm. What is the smallest possible value of its perimeter?

(Solution)

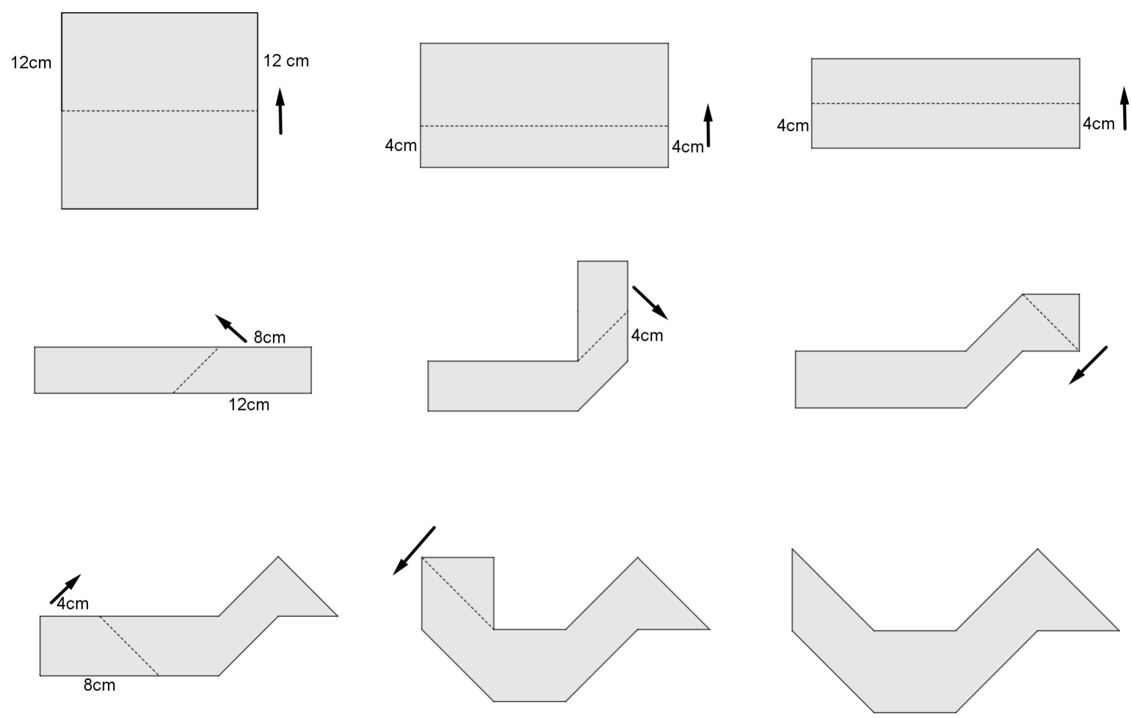
10. The pair of positive integers (a, b) is such that a does not divide b, b does not divide a, both numbers are at most 100, and they have the maximal possible number of common divisors. What is the largest possible value of $a \cdot b$?

(Solution)

11. In rectangle $ABCD$, diagonal AC is met by the angle bisector from B at B' and the angle bisector from D at D'. Diagonal BD is met by the angle bisector from A at A' and the angle bisector from C at C'. The area of quadrilateral $A'B'C'D'$ is $\frac{9}{16}$ the area of rectangle $ABCD$. What is the ratio of the longer side and shorter side of rectangle $ABCD$?

(Solution)

12. Csongi taught Benedek how to fold a duck in 8 steps from a 24 cm × 24 cm piece of paper. The paper is meant to be folded along the dashed line in the direction of the arrow. Once Benedek folded the duck, he undid all the steps, finding crease lines on the square sheet of paper. On one side of the paper, he drew in blue the folds which opened towards Benedek, and in red the folds which opened toward the table. How many cm is the difference between the total length of the blue lines and the red lines?



(Solution)

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- each number is equal to how many numbers there are in edge-adjacent fields,
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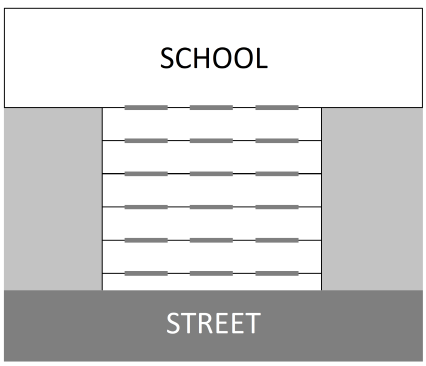
What is the sum of numbers in the resulting table?

					1
1			3		
		4			2
	2				
					2
1					

(Solution)

14. In Dürer's duck school, there are six rows of doors, as seen on the diagram; both rows are made up of three doors. Dodo duck wishes to enter the school from the street in a way that she uses all six doors exactly once. (On her path, she may go to the street again, or leave the school, so long as she finishes her path in the school.) How many ways can she perform this?

Two paths are considered different if Dodo takes the doors in a different order.



(Solution)

15. Doofy duck buy tangerines in the store. All tangerines have equal weight and are divided into 9, 10, 11, 12, or 13 equal wedges, although this cannot be seen without peeling them. How many tangerines does Doofy duck need to buy if he wishes to eat exactly one tangerine's worth while eating at most one wedge from every tangerine?

Doofy duck only peels the tangerines at home.

(Solution)

16. The number 60 is written on a blackboard. In every move, Andris wipes the numbers on the board one by one, and writes all its divisors in its place (including itself). After 10 such moves, how many times will 1 appear on the board?

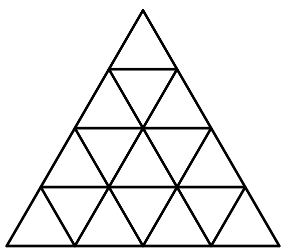
(Solution)

2.4.4 Category E$^+$

1. How many 10-digit sequences are there, made up of 1 four, 2 threes, 3 twos, and 4 ones, in which there is a two in between any two ones, a three in between any two twos, and a four in between any two threes?

(Solution)

2. At least how many regular triangles are needed to cover the lines of the following diagram? (Only the perimeter of the triangles is involved in the covering, and the entire perimeter need not be incident on the diagram.)



(Solution)

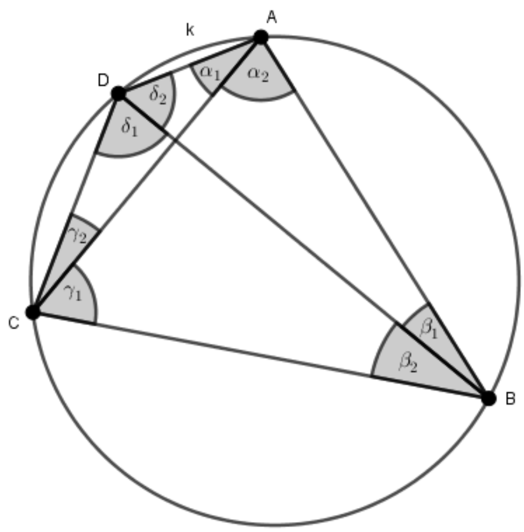
3. Three palaces, each rotating on a duck leg, make a full round in 30, 50, and 70 days, respectively. Today, at noon, all three palaces face northwards. In how many days will they all face southwards?

(Solution)

4. In Kacs Aladár street, houses are only found on one side of the road, so that only odd house numbers are found along the street. There are an odd number of allotments, as well. The middle three allotments belong to Scrooge McDuck, so he only put up the smallest of the three house numbers. The numbering of the other houses is standard, and the numbering begins with 1. What is the largest number in the street if the sum of house numbers put up is 3133?

(Solution)

5. On a circle k, we marked four points (A, B, C, D) and drew pairwise their connecting segments. We denoted angles as seen on the diagram. We know that $\alpha_1 : \alpha_2 = 2 : 5$, $\beta_1 : \beta_2 = 7 : 11$, and $\gamma_1 : \gamma_2 = 10 : 3$. If $\delta_1 : \delta_2 = p : q$, where p and q are coprime positive integers, then what is p?



(Solution)

6. Every side of a right triangle is an integer when measured in cm, and the difference between the hypotenuse and one of the legs is 75 cm. What is the smallest possible value of its perimeter?

(Solution)

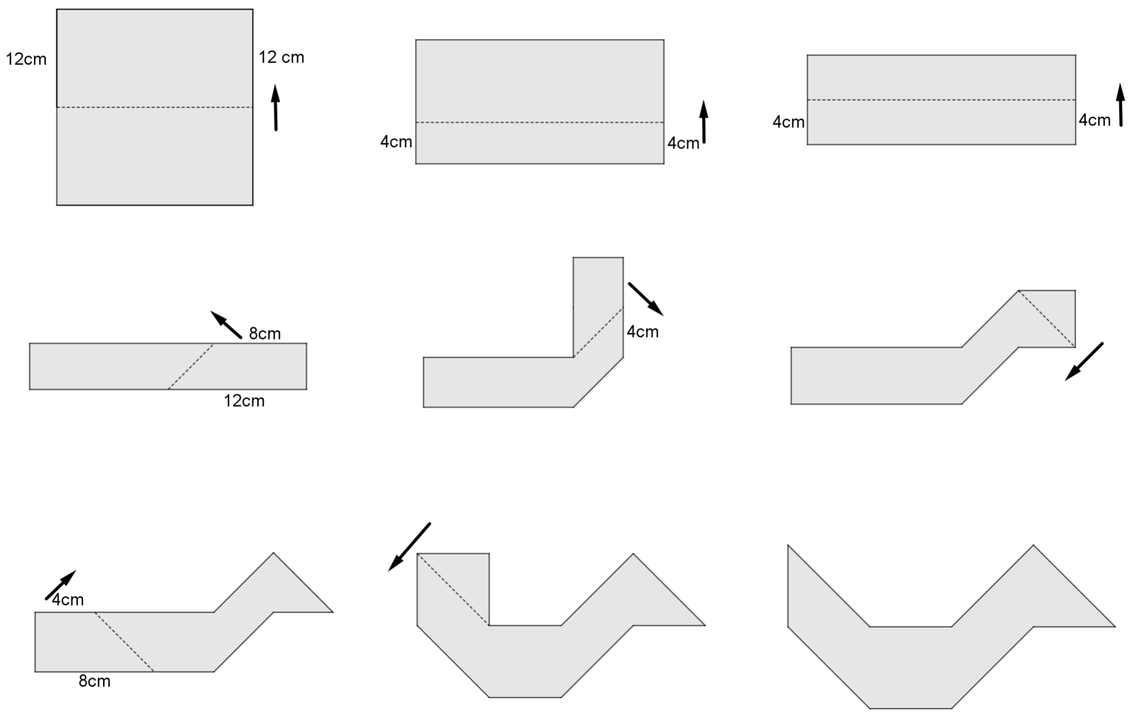
7. The pair of positive integers (a, b) is such that a does not divide b, b does not divide a, both numbers are at most 100, and they have the maximal possible number of common divisors. What is the largest possible value of $a \cdot b$?

(Solution)

8. In rectangle $ABCD$, diagonal AC is met by the angle bisector from B at B' and the angle bisector from D at D'. Diagonal BD is met by the angle bisector from A at A' and the angle bisector from C at C'. The area of quadrilateral $A'B'C'D'$ is $\frac{9}{16}$ the area of rectangle $ABCD$. What is the ratio of the longer side and shorter side of rectangle $ABCD$?

(Solution)

9. Csongi taught Benedek how to fold a duck in 8 steps from a 24 cm × 24 cm piece of paper. The paper is meant to be folded along the dashed line in the direction of the arrow. Once Benedek folded the duck, he undid all the steps, finding crease lines on the square sheet of paper. On one side of the paper, he drew in blue the folds which opened towards Benedek, and in red the folds which opened toward the table. How many cm is the difference between the total length of the blue lines and the red lines?



(Solution)

10. Write some positive integers in the following table such that

- there is at most one number in each field
- each number is equal to how many numbers there are in edge-adjacent fields,
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What is the sum of numbers in the resulting table?

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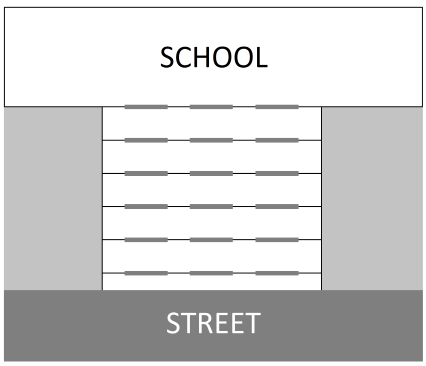
		4			2
	2				
					2
1					

(Solution)

11. In Dürer's duck school, there are six rows of doors, as seen on the diagram; both rows are made up of three doors. Dodo duck wishes to enter the school from the street in a way that she uses all six doors exactly once. (On her path, she may go to the street again, or leave the school, so long as she finishes her path in the school.) How many ways can she perform this?

Two paths are considered different if Dodo takes the doors in a different order.



(Solution)

12. Doofy duck buy tangerines in the store. All tangerines have equal weight and are divided into 9, 10, 11, 12, or 13 equal wedges, although this cannot be seen without peeling them. How many tangerines does Doofy duck need to buy if he wishes to eat exactly one tangerine's worth while eating at most one wedge from every tangerine?

Doofy duck only peels the tangerines at home.

(Solution)

13. Circle k_1 has radius 10, externally touching circle k_2 with radius 18. Circle k_3 touches both circles, as well as the line e determined by their centres. Let k_4 be the circle touching k_2 and k_3 externally (other than k_1) whose center lies on line e. What is the radius of k_4?

(Solution)

14. Benedek scripted a program which calculated the following sum: $1^1+2^2+3^3+\ldots+2021^{2021}$. What is the remainder when the sum is divided by 35?

(Solution)

15. An ant crawls along the grid lines of an infinite quadrille notebook. One grid point is marked red, this is its starting point. Every time the ant reaches a grid point, it continues forward with probability $\frac{1}{3}$, left with probability $\frac{1}{3}$, and right with probability $\frac{1}{3}$. What is the chance that it is after its third turn, but not after its fourth turn that it returns to the red point?

If the answer is $\frac{p}{q}$, where p and q are coprime positive integers, then your answer should be $p+q$.

The steps of the ant are independent.

(Solution)

16. The number 60 is written on a blackboard. In every move, Andris wipes the numbers on the board one by one, and writes all its divisors in its place (including itself). After 10 such moves, how many times will 1 appear on the board?

(Solution)

3 Solutions – primary school categories

3.1 Regional round

3.1.1 Tables

#	ANS	Problem	P
A-1	9	Anna and Panna	3p
A-2	19	Bonnie has a fair die on her desk.	3p
A-3	16	At the market of Piripócs, the tradition of trading persists.	3p
A-4	6	A six member duck family,	3p
A-5	357	The notation of the date today (2021.11.19.)	4p
A-6	22	Sári wrote a message to her friend.	4p
A-7	50	The fifth and sixth grade of the Quackton elementary school	4p
A-8	5	Andris and Anett went on a 10-kilometer tour.	4p
A-9	11	The ship Graceful Hippo	5p
A-10	7	Flea Florence likes to hop on the plane.	5p
A-11	23	Fill the circles with the numbers from 1 to 10	5p
A-12	9	Mickey Mouse, Minnie Mouse and Donald Duck	5p
A-13	56	In Duck School,	6p
A-14	21	At a chess contest where everyone played everyone exactly once,	6p
A-15	15	The diagram shows the 8 islands of the country Oxis,	6p

#	ANS	Problem	P
B-1	6	Anna, Hanna, and Panna	3p
B-2	6	A six member duck family,	3p
B-3	357	The notation of the date today (2021.11.19.)	3p
B-4	50	The seventh and eighth grade of the Quackton elementary school	3p
B-5	4	Princess Atonce would like to build a castle in the air.	4p
B-6	5	Andris and Anett went on a 10-kilometer tour.	4p
B-7	11	The ship Graceful Hippo	4p
B-8	7	Flea Florence likes to hop on the plane.	4p
B-9	23	Fill the circles with the numbers from 1 to 10	5p
B-10	9	Mickey Mouse, Minnie Mouse and Donald Duck	5p
B-11	56	In Duck School,	5p
B-12	30	At a chess contest where everyone played everyone exactly once,	5p
B-13	15	The diagram shows the 8 islands of the country Oxis,	6p
B-14	72	In triangle ABC,	6p
B-15	240	Sophie glued three identical standard dice together.	6p

3.1.2 Category A

1. In the table, the colored fields are those which are in odd rows or even columns, so the fields without a color are in even rows and odd columns. There are three choices for the row number (2, 4, 6) and three choices for the column number (1, 3, 5). Therefore, there are $3 \cdot 3 = 9$ fields without a color.

(Back to problems)

2. The number of dots on the die is $3 \cdot 7 = 21$. In the beginning only the bottom face is not visible, which therefore has $21 - 16 = 5$ dots. Which means the top face has $7 - 5 = 2$ dots. Hence when turned around the bottom face will have 2 dots, so $21 - 2 = 19$ dots will be visible.

(Back to problems)

3. Aunt Kati can trade her two ducks for two geese and four hatchlings. Then, she will have four geese and four hatchlings. The four geese she can trade for $4 \cdot 3 = 12$ hatchlings. Therefore, Aunt Kati can obtain 16 hatchlings with the trades.

(Back to problems)

4. Either the mother or the father can be at the front of the line, with the other parent at the end, that's already 2 options. In the other 4 spots, right after the yellow duckling comes the white one, so the yellow duckling can be at 3 spots from these remaining 4, since he can't be on the last one. Now the white duckling has only 1 option, being right behind the yellow one. In the two final spots the black and the grey ducklings can only be in order, since the black one is ahead of the grey one. Considering all this we get a $3 \cdot 2 = 6$ ways to order the family.

(Back to problems)

5. In 2021, there are no more such dates, because in the 11th month, the days begin with 2 or 3 from now on. In the 12th month, at most 3 equal digits can occur consecutively. In 2022, the digit 2 cannot occur 4 times, because there is no 22nd month, but the digit 1 can, with the date 2022.11.11. This is the only such possibility in 2022, as the month and day numbers must have the equal digits, and only the 11th month is suitable. Therefore, the soonest such date is 2022.11.11, which occurs in $365 - 8 = 357$ days (as 2022 is not a leap year).

(Back to problems)

6. Sári typed $60 \cdot 2 = 120$ keys, but only sent 76 of these. Thus, she typed $120 - 76 = 44$ extra times, during which she deleted and typed a character the same number of times. Sári typed the delete button $44 \cdot \frac{1}{2} = 22$ times.

(Back to problems)

7. A room goes swimming if at least 3 of the 5 children prefer to. Therefore, at most $30 \cdot \frac{1}{3} = 10$ rooms decide to swim. This is indeed possible, when there are 10 rooms with 3 swimmers and 2 ice cream consumers, and 2 more rooms with 5 ice cream consumers. Therefore, at most $10 \cdot 5 = 50$ children go swimming in the afternoon.

(Back to problems)

8. The third sign is surely false, because it ought to be followed by three positive integers, but there is only one positive integer smaller than 2. The fourth sign is surely false, because if it were good, then only 10, 9, 8 could precede it, but then the first three signs are false, and then the final two signs are correct, yet 4 cannot follow 3, absurd. Indeed, either the fifth or the sixth sign is false. This means the first and second signs are correct. Then the sixth sign cannot be correct, as then between 6 and 4, there should be 3 integers. So it is the first, second, and fifth sign which are correct. Then, the third and fourth sign have distance 5 and 4 from the target, as there is 6 on the second board, and 3 on the fifth. This way, at the third board, Andris and Anett were 5 kilometers from the target.

(Back to problems)

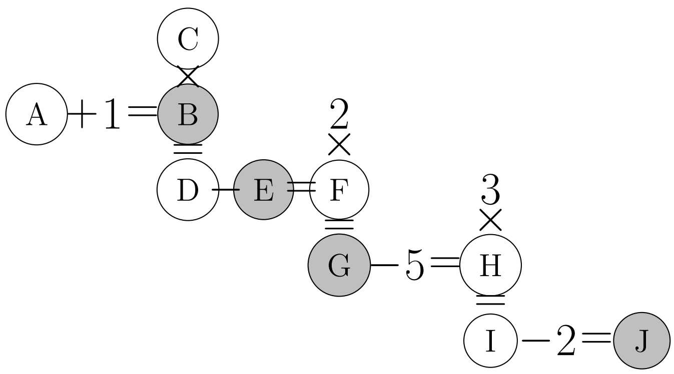
9. If the captain is a knight, then one who debarked before him lied, as there was 1 knight on the vessel. So it was a knave, and the previous one who debarked also lied, as there was 1 knight and 1 knave on board, so it was a knave as well. But then there were 1 knight and 2 knaves on board, so a knight debarked before that. From then on, it is clear that knaves and knights debarked the ship alternately. Therefore, in this case, there were 11 knaves and 10 knights on board. If the captain is a knave, then the one who debarked before him told the truth, as there was 1 knave on board. The one before that lied, as there were 1 knight and 1 knave. Similarly to before, knaves and knights alternate, so there were 11 knaves and 10 knights in this case, as well.

(Back to problems)

10. Flea Florence jumped an even number of metres, as she jumped east and west the same number of times, and north and south by the same amount, so she jumped an even number of metres in total. If her final jump was n metres, then she jumped $\frac{n(n+1)}{2}$ metres in total, so n or $n+1$ is divisible by 4. If $n=3$ or $n=4$, she could not jump back to her starting point, as she needs to jump in both directions, but the jump lengths 3 or 4 do not cancel the lengths 1 and 2. The next possibility is $n=7$, in which case she can return as follows: E1, N2, S3, S4, N5, E6, W7. These jumps cancel out, because $1+6-7=0$ and $2-3-4+5=0$. Thus, Flea Florence needed to jump at least 7 times.

(Back to problems)

11.



With the notation above, let's consider first the equations $G - 5 = H$ and $2 \cdot F = G$. As every circle must contain a number from 1 to 10, $10 \geq G \geq 6$, so $5 \geq F \geq 3$. The value of G can be 6, 8, 10; the value of H can be 1, 3, 5. However, because $3 \cdot H = I$, $H \neq 5$, as then $I = 15$ would be the case. If $H = 1$, then $I = 3$, $J = I - 2 = 1$, and we used a number twice, so this case cannot hold. It follows that $H = 3$, $I = 3 \cdot H = 9$, $J = I - 2 = 7$, $G = H + 5 = 8$, $F = \frac{G}{2} = 4$.

The numbers we haven't used so far: 1, 2, 5, 6, 10. None of A, B, C, E can be 10, as then the remaining circles cannot be filled according to the rules. So $D = 10$. This also means that C and B are 5 and 2, by equation $C \cdot B = D$. Now B cannot be 5, because then $A = 5 - 1 = 4 = F$ would be the case. Thus, $B = 2$, $A = 1$, $C = 5$, $E = D - F = 6$.

Therefore, the numbers written in gray circles sum to 2+6+8+7=23.

(Back to problems)

12. If Donald lives between Mickey and Minnie, then since there are 11 houses between Mickey and Minnie, one of these belongs to Donald, with n between Mickey and Donald, $2n$ between Donald and Minnie. This yields the equation $3n = 10$, which does not have an integer solution, so this case can be ruled out. As Donald lives closer to Mickey, Donald lives further on than Mickey along the street. If there are n houses between them, then there are $2n$ houses between Donald and Minnie, which is precisely $11 + 1 = 12$ more than n, counting the houses between Mickey and Minnie, as well as Mickey's house. Thus, $n + 12 = 2n$, so $n = 12$. Hence, there are 12 houses between Mickey and Donald, and these are from 33 down to 11, so Donald lives under house number 9.

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13.

If Donald Duck got the green distinction, then he couldn't have reached rank 10 egg laying, since then he would either have to have rank 7 quacking, which already is a total of 17 ranks, and this cannot be, or rank 4 flying, in which case he has 2 ranks left, so he couldn't have reached rank 3 quacking, so he had to get 3 distinctions just from egg laying and flying, for which he would need rank 8 flying, but that's already 18 ranks, which cannot be either. So if Donald Duck got the green distinction, then he achieved rank 7 quacking and rank 4 flying. If so, if he also got the yellow disctinction, he could only have done that with rank 3 egg laying, since he couldn't be rank 11 flying. This is already 14 ranks in total, and for the red distinction he would need at least 4 more (from egg laying or quacking), so he can't have 3 distinctions this way. This means that if he got the green one, he couldn't have the yellow one, so he had to get the red and the blue ones. If so he had to earn the red distinction with egg laying and flying, since if he were rank 11 from quacking he would have 1 rank left, which he needs at egg laying for the red distinction, but in this case he couldn't get the blue one. So this way he's rank 1 egg laying, rank 8 flying and rank 7 quacking, and he got the blue, red and green distinctions. If Donald Duck didn't get the green disction, then he got the blue, red and yellow ones. He earned the yellow distinction from quacking and egg laying, since if he were rank 11 flying, and rank 3 from something else to acompany that, then he'd have 2 ranks left, and as such he wouldn't have the required 3 extra ranks needed for the blue distinction. So from quacking and egg laying he's at least rank 3. This means that he got the red distinction with egg laying and flying, since if he was rank 11 quacking, he'd have 2 ranks left, but with that he couldn't get blue, because he'd need 3 more ranks of egg laying. As such he's at least rank 8 flying considering the red distinction. This way he's got $3 + 3 + 8 = 14$ ranks already, so 2 are left for the blue distinction, but for that he'd need 3 from quacking or egg laying. So this case isn't possible. The only solution then is that he's rank 1 egg laying, rank 8 flying and rank 7 quacking, so the product of his ranks must be $8 \cdot 1 \cdot 7 = 56$.

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14. They played a total of $\frac{6 \cdot 5}{2} = 15$ games, and 2 points were given out at every game, so a total of 30 points were awarded. Of these, 2 points were given at the game of the fifth and sixth player, so the first four gained at most 28 points in total. Of these, the second, third, and fourth gained at most 21, as with 22 or more, the first player would have at most 6 points, and so the four of them would have at most $4 \cdot 6 = 24$, contradiction. Moreover, 21 points can be attained, when the first four players won versus the fifth and sixth, while playing draws with one another. Then, each of the first four players had $3 + 2 \cdot 2 = 7$ points.

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15. One must check for which days there exists a journey where Adventurer sleeps on island X that day. Let's call these days admissible. If a day is admissible, then the fourth day after is, too, by traversing the small cycle once more. It is sufficient, therefore, to examine according to residue when dividing by 4, which is the earliest admissible day. The residue only changes when walking around the large cycle, so the earliest admissible days are found with respect to the least number of such rounds. For days of the form $4k + 3$, 7 can be reached with one round, so the wife recorded 1 such day. For days of the form $4k + 2$, 14 can be reached with two rounds, so the wife recorded 3 such days. For days of the form $4k + 1$, 21 is admissible earliest, and the wife records 5 days. For days of the form $4k$, 28 is admissible, and 6 days are recorded. Therefore, the number of recorded days is $1 + 3 + 5 + 6 = 15$.

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3.1.3 Category B

1. The fields in the table colored by either Anna or Hanna are in odd rows or even columns, so they do not color the fields in even rows and odd columns. There are three possible row numbers (2, 4, 6) and three possible column numbers (1, 3, 5). This yields $3 \cdot 3 = 9$ fields not colored by them. Panna colors the fields in the diagonal as well, one of which has equal row and column numbers, and none of our fields are like this. The other diagonal has the fields with row and column numbers adding up to 7. Of these, it is the even row and odd column numbers which are newly colored, so $(2, 5)$, $(4, 3)$, $(6, 1)$. Therefore, $9 - 3 = 6$ fields are left without a color.

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2. For the solution, see Category A Problem 4.

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3. For the solution, see Category A Problem 5.

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4. A room goes swimming if at least 3 of the 5 children prefer to. Therefore, at most $30 \cdot \frac{1}{3} = 10$ rooms decide to swim. This is indeed possible, when there are 10 rooms with 3 swimmers and 2 ice cream consumers, and 2 more rooms with 5 ice cream consumers. Therefore, at most $10 \cdot 5 = 50$ children go swimming in the afternoon.

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5.

One half of the third of the required amount is one sixth of the required amount, the gnomes carry this much away the first night, with the same amount remaining for tommorrow. On the second day the workers bring the double of this, 2 sixths of the required amount, so in the evening 3 sixths, half of the required amount will be on site. The evil gnomes take half of this on the second night, so one fourth of the required amount, leaving this much there. On the third day the workers bring the double of this, 2 fourths of the required amount, so in the evening 3 fourths of the required amount will be on site. The evil gnomes take half of this on the third night, 3 eights of the required amount, also leaving this much there. So next day the double of this, six eights of the required amount is brought there, so more than the required amount, 9 eights of it will be on site. All this means that Princess Atonce has to wait 4 days.

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6. For the solution, see Category A Problem 8.

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7. For the solution, see Category A Problem 9.

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8. For the solution, see Category A Problem 10.

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9. For the solution, see Category A Problem 11.

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10. For the solution, see Category A Problem 12.

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11. For the solution, see Category A Problem 13.

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12. At total of $\frac{8\cdot 7}{2} = 28$ games were played, and there were 2 points given out at each, a total of 56 points. Of these, $\frac{3\cdot 2}{2} \cdot 2 = 6$ points were given out at the games of the sixth, seventh, and eighth players. The first 5 gained at most 50 points, of which the third, fourth, and fifth gained at most $50 - 2 \cdot \frac{1}{5} \cdot 50 = 30$. The reason for this is that if they gained at least 31, then the first and second have at most 19 points, so the second has at most 9, as well as the third through fifth players, totalling at most $19 + 9 \cdot 3 < 50$ points, contradiction. Moreover, 30 points are attainable if the first five players won versus the final three, while playing draws with one another. Then, each of the first five players had $4 + 3 \cdot 2 = 10$ points.

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13. For the solution, see Category A Problem 15.

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14. Let $ABD\angle = \alpha$, then $CBD\angle = \alpha$. As BC and DE are parallel, $BDE\angle = \alpha$, equal to $CBD\angle$. Hence, $ADE\angle = \alpha$. Next, since BC and DE are parallel, $BCA\angle = \alpha$, equal to $ADE\angle$. Let $AD = x$, then $BE = x$. As $BDE\angle = DBE\angle = \alpha$, triangle BDE is isosceles, so $DE = x$. Hence, ADE is also isosceles, whence $DAE\angle = \frac{180^\circ - \alpha}{2} = 90^\circ - \frac{\alpha}{2}$. Thus, in triangle ABC, the sum of angles is $\alpha + 2\alpha + 90^\circ - \frac{\alpha}{2} = 180^\circ$. From here, $\alpha = 36^\circ$, so $CAB\angle = 90^\circ - \frac{36^\circ}{2} = 72^\circ$.

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15. We'll choose one die to glue the other two onto. Two faces of this die can be choosen (disregarding the order) in $\frac{6\cdot 5}{2} = 15$ ways. We have to glue the other two dice to these with their given faces, but we can do this in 4 ways for bot dice, since we can rotate them. This way we get each construction exactly once, so the number of possibilities is $15 \cdot 4 \cdot 4 = 240$.

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3.2 Final round – day 1

3.2.1 Category A

1. In the Ducko Doble Dance School, three drakes (Duckevin, Richarduck, Duckmond) and four ducks (Aduck, Rhoduck, Duckatie, Duckota) learn to dance in pairs. Over the course of the evening, there are three dances, and with every dance, each pair consists of a duck and a drake. (This way, with each dance, a duck is left without a pair.)

In the first dance, the pairs were Duckevin-Aduck, Richarduck-Rhoduck, Duckmond-Duckatie. In the second dance, Duckevin-Duckatie, Richarduck-Aduck, Duckmond-Duckota were the pairs. Find a possible pairing for the third dance, if we know that Duckmond does not wish to dance with Aduck, nor would two ducks be in the same pair twice. *It is not necessary to find all cases, but merely to find one possible pairing.*

Let's examine who could be the pair of Duckmond in the third dance. As he already danced with Duckatie and Duckota, they cannot be the pair. Other than this, he does not wish to dance with Aduck, so his pair can only be Rhoduck.

After this, we examine the possible pair of Duckevin. We know it cannot be Rhoduck, as Duckmond shall choose her. Other than this, his pair in the first two dances were Aduck and Duckatie, so he cannot dance with them at this point. Only one possibility remains: Duckevin's pair is Duckota.

Finally, we examine who Richarduck could be paired with. We know it cannot be Rhoduck or Duckota, as they are already paired. Nor can it be Aduck, from his second dance. Richarduck and Duckatie haven't danced together before, so they could be the final pair.

Thus, we have found the only possible pairing: Duckmond-Rhoduck, Duckevin-Duckota, Richarduck-Duckatie.

Comment: The solution described here shows how to find a solution and see that it's unique. Naturally, we gave a full score for just the correct pairing.

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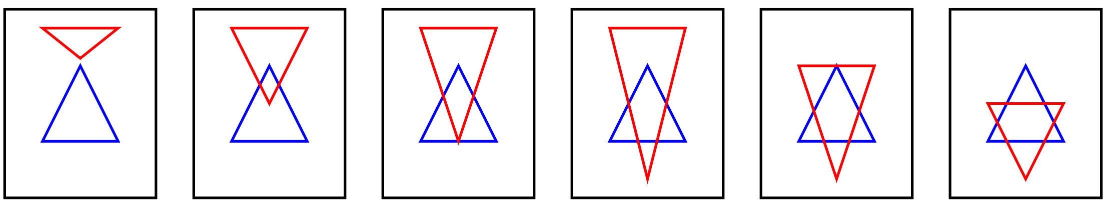
2. Each of the cases 3, 4, 5, 6, 7, and 8 are possible:



Although the task did not ask for it, for sake of completeness, we further show that no other value is possible.

It is clear that the two triangles divide the plane into at least three parts.

Let's verify that no more than 8 parts are possible. First, let's see how many parts another triangle divides the inside of, say, the blue triangle. This is at most 4 parts, because the red triangle is made up of three segments which do not intersect, and each can increase the number of parts by at most 1. Similarly, the blue triangle can divide the interior of the red triangle into at most 4 parts.

If the two triangles do not intersect, they divide the plane into 3 parts. If they are intersecting, there is a part contained in both triangles, counted twice in the sum $4 + 4$. As the exterior region is not counted here, the number of parts is therefore at most $4 + 4 - 1 + 1 = 8$.

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3. a) If not a single page is missing from a book, then the sum of two page numbers changes by 4 upon turning the page, as both numbers increase by 2. When some sheets fall out, it is one of these numbers which increases by the number of pages lost. As the difference is $71 - 49 = 22$, the number of lost pages is $22 - 4 = 18$.

b) There are two possibilities according to which place the pages fell from.

If it is between the pages of sum 49, then $71 = 35 + 36$ corresponds to page numbers 35 and 36. The other side of page 35 is 34, based on which the other page number is $49 - 34 = 15$.

If the lost pages are between the pages of sum 71, then $49 = 24 + 25$ corresponds to page numbers 24 and 25. The other side of the 25 page has number 26, after which comes the page number $71 - 26 = 45$.

Based on these, there are two solutions, except that in the first case, the even numbers are on the right, not the left. So only the latter case is possible.

Therefore, it can be uniquely determined that the page numbers from 27 to 44 are missing, so the smallest missing page number is 27.

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4. There are numerous solutions for each part of the question. We've provided 6 solutions for each.

a) Például: $7 + 6 + 5 - 4 + 3 + 2 + 1 = 20$
$7 + 6 + 5 + 4 - 3 + 2 - 1 = 20$
$7 + 6 + 5 - 4 + 3 \cdot 2 \cdot 1 = 20$
$7 \cdot (6 - 5 - 4 + 3 \cdot 2) - 1 = 20$
$7 \cdot 6 \;/\; (5 - 4 + 3 - 2) - 1 = 20$
$- 7 + 6 \cdot 5 - 4 + 3 - 2 \cdot 1 = 20$

b) Például: $8 + (7 \cdot 6 \cdot 5) - 4 \cdot (3 + 2 - 1) = 202$
$8 \cdot (7 + 6 + 5 + 4 + 3) + 2 \cdot 1 = 202$
$- 8 + 7 + (6 \cdot 5 + 4) \cdot 3 \cdot 2 - 1 = 202$
$- 8 + 7 \cdot 6 \cdot 5 \cdot (4 - 3) \cdot (2 - 1) = 202$
$8 \cdot (7 \cdot (6 + 5) \;/\; 4 + 3 + 2 + 1) = 202$
$8 \cdot (7 \cdot (6 + 5) \;/\; 4 + 3 \cdot 2 \cdot 1) = 202$

c) Például: $9 \cdot 8 \cdot 7 \cdot (6 - 5) \cdot 4 + 3 + 2 + 1 = 2022$
$9 \cdot (8 + 7 \cdot 6 \cdot 5 + 4 + 3) - 2 - 1 = 2022$
$9 \cdot 8 \cdot 7 \cdot (6 - 5) \cdot 4 + 3 \cdot 2 \cdot 1 = 2022$
$(9 \cdot 8 \cdot 7 + 6 - 5) \cdot 4 + 3 - 2 + 1 = 2022$
$9 + 8 \cdot 7 \cdot 6 \cdot (5 + 4 - 3) - 2 - 1 = 2022$
$9 \cdot (8 + 7) \cdot (6 + 5 + 4) - 3 \cdot (2 - 1) = 2022$

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5. The grandchildren can be grouped into families according to whether they are siblings.

The number of grandchildren could be 7, 8, 9, 10, 12, 13, or 14. We will show an example for each. For instance, the third example means three families have two children and one family has three children.

- 7 unoka: 1+1+1+1+1+1+1
- 8 unoka: 2+2+2+2
- 9 unoka: 2+2+2+3
- 10 unoka: 3+3+4
- 12 unoka: 6+6
- 13 unoka: 6+7
- 14 unoka: 7+7

Other numbers are not possible, and we will now prove this.

If the number of grandchildren were at most 6, then any grandchild could have at most 5 cousins, so this is not possible.

If the number of grandchildren were 15 or more, then choosing a member of two different families, say Anett and Bence, it is clear that all grandchildren are the cousin of either Anett or Bence, so in total, there are at most $7 + 7 = 14$ grandchildren, impossible.

Moreover, if the number of grandchildren were 11, then every grandchild has 3 siblings (if they said 7) or 4 siblings (if they said 6). However, 4-member and 5-member families cannot add up to 11, so this case is also impossible.

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6. Let the other player begin. AFter they made a move, there will surely be a coin occurring an even number of times, as well as a coin occurring an odd number of times. If only one type is odd, then let's take away one of these coins without replacing it.

If two types are odd, then let's take away one of the larger denomination, and add one to the smaller type.

After our move, there will surely be an even number of every coin.

From now on, let's copy the opponent's move and always make the same move they made. Thus, after our moves, there will be an even number of every coin. We can always make this move, because there will be an odd number of coins in the denomination the opponent took from. It is clear that with this strategy, only we could take the final coin, so we will surely win.

Comment: From the solution above, it is easy to read off who has a winning strategy and when in general, that is, when we start out not with 3 ones, 5 twos and 7 threes, but with an arbitrary other starting state. All this is explained in detail in the solution for the B version of the game.

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3.2.2 Category B

1. For the solution, see Category A Problem 2.

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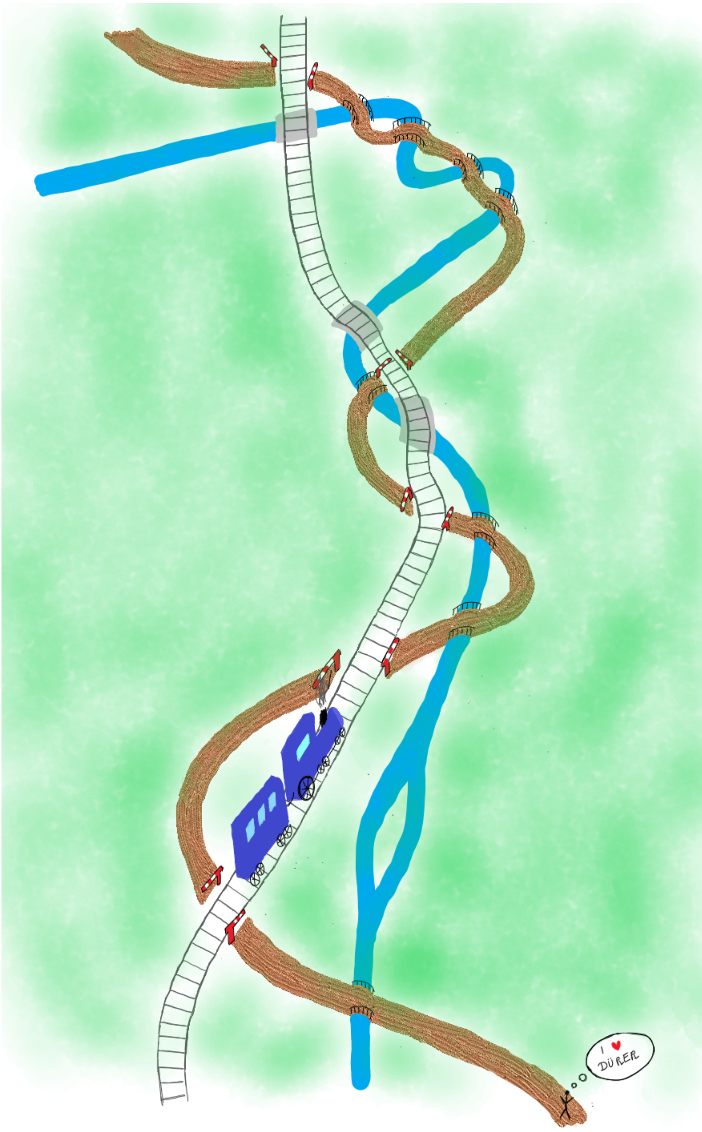
2. The main observation is that we reach the other side of the stream if and only if we crossed it an odd number of times. Indeed, when we cross it twice, we go back to the starting side, so an odd number of crossings is the same as crossing it once, and an even number of crossings is the same as not crossing it. The same can be said for the railway.

We know that we crossed the stream eight times, so at the end, we remain on the same side as before, with the stream to the left of the path. We crossed the railway five times, so it will be on our other side at the end, to the right of the path. Therefore, the stream is to our left, the railway is to our right, this determines the order. One possible example can be seen on the diagram.



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3. For the solution, see Category A Problem 4.

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4. For the solution, see Category A Problem 5.

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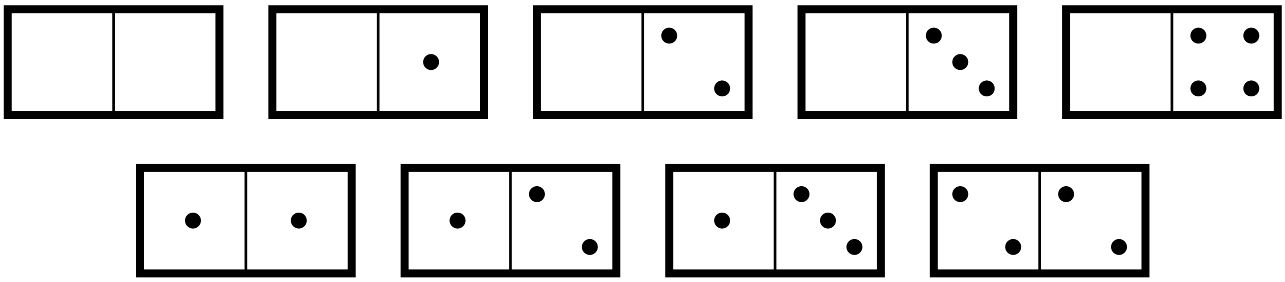
5. a) Dominika chose the following dominos:



This is a total of 9 pieces.

b) Let's denote the dominos according to their number of dots. For example, the pair $(3, 1)$ means one side has 3, the other side has 1 dot. First, let's count the cases where the dominos $(0, 0)$, $(1, 1)$, and $(2, 2)$ are omitted in the ordering.

Let's notice that apart from the numbers at the ends, every number appears in pairs. This means that if a number appears an odd number of times, then it occurs at an end. By looking, it is 1 and 4 which appears an odd number of times, so these must occur at the ends. First, let's count those cases where 4 is at the left ending.

The remaining dominos $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 2)$, and $(1, 3)$ are to be placed such that the left ending is 0, the right ending is 1. Observe that dominos $(0, 2)$ and $(1, 2)$ are to meet at 2, while $(0, 3)$ and $(1, 3)$ at 3. Combining, we are to place $(0, 1)$, $(0, 2)(2, 1)$, and $(0, 3)(3, 1)$. All three sequences have 0 on one end and 1 on the other, so they can be arranged in any order, in $3! = 6$ ways.

As the cases where 4 is at the right ending are the same, there are $2 \cdot 6 = 12$ regular ways to order the dominos, if $(0, 0)$, $(1, 1)$, $(2, 2)$ are discarded.

Finally, in each of these cases, $(2, 2)$ can be inserted in exactly one place, while $(1, 1)$ and $(0, 0)$ can be inserted in two ways. Every possible ordering is produced this way – a total of $12 \cdot 2 \cdot 2 = 48$ regular orderings.

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6. Let's say that a state is a winning position if upon producing it, we win. For instance, the state with 0 of all coins is a winning position. Another winning position is, for example, when there are 2 ones remaining, as then the other player can only remove one of these, after which we win by removing the last one. Our aim is to find all the winning positions.

By trial and error, we arrive at the belief that the winning positions are exactly those with an odd or an even number of all coins. Let's set out to prove this.

If we just produced a winning position, it is clear that the opponent cannot move to a winning position. But from a non-winning position, where there are either one or two odd coin counts, we may either remove a coin from an odd pile, or replace a coin from one odd pile to the other, resulting in a position with all even numbers of coins.

Based on these, we need to let the opponent begin when all denominations occur with the same parity. Otherwise, we need to begin. Our strategy is to keep producing winning positions, as described above.

Comment: From the above, one can see that the winning player make their first move such that in that position, all denominations occur an even number of times. From here on, the strategy of copying the opponent's move shall work.

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3.3 Final round – day 2

3.3.1 Tables

#	ANS	Problem	P
A-1	13	Csongi duck dined on a dozen dozen lawn larvae	3p
A-2	7	Three ducks, Huey, Dewey, and Louie,	3p
A-3	19	Three girls and three boys participate in a lottery game.	3p
A-4	5	The sheriff's son collected sticks to duel a catus.	3p
A-5	31	Waddles, the clever duck,	4p
A-6	25	Five friends,	4p
A-7	279	Gábor collected all four-digit numbers	4p
A-8	16	Mister Hamster's favorite chocolate is the one with holes in it.	4p
A-9	14	Adam mows the lawn in his garden.	5p
A-10	16	Adam, Benedict, Calvin, Daniel, and Hannah	5p
A-11	37	Bugs Bunny thought of a positive integer less than 20,	5p
A-12	60	Lucy made the tower of cubes	5p
A-13	90	On the railway from Duckton to Duckaster,	6p
A-14	8	A 500-calory packet of mixed nuts	6p
A-15	82	100 chameleons sit in a row on a long tree branch,	6p
A-16	96	The basic design of a house is a 2×9 rectangle	6p

#	ANS	Problem	P
B-1	13	Csongi duck dined on a dozen dozen lawn larvae	3p
B-2	19	Three girls and three boys participate in a lottery game.	3p
B-3	5	The sheriff's son collected sticks to duel a catus.	3p
B-4	31	Waddles, the clever duck,	3p
B-5	72	On the diagram, the spider moves 12 centimetres	4p
B-6	20	Rainville, Cloudford, and Stormland	4p
B-7	16	Mister Hamster's favorite chocolate is the one with holes in it.	4p
B-8	14	Adam mows the lawn in his garden.	4p
B-9	16	Adam, Benedict, Calvin, Daniel, and Hannah	5p
B-10	60	Lucy made the tower of cubes	5p
B-11	90	On the railway from Duckton to Duckaster,	5p
B-12	8	A 500-calory packet of mixed nuts	5p
B-13	82	100 chameleons sit in a row on a long tree branch,	6p
B-14	96	The basic design of a house is a 2×9 rectangle	6p
B-15	1429	Anett has a 1000-page novel to read over the summer vacation.	6p
B-16	945	We have three large boxes of cookies:	6p

3.3.2 Category A

1. A dozen dozen is 12 dozen, and a dozen is 1 dozen. In total, $12 + 1 = 13$ dozen.

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2. On Dewey's and Louie's hats, there are as many feathers in total as on Huey's, namely 10. On Dewey's hat, there are four more feathers than on Louie's, so there are 3 feathers on Louie's and 7 on Dewey's.

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3. As there are 3 girls and 3 boys, and an equal number of girls and boys won presents, the following three cases are possible:

- If 1 girl and 1 boy won a present, then there are 3 possibilities for the girl, 3 for the boy, $3 \cdot 3 = 9$ possibilities.
- If 2 girls and 2 boys won a present, let's rather list those who didn't win anything. Again, there are $3 \cdot 3 = 9$ possibilities for that.
- If 3 girls and 3 boys won a present, then everybody won, so that's 1 possibility.

In total, there are $9 + 9 + 1 = 19$ possibilities for the list of winners.

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4. Whenever the boy hits the cactus, the number of his sticks increases by 3. Which means if he his it only once then originally he had $17-3=14$ sticks; if he hit it twice then $17-2\cdot3=11$ and so on. So the number he collected originally can be 14, 11, 8, 5 or 2, which is 5 possibilities.

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5. The first observation is that on the weekends, he does not swim a single metre, and so the following week, he will swim 0 metres, always twice the previous day's amount. On the five weekdays of the first week, he swims a total of $1+2+4+8+16=31$ metres. Therefore, Waddles swims 31 metres in all.

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6. Let's think in reverse! There were 2 pebbles left for Zsófi. In front of Kristóf, there had to be 8 pebbles, as eight is the number whose quarter is 2. Before Kartal's toss, there were $8+4=12$ pebbles. Before Anett's toss, there were as many pebbles as $\frac{2}{3}$ the number of pebbles left after her toss, which is 12, so there were 18 pebbles before her toss. Before Andris's toss, there were $18+7=25$ pebbles, so the answer is 25.

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7. If 15 is the first two digits of the number, then there are $10\cdot10$ possibilities for the other two. If 15 is the second and third digit, then the first digit has 9 possible values, the last has 10, so there are $9\cdot10=90$ possibilities in total. If 15 is the third and fourth digit, then similarly, there are $9\cdot10=90$ possibilities. Therefore, we have counted every number this way, in fact, there is a certain number we have counted twice, namely 1515. So the answer is $100+90+90-1=279$.

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8. The perimeter of the larger rectangle is $2\cdot(12+40)=104$ centimetres, so the small rectangle has a 52-centimetre perimeter. The area of the larger one is $12\cdot40=480$ cm^2, so the smaller one has area 160 cm^2. Thus, if the side lengths of the small rectangle in centimetres are a and b, then $a+b=52/2=26$ and $a\cdot b=160$. It is easy to check that the only solution for this is $a=10$, $b=16$ or vice versa. Therefore, the side length x can only be 16 cm, because the shorter side length of the large rectangle is 12 cm, so the side parallel to that could only be shorter.

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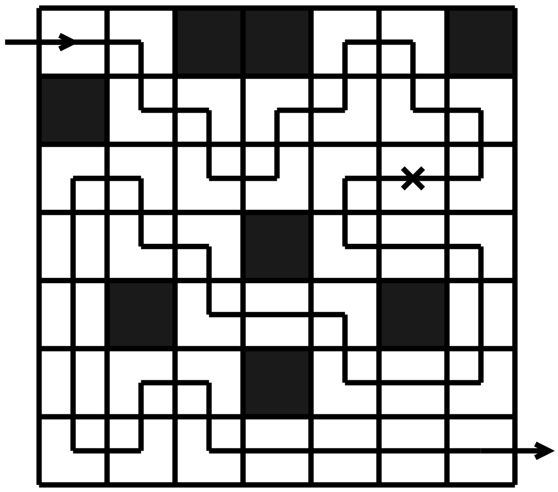
9. Thinking it through, we see that Adam could only proceed along the following path. Thus, the field with an X is 14th.



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10. After Benedict, there are two possible orders: Adam-Benedict and Benedict-Adam. After this, every contestant doubles the number of possible orders by taking the place before or after them. There were three more contestants, so there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways they could finish.

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11. Dodo duck asks every number twice. If he gets two no answers, that number surely isn't it. But if he gets an answer yes, he asks a third time. If the answer is yes, then that's the number, and if no, then he moves on. With this, Bugs Bunny already lied, so it is enough to ask about the remaining numbers once, excluding the final number. Thus, Dodo asks $2 \cdot 18 + 1 = 37$ questions at most.

It turns out that 36 questions could be insufficient for guessing the right number. If there are two numbers he asks about at most once, then it could be that Bugs Bunny replied no for each question, and it is unclear which of these two numbers is it. If all but one number was asked about at least twice, then all numbers were asked about twice, and one number was not asked about. In this case, it could be that Bugs Bunny replied yes to the final question, so Dodo cannot tell between that number and the one he didn't ask about.

Therefore, 37 questions are indeed necessary.

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12. There are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ distinct ways to place six cubes if there is no extra condition. However, with the tower of height 2, the bottom cube must be placed first. This means that whichever cubes are used in this tower, for every 2 ways, only 1 way is valid. Similarly, with the tower of height 3, there is 1 out of $3 \cdot 2 \cdot 1 = 6$ possible ways which is valid. Therefore, the gross number of towers divided by $2 \cdot 6$ is the answer. The cubes can be placed in $\frac{720}{12} = 60$ different ways.

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13. Tommy duck sits at the station in Duckchurch, watching the trains, not travelling anywhere. If Tommy observes how much time one waits between a train from Duckton and a train from Duckaster, the time is always the same, as trains depart hourly. The same is true in the opposite direction.

Further, Tommy may notice that between two arrivals from Duckton, which is an hour, exactly one train from Duckaster arrives in Duckchurch. If he writes down the two times, then their sum is precisely 60 minutes.

Tony stepped off a train from Duckton and boarded the opposite direction, and Toby stepped off a train from Duckaster and also boarded the opposite direction. Thus, they waited a total of one hour at Duckchurch.

From the text, we know that the travel from Duckton to Duckchurch and back took 100 minutes with Tony's wait, and the travel from Duckaster to Duckchurch and back took 140 minutes with Toby's wait. If we add all these, we find that the travel from Duckaster to Duckchurch and back is 240 minutes, together with Tony's and Toby's wait. As we know they waited a total of 60 minutes, the total ride there and back is $240 - 60 = 180$ minutes. Thus, the train from Duckton to Duckaster makes the trip in $\frac{180}{2} = 90$ minutes.

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14. Since a cashew nut has as many calories as a peanut and a raisin in total, there are 500 calories in $51 + 16 = 67$ peanuts and $33 + 16 = 49$ raisins, as well. If we find the amount of calories in a peanut and a raisin, we obtain the calory content of a cashew nut as their sum.

We know that the peanuts and the raisins have a positive integer calory. The calory content of a peanut can be between 1 and 7, because $8 \cdot 67 = 536$ is already more than 500.

Let's investigate these 7 cases. If there is 1 calory in a peanut, then there must be $500 - 1 \cdot 67 = 433$ calories in the raisins. This is not divisible by 7, nor by 49, so a peanut is not 1 calory.

Similarly, $500 - 2 \cdot 67 = 366$, $500 - 3 \cdot 67 = 299$, $500 - 4 \cdot 67 = 232$, $500 - 5 \cdot 67 = 165$ are not divisible by 7, nor by 49. So the only possibility is that a peanut is 6 calories, as $500 - 7 \cdot 67$ is also not divisible by 7.

In this case, $500 - 6 \cdot 67 = 98$, so a raisin is 2 calories, and a cashew is $6 + 2 = 8$. Thus, a cashew contains 8 calories.

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15. Let's see what happens with two chameleons, let's say, Caroline and Candice, who sit on the branch consecutively, looking eye to eye. If they are both the same color at first, say, blue, then after a bird flies over the tree, they remain blue, and so on. If a few chameleons sit behind Candice looking the same way, then after a while, they will also be blue, and the same goes for chameleons behind Caroline.

If Caroline was red at first and Candice blue, then after a bird flies over the tree, it is Candice who is red and Caroline who is blue, and after the next bird, this is swapped again, and so on. Similarly, the chameleons behind Candice or Caroline will switch colors red and blue after a while.

Let's divide the chameleons into blocks by drawing a line between chameleons sitting with their back to one another. Then every block appears as a few chameleons facing right, followed by a few chameleons facing left. Exceptionally, at the two ends of the branch, there need not be a chameleon facing right or left, respectively. Previously, we have noted that after many birds flying over the tree, an entire block will be the color of the two chameleons facing one another. The exceptional blocks at the two ends, if any, will take on the color of the one at the end. Therefore, after 100 birds fly over the tree, every block will have one or two colors.

After 100 birds, there are 37 different colors, and every block is at most two colors, so at least 19 blocks are needed. With at most one exception, there must be a chameleon facing left in every block, so there are at least 18 chameleons facing left. Thus, at most 82 chameleons face right.

This is indeed possible. If there are 18 blocks of chameleons facing one another, followed by a row of chameleons facing right, and every chameleon has its own color at first, then by previous observations, the first 18 blocks have two colors after the birds fly over, and the final block is the same color, all different. Thus, in this case, the chameleons will sport 37 colors indeed.

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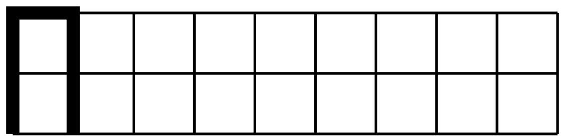
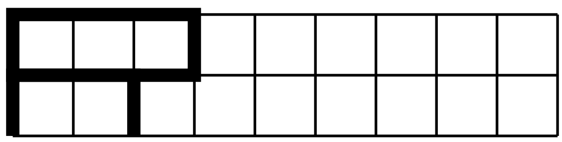
16. Let's solve this question for n rather than for 9. Let L_n denote the number of possible partitions of the $2 \times n$ rectangle. We think recursively, finding L_n from the values L_k for $k < n$.

The main idea is to seek from the left edge of the $2 \times n$ rectangle the first vertical line of length 2 which is the complete right wall of an apartment. We will examine cases according to what shaped apartment the bottom left room belongs to.

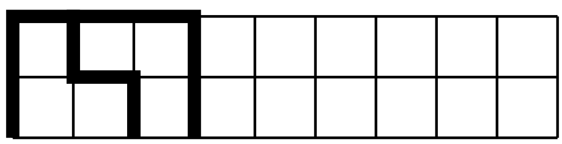
In the first case, the partition begins with a 2×1 rectangle (see diagram). In this case, the number of partitions equals the number for a $2 \times (n-1)$ rectangle, namely L_{n-1}.



In the second case, a horizontal 1×2 rectangle contains the bottom left room (see diagram). But now, the top left room cannot reach the corridor, so there is no partition in this case.

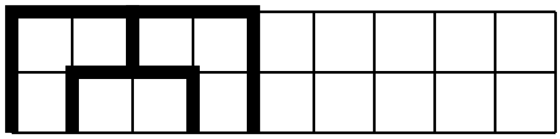


In the third case, there is an L-shape apartment containing the bottom left room, like on the diagram. For the second top room to reach the corridor, it must be in an apartment shaped like an inverted L. Thus, in this case, the number of partitions equals the number for a $2 \times (n-3)$ rectangle, namely L_{n-3}.

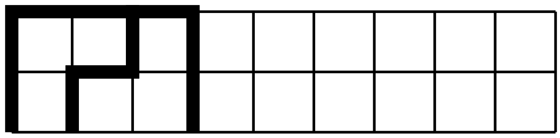


In the fourth case, an inverted L shaped room contains the bottom left room. There are two subcases (see diagrams below).

If a 2-room apartment contains the second bottom room, then it must be a 1×2 rectangle. Then, the third top room can only reach the corridor in an inverted L shaped apartment. But then the number of partitions is given by those of the remaining $2 \times (n-4)$ rectangle, namely L_{n-4}.



In the second subcase, the second bottom room is contained in a reflected L shaped apartment. So the partitions correspond to partitions of a $2 \times (n-3)$ rectangle, and there are L_{n-3} possibilities.



All in all, we have found that $L_n = L_{n-1} + 2 \cdot L_{n-3} + L_{n-4}$.

It is easy to calculate that $L_1 = 1$, $L_2 = 1$, $L_3 = 3$, $L_4 = 6$. From here on, using the formula we obtained, $L_5 = 6 + 2 \cdot 1 + 1 = 9$, $L_6 = 16$, $L_7 = 31$, $L_8 = 55$, and $L_9 = 96$. Therefore, the answer to the question is 96.

Comment: From the solution above, it is clear that the apartments can be arranged into 2×1 regions, two kinds of 2×3 regions, and 2×4 regions. Based on this, the answer may be obtained through casework.

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3.3.3 Category B

1. For the solution, see Category A Problem 1.

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2. For the solution, see Category A Problem 3.

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3. For the solution, see Category A Problem 4.

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4. For the solution, see Category A Problem 5.

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5. On the thickened line, there are three different lengths, denoted $a > b > c$. We may observe that $a = b + c$, as both are sides of the same regular triangle. At the end, the spider is at a distance $b + c$, i.e., $a = b + c = 12$. From the centre of the path, the lengths are $b, c, a, c, b, b, c, a, a$. Therefore, the length of the path is $3a + 3b + 3c = 3a + 3(b + c) = 6a = 72$ centimetres.

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6. If Cloudford is 10 km from Rainville and Stormland, and these are along a line, then Stormland is 20 km from Rainville. If the raincloud moves at a rate of 20 km per hour, and it is at a distance 20 km from Rainville, then it arrives there in exactly 1 hour, or 60 minutes. Ann must travel 10 km at a speed of 15 km per hour, which takes 40 minutes. Therefore, Ann has 20 minutes before she needs to head home.

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7. For the solution, see Category A Problem 8.

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8. For the solution, see Category A Problem 9.

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9. For the solution, see Category A Problem 10.

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10. For the solution, see Category A Problem 12.

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11. For the solution, see Category A Problem 13.

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12. For the solution, see Category A Problem 14.

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13. For the solution, see Category A Problem 15.

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14. For the solution, see Category A Problem 16.

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15. Notice that the consecutive page numbers cannot be both circled in red, as that would mean they are divisible by 7. Similarly, they cannot be both circled in green, as that would mean they are divisible by 11. The page number not circled in red must be circled in the three remaining colors, as is the case with the page number not circled in green. Hence, the digits 1 and 7 occur in both page numbers. If either digit occurs only in the ones digit, then the preceding digits are all the same, as no tens are carried. But then our digit will not occur in both numbers. Therefore, the 1 and 7 both occur in the tens and hundreds digit, so the page numbers are between 170 and 180 or between 710 and 720. The only multiple of 11 between 170 and 180 is 176, which has a neighboring multiple of 7, namely 175. The only multiple of 11 between 710 and 720 is 715, which has a neighboring multiple of 7, namely 714. However, according to the question, the even page numbers are on the left, so it is the even number which is smaller. Therefore, the two numbers could only be 714 and 715, and their sum is 1429.

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16. There are 3 possible choices for the centre, yielding a multiplier 3. Given the central cookie, there are 3^7 ways to place the crackers without rotations. But if the 7 crackers are not the same, then all 7 rotations are different, and are to be counted once. There are $3^7 - 3$ such choices, which we count $\frac{3^7-3}{7} = 312$ times. The arrangements with the same flavors are all distinct, so in total, there are $3 \cdot (312 + 3) = 945$ different flowers.

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4 Solutions – high school categories

4.1 Online round

4.1.1 Tables

#	ANS	Problem	P
C-1	360	Imagine a building with a ground floor and five further floors.	3p
C-2	3	Benedek went to the meadow looking for 4-leaf clovers.	3p
C-3	29	Anett has a diary,	4p
C-4	9	Csenge draw four lines on a paper.	4p
C-5	200	Anett, Andris, and Orsi went for a bike ride.	4p
C-6	135	How many degrees is the angle marked on the picture?	5p
C-7	8896	The sum of a single-digit number divisible by one,	5p
C-8	10	Máté is rolling a die on a checkered paper,	6p
C-9	2124	Scrooge McDuck keeps his wealth in a big safe,	6p

#	ANS	Problem	P
D-1	3	Benedek went to the meadow looking for 4-leaf clovers.	3p
D-2	29	Anett has a diary,	3p
D-3	200	Anett, Andris, and Orsi went for a bike ride.	4p
D-4	8	On Béluska's keyboard,	4p
D-5	8896	The sum of a single-digit number divisible by one,	4p
D-6	90	The 10 members of a duck family are standing in a circle,	5p
D-7	81	The side lenght of square $ABCD$ is 3 units.	5p
D-8	864	Samantha Swan, Susie Swan, Giselle Goose,	6p
D-9	2124	Scrooge McDuck keeps his wealth in a big safe,	6p

#	ANS	Problem	P
E-1	29	Anett has a diary,	3p
E-2	200	Anett, Andris, and Orsi went for a bike ride.	3p
E-3	8	On Béluska's keyboard,	4p
E-4	8896	The sum of a single-digit number divisible by one,	4p
E-5	90	The 10 members of a duck family are standing in a circle,	4p
E-6	81	The side lenght of square $ABCD$ is 3 units.	5p
E-7	864	Samantha Swan, Susie Swan, Giselle Goose,	5p
E-8	36	Csongi really likes watching the ducks at the lakeside.	6p
E-9	2124	Scrooge McDuck keeps his wealth in a big safe,	6p

4.1.2 Category C

1. Counting from the bottom to the top, each of the floors have 19, 21, 19, 21, 19, and 21 rooms respectively. This means $3 \cdot (19 + 21) = 3 \cdot 40 = 120$ rooms in total, and every room has 3 windows, so there are $120 \cdot 3 = 360$ windows in total.

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2. The remainders of the heaps modulo 3 are 0, 1, and 2 respectively. The remainder of 4 divided by 3 is 1, so the first heap could consist of exclusively 3-leaf clovers, the second heap has at least one 4-leaf clover, and the third heap has at least two. This is possible: the first heap can consist of 14 3-leaf clovers, the second heap can consist of one 4-leaf and 11 3-leaf clovers, and the third heap can have two 4-leaf and 8 3-leaf clovers. Hence, Benedek found at least 3 4-leaf clovers.

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3. Let us write the days and weeks in the following table. If two marked days can be on the same week, write them in the same week. It is easy to see that this way we get the shortest time period.

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Week 1			x	x			
Week 2		x			x	x	
Week 3			x				x
Week 4						x	
Week 5				today			

We can see that at least 4 weeks and 1 day passed until the Dürer online round, so Anett started her diary at least $4 \cdot 7 + 1 = 29$ days ago.

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4. If there is a point which all 4 lines go through, then for all other points of the plane there is at most 1 line going through it, thus for the 4 points the sum of numbers is at most $4+1+1+1 = 7$. If there is no point, for which there are 3 lines going through it, then the sum can be at most $4 \cdot 2 = 8$. The last case if, if there is a point, through which there are exactly 3 lines. In this case, there is no other point, which has 2 or more out of these 3 lines going through it, so there are at most 2 lines going through any other point. Hence for the 4 points, the sum is at most $3 + 2 + 2 + 2 = 9$. Because of this the largest possible sum is 9, as we have considered all possible cases. This sum of 9 can be achieved, if there are 3 lines going through a common point, and the fourth line intersects the other three in three different points, and Csongi picks the four points of intersection. Hence the answer is 9.

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5. The bicycles have be assigned in a way, which ensures that the slowest person's velocity is as large as possible. This can be achieved by giving the fastest bicycle to the slowest cycling person, the second fastest bicycle to the second slowest cycling person, and the slowest bicycle to the fastest cycling person.

Now let's calculate the slowest person's velocity with this assignment: Andris's velocity is $2 \cdot 2.5\frac{\text{m}}{\text{s}}$, Anett's velocity is $3 \cdot 1.5\frac{\text{m}}{\text{s}}$, and Orsi's velocity is $5 \cdot 1\frac{\text{m}}{\text{s}}$. Thus, the slowest of them has a velocity of $4.5\frac{\text{m}}{\text{s}}$, which means they take $\frac{54000}{4.5}$ s $= 12000$ s $= 200$ minutes to complete the route.

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6. Let the intersection of the circle with center A and the AC segment marked with gray be P. Furthermore, let the point at the green angle be Q. The ADP triangle is isosceles, as all radii of the circle have equal length. The ADP triangle's inner angle at A is 45°, so the other two angles are 67.5° each. Thus $QDC\sphericalangle = PDC\sphericalangle = 90° - 67.5° = 22.5°$, and likewise $QCD\sphericalangle = 22.5°$. From this, it follows, that $DQC\sphericalangle = 180° - 22.5° - 22.5° = 135°$. The angle marked with green forms vertical angles with this, so it is 135° as well.

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7. Let a be the single-digit number, b be the 2-digit number, c be the 3-digit number, and d be the 4-digit number. $a+b+c+d$ has five digits, so it is at least 10000. c has 3 digits and it is divisible by 3, so $c \leq 999$. b has 2 digits and it is divisible by 2, so $b \leq 98$, and $a \leq 9$.

Therefore $d \geq 10000 - 999 - 98 - 9 = 8894$. And d is divisible by 4, so in fact $d \geq 8896$, because this is the smallest number divisible by 4 after 8894. This can be optained: Let $d = 8896$, $c = 999$, $b = 98$, $a = 7$. These satisfy the conditions and their sum is $8896 + 999 + 98 + 7 = 10000$, which is a 5-digit number divisible by 5. So the solution is 8896.

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8. We can assume that the die is numbered in the usual way: 1 is on the side opposite to the side with 6, 2 is on the side opposite to the side with 5, and 3 is on the side opposite to the side with 4. Máté can roll the dice over to any side except for the one opposite to the current one. Thus the only numbers he can't write down are the ones which have two adjacent digits which are on opposite sides of the die.

The first digit is 1. Let's assume that the second number is 2 and let's count the number of legal final numbers we can end up with. We will have to multiply this count by 4 at the end to get the result, as because of symmetry there will be the same number of final numbers if the second digit is 3, 4 or 5. We split the counting into four cases, based on where the digit 6 appears in the final number.

Case 1: 6 is the third digit. In this case, as 3 and 4 aren't next to each other, they are the fourth and sixth digits in some order, and 5 is the fifth digit. These are 2 possibilities.

Case 2: 6 is the fourth digit. In this case, the third digit is either 3 or 4, these are 2 possibilities. In both cases the last two digits can be in arbitrary order, so these are another 2 possibilities, for a total of $2 \cdot 2 = 4$ different final numbers.

Case 3: 6 is the fifth digit. As 3 and 4 aren't next to each other, they can't be the third and fourth digit. However, the third digit can't be 5 either, as it can't be next to 2 which is the second digit, thus 5 must be the fourth digit. 3 and 4 can be in arbitrary order at the third and sixth digits, so these are 2 possible ways.

Case 4: 6 is the last digit. In this case, as 3 and 4 can't be next to each other, they must be the third and fifth digits in some order, and hence 5 must be the fourth digit. These are 2 possibilities as well.

Thus in total there are $2 + 4 + 2 + 2 = 10$ final numbers in which 2 is the second digits, so there are $4 \cdot 10 = 40$ different numbers Mát could have written down.

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9. Let us denote the empty squares by A, B, $\ldots I$ the following way:

A	/	B	×	C	=	3
+		×		+		
D	/	E	×	F	=	28
+		/		−		
G	×	H	+	I	=	29
=		=		=		
20		1		4		

Look at the middle row. Since $D \times F/E = 28$ and $D \times F \leq 72$, we get $E \leq 2$. If $E = 1$ then because of the middle row $B/H = 1$, thus $B = H$, which is not possible. Therefore $E = 2$ and $D \times F = 56$. It is only possible if the two numbers are 7 and 8 in some order.

Now look at the middle column. $B \times 2/H = 1$, thus $2B = H$. The numbers that are still available are the following: 1, 3, 4, 5, 6, 9. The only possible solution is $B = 3$ and $H = 6$.

Left hand side column: Since $G = 4$, we get $A + D = 16$. The only way to reach this is if the two numbers are 7 and 9. Now we obtain all the prevously missing values: $A = 9$, $C = 1$, $D = 7$ and $F = 8$.

9	/	3	×	1	=	3
+		×		+		
7	/	2	×	8	=	28
+		/		−		
4	×	6	+	5	=	29
=		=		=		
20		1		4		

From this, the secret code is $931 + 728 + 465 = 2124$.

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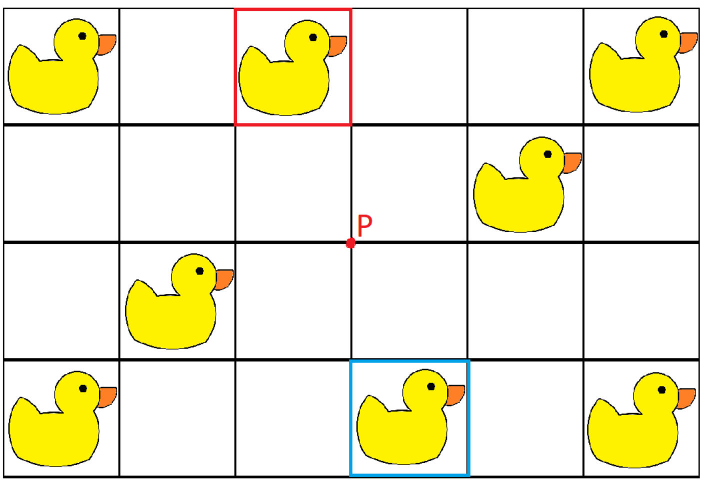
10. The second player can always win. Their strategy should be the following: wherever the first player puts a duck (red square), they should reflect this square through P, the centre of the board, and put a duck in this resulting square (blue square). This will always be a legal move, as if there already were a duck in one of the 4 neighbouring squares, then there would be a duck next to the first player's duck as well, as the figure is centrally symmetric. Thus, the second player will always be able to make a move, after the first player made a move, so they have a winning strategy, as after a finite time there will be no more possible moves.



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4.1.3 Category D

1. For the solution, see Category C Problem 2.

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2. For the solution, see Category C Problem 3.

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3. For the solution, see Category C Problem 5.

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4. Let's count the possible numbers based on their number of digits.

One-digit numbers are all easy-to-type, among them 2, 3, 5, and 7 are primes, so 4 numbers.

The two-digit numbers can all be checked: in the following table we have written down which ones are prime, and for the ones that are not, we have also written down a proper factor of it as well.

Number	Is prime?	Number	Is prime?
12	no (2)	10	no (2)
23	yes	21	no (3)
34	no (2)	32	no (2)
45	no (5)	43	yes
56	no (2)	54	no (2)
67	yes	65	no (5)
78	no (2)	76	no (2)
89	yes	87	no (3)
		98	no (2)

Hence, among two-digit numbers, there are 4 easy-to-type primes.

Among three-digit numbers there are no solutions, as if a number consists of 3 adjacent digits, ($x-1$, x, and $x+1$) then its sum of digits is $3x$, which is divisible by 3. If a number's sum of digits is divisible by 3, then so is the number itself, thus (as the number is bigger than 3) it can't be a prime.

Hence Béluska can send $4+4=8$ different numbers in total.

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5. For the solution, see Category C Problem 7.

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6. Notice, that the duck standing in place 0 originally will be a part of all place swaps. Hence after every 10-th swap it will be back to place 0, as after n swaps, it will stand on the place marked with the remainder of n modulo 10.

We can also notice, that if we complete 10 swaps starting with the $0-1$ swap, then other than the duck at place 0, everyone will be standing at a place with a number 1 smaller than their place before the 10 swaps, except for the duck at place 1, who will stand at place 9. We can imagine this as the 9 ducks at the non 0 places rotating by one: place 2 moves to place 1, place 3 to 2, ..., 9 to 8 and 1 to 9. Thus after iterating this process of 10 swaps 9 times, everyone will get back to their own places, so after 90 swaps everyone will be back at their original place.

From the though process above, we can also see that less swaps than this are not sufficient, as the number of swaps has to be divisible be by 10, in order for the duck at place 0 to be back to its original place, and for a numbers divisible by 10, less than 90, the other family members won't be back to their original places. Hence, the answer is 90.

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7. From the Pythagorean theorem, the diagonal of a square is $\sqrt{2}$ times the side lenght. Therefore $AC = 3\sqrt{2}$. Note that $AL + JC = AC + JL$, thus $JL = 2 \cdot 3 - 3\sqrt{2}$, so we get the side lenght of the little square from dividing this by $\sqrt{2}$; it is $3\sqrt{2} - 3$. Also, $FH = FA + AC + CH = 3 + 3\sqrt{2} + 3 = 6 + 3\sqrt{2}$, so divide this by $\sqrt{2}$ to get the side lenght of the big square, therefore it is $3\sqrt{2} + 3$. The product of the areas of the two square is

$$(3\sqrt{2}-3)^2(3\sqrt{2}+3)^2 = ((3\sqrt{2}-3)(3\sqrt{2}+3))^2 = (18-9)^2 = 81.$$

$$(3\sqrt{2}-3)^2(3\sqrt{2}+3)^2 = ((3\sqrt{2}-3)(3\sqrt{2}+3))^2 = (18-9)^2 = 81.$$

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8. The 3 ducks must stand in the front row, the 2 swans must stand in the back row, so there will be 1 goose in the first, and 2 geese in the back row. Behind the goose in the front row there must be a swan standing, but other than this the only restrction is that the ducks stand in the front, the rest in the back, and this guarantees that everyone is visible.

There are 3 ways of choosing which goose stands in the first row, and there are 2 ways to choose the swan behind them. Furthermore, there are four ways to choose which column this goose-swan pair stands in. This means $3 \cdot 2 \cdot 4 = 24$ different possibilities. Besides this, the 3 ducks in the first row can stand in 6 ways, and in the back row the remaining 2 geese and 1 swan can stand in 6 different ways as well. Thus, in total, there are $24 \cdot 6 \cdot 6 = 864$ possibilities.

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9. For the solution, see Category C Problem 9.

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10. The first player has a winning strategy if and only if the number of stones in the heap is not divisible by 3. In this case their strategy is to always remove a number of stones, so that the remaining number is divisible by 3. They are always able to do this by removing 1 or 2 stones, if the original number of stones in the heap is not divisible by 3. In this case, the second player always gets a heap with a number of stones which is divisible by 3, and whatever power of 2 they choose to remove, there remaining count will never be divisible by 3. Hence, if the first player follows this strategy, they will never lose, as after any move of the second player, there will be a number of stones not divisible by 3, thus greater than 0. Because of this, the first player will definitely win, as the game ends in a finite number of moves.

In case the original number of stones is divisible by 3, the second player has a winning strategy, as after the first player's move there will be a number of stones not divisible by 3, so the second player is able to implement the strategy described above.

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4.1.4 Category E

1. For the solution, see Category C Problem 3.

(Back to problems)

2. For the solution, see Category C Problem 5.

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3. For the solution, see Category D Problem 4.

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4. For the solution, see Category C Problem 7.

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5. For the solution, see Category D Problem 6.

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6. For the solution, see Category D Problem 7.

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7. For the solution, see Category D Problem 8.

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8. Let's notice, that Csongi will see the head of every duck exactly 3 times and the duck will be underwater 2 times. If n is the number of ducks, then we have the following equation:

$$3 \cdot n = 24 + 22 + 20 + 17 + 25 = 108$$

Thus there are $n = 36$ ducks in the

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9. For the solution, see Category C Problem 9.

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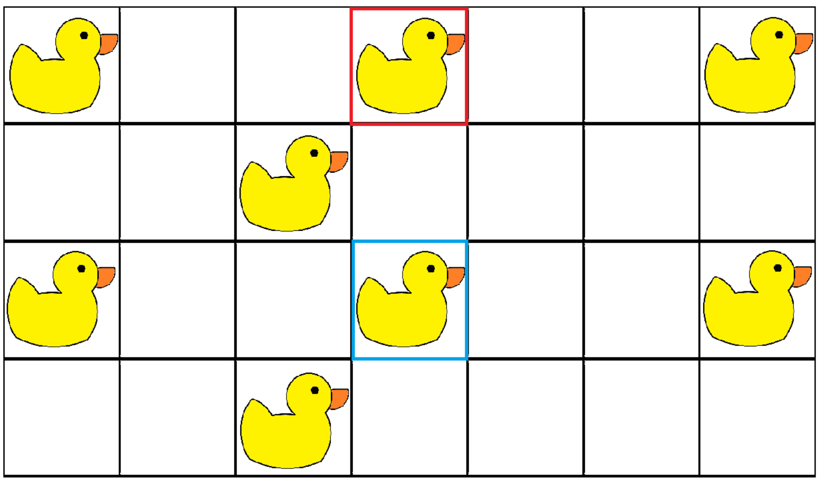
10. The second player can always win with the following strategy: If the first player puts a duck into the (red square) first or third row, into the k-th column, the second player should put a duck into the third or first row respectively, into the k-th column. Likewise, if the first player puts a duck into the second or the fourth row, into the k-th column, then the second player should put a duck into the fourth or the second row. This will always be a legal move, as if there were another duck next to the one placed by the second player, then there would have been one next to the first player's duck as well. This is because the first and third rows, and the second and fourth rows will look the same after every move of the second player. Hence, the second player will always be able to make a move, so they have a winning strategy, as there can only be a finite number of steps, so the game will end at some point.



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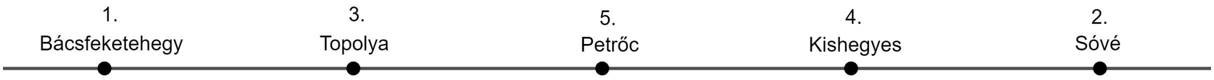
4.2 Regional round

4.2.1 Category C

1. a) From the final point, we know that Topolya is the second town. Let's see what we know about the locations of Bácsfeketehegy, Petrőc, and Sóvé. From the first point, it is clear that

Bácsfeketehegy is either the first or the fifth along the road. Based on the third point, Petrőc, with the smallest army, can send a message to Topolya, which means that Topolya can only be its neighbour, making Petrőc first or third along the road. Based on the fourth claim, Sóvé is either in third or fifth place. So far, we know the following. Bácsfeketehegy: 1st or 5th, Kishegyes: ?, Petrőc: 1st or 3rd, Sóvé: 3rd or 5th, Topolya: 2nd. Since Bácsfeketehegy, Petrőc, and Sóvé occupy the first, third, or fifth location, the fourth city is surely Kishegyes. By the second point, Kishegyes cannot message Bácsfeketehegy, so they cannot be neighbours, and Bácsfeketehegy can only be the first city along the road. For this reason, the fifth city can only be Sóvé, and the third is Petrőc. The cities are located in the following order: Bácsfeketehegy, Topolya, Petrőc, Kishegyes, Sóvé.

b) We know that Kishegyes has 4830 soldiers and that Topolya has 16171. According to the third point, Petrőc, the third city, has the smallest army, so its army numbers less than 4830. According to the fourth claim, Bácsfeketehegy and Sóvé, the two extreme cities can mutually send messages, so both their armies are larger than 16171. Of the two, due to the first point, Bácsfeketehegy has the larger army. Thus, the towns in order of decreasing army sizes: Bácsfeketehegy, Sóvé, Topolya, Kishegyes, Petrőc.



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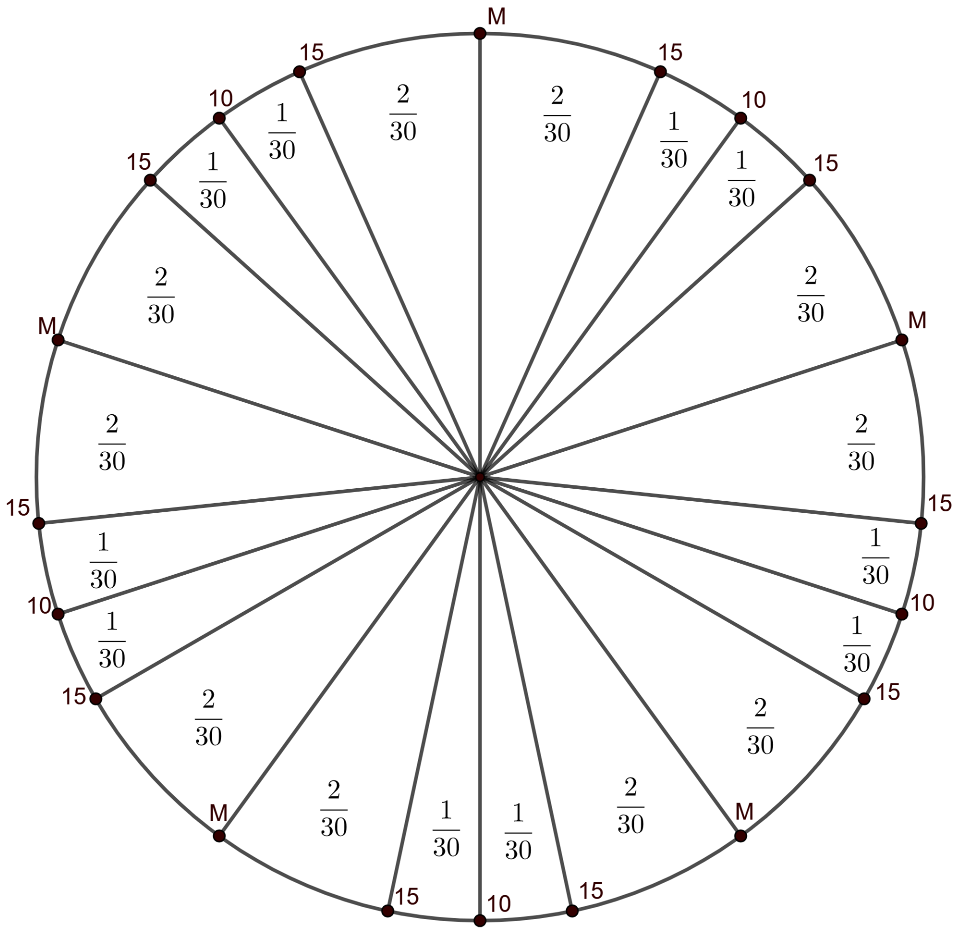
2. Yes, they can. On the figure the cuts which are part of both splits are marked with an M, and the cuts which only appear in the 10 and 15 piece splits are marked with a 10 and 15 respectively. Note that 3 adjacent slices from the 15 piece split have the same size as 2 adjacent slices from the 10 piece split, as they are both one fifth of the cake, and hence there are 5 cuts appearing in both splits. For sake of simplicity, let's only consider the cake as a circle, and assume the area of the cake is 1. We will see, that all the slice sizes are multiples of $\frac{1}{30}$, so we will always count in one-thirtieths.



Let's look at two adjacent cuts marked by M. The circular sector between these two is split into 3 equal parts by the 15-cuts and the 10-cut splits the middle piece into two equal pieces. Hence between the two M cuts there are 2 slices with size $\frac{2}{30}$ and 2 with size $\frac{1}{30}$. The cake consists of 5 blocks like this, thus in total the cake has been split into 10 slices of size $\frac{2}{30}$, and 10 slices of size $\frac{1}{30}$.

If 6 people want to share the cake fairly, everyone has to get $\frac{5}{30}$ of the cake. This is indeed achievable: 4 people get 2 big (with size $\frac{2}{30}$), and 1 small slices (with size $\frac{1}{30}$), while the remaining two people get 3 small, and one big slices.

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3. **a)** For instance, triple $a = 2$, $b = 3$, $c = 6$ is a solution, because $2 \mid 3 \cdot 6$, $3 \mid 2 \cdot 6$, $6 \mid 2 \cdot 3$ are all true, and their sum $2 + 3 + 6 = 11$ is a prime number.

b) Let p be an arbitrary prime which divides at least one of a, b, c. Denote by x, y, z the exponent of p in the canonical prime factorizations of a, b, c, respectively. Clearly, x, y, z are nonnegative integers.

It is impossible that x, y, z are all positive, because then a, b, c are all divisible by p, so p divides $a+b+c$. But $a+b+c \geq p+p+p$ is larger than p, which contradicts the requirement that $a+b+c$ is prime. Hence, at least one of a, b, c has exponent 0. Let's suppose this number is a. (If it isn't a, but rather c, then we swap the values of a and c, still retaining the same criteria.)

Now, let us examine what the divisibility requirements are in the task. It is true in general that the exponent of the prime p in a product $a \cdot b$ is the sum of the exponents in a and in b. The exponent of p in a is 0, in b it is y, in c it is z, so in $a \cdot b$ it is $0 + y = y$, in $a \cdot c$ it is $0 + z = z$, and in $b \cdot c$ it is $y + z$. As $b \mid ac$, the exponent of p in b is at most the exponent in ac, so $y \leq z$. Similarly, $c \mid ab$ tells us that $z \leq y$. Thus, $y = z$.

We obtained that any prime p which divides at least one of a, b, c will not divide one of the three, while occurring with the same exponent in the prime factorizations of the other two. So in the prime factorization of abc, p occurs with exponent $x + y + z = 0 + y + y = 2y$, which is even. Every prime factor of abc divides at least one of a, b, c, so every prime factor of abc occurs with an even exponent. This is precisely the condition for abc to be a square number.

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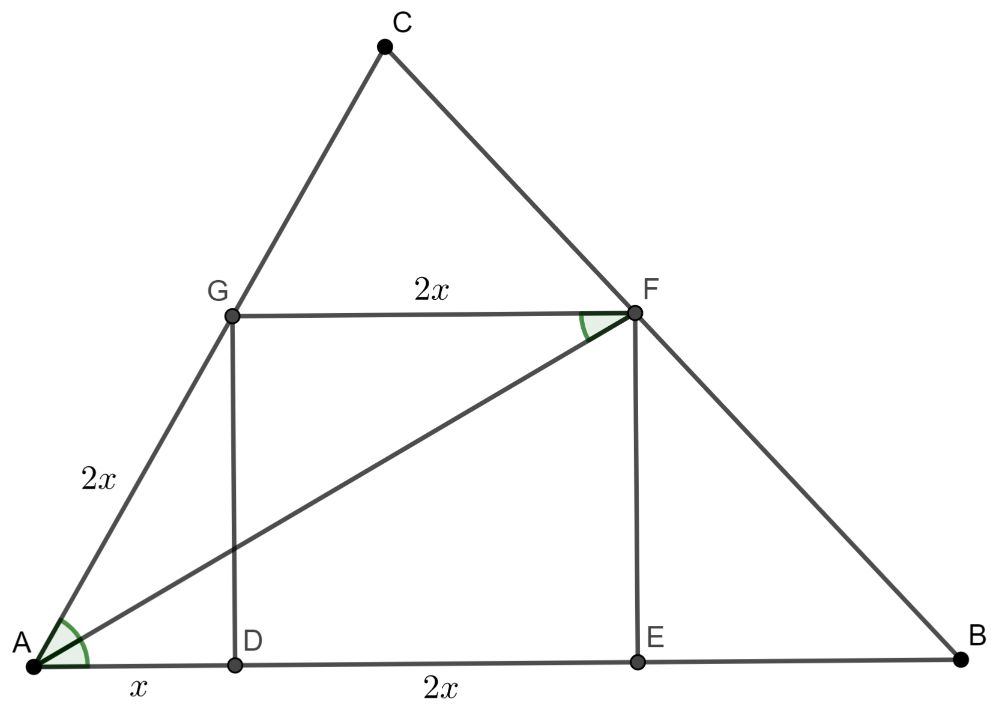
4. In this solution, the length of a segment is denoted with an overline. Let $\overline{AD} = x$, so that the conditions imply $\overline{DE} = 2x$. As quadrangle $DEFG$ is a rectangle, its opposite side have equal length, whence $\overline{GF} = 2x$. As AF is an angle bisector, $\angle GAF = \angle FAE$. By parallels, this angle also appears as $\angle FAE = \angle AFG$. Due to the equal angles, triangle AGF is isosceles, so $\overline{AG} = 2x$. Now, let's examine triangle ADG: it has a right angle at D, and its hypotenuse is exactly double one of its legs. Hence, this is a regular triangle folded in half, meaning its angle at A is precisely $60°$. Therefore, the answer to the question is $\angle CAB = 60°$.



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5. a) The following table is suitable.

1	2	5
1	4	7
3	6	7

To check this, write down the horizontal sums: $1 + 2 = 3$, $2 + 5 = 7$, $1 + 4 = 5$, $4 + 7 = 11$, $3 + 6 = 9$, and $6 + 7 = 13$. Vertically: $1 + 1 = 2$, $1 + 3 = 4$, $2 + 4 = 6$, $4 + 6 = 10$, $5 + 7 = 12$, and $7 + 7 = 14$. These are all distinct, indeed.

b) No, there is no suitable way to fill the table. The smallest possible two-term sum is $1 + 1 = 2$, while the largest is $7 + 7 = 14$. Thus, these 12 sums are between 2 and 14.

The sum of numbers in the table is 36, so the average of row sums is 12. This means there is a row with sum at most 12, as the average exceeds the smallest term. Similarly, there is a column whose sum is at most 12.

Therefore, the 12 sums of pairs, as well as at least one row sum and one column sum is between 2 and 14. There are only 13 distinct numbers between 2 and 14, so these 14 sums cannot be all distinct.

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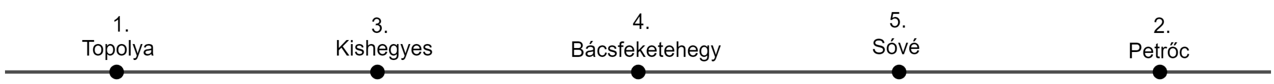
4.2.2 Category D

1. a) From the first point, we know that Topolya could be the first or fifth town. From the fourth point, we know that Petrőc is the third or fifth. The army of Bácsfeketehegy is smaller than that of Petrőc (fourth point), but it can send a message to Kishegyes (third point), so both towns are on the same side of Petrőc. If Petrőc were third, then by the previous observation, Kishegyes and Bácsfeketehegy are neighbors in some order at the first and second, or the fourth and fifth locations. However, this is not possible, since Sóvé would neighbor Topolya, so it could send it a message, contradicting the second point. From this, we know that Topolya is the first and Petrőc is the fifth city. Sóvé does not neighbor Topolya, nor Kishegyes, nor is it at one of the ends, so its two neighbors could only be Petrőc and Bácsfeketehegy. Thus, Sóvé is fourth along the road and Bácsfeketehegy is third. Kishegyes remains to be the second town. The final order: Topolya, Kishegyes, Bácsfeketehegy, Sóvé, Petrőc.

b) Topolya and Petrőc are at the two ends, but they can mutually send messages (fifth point). Therefore, these are the two cities with the largest armies, and it is that of Topolya which is larger (first point). The rest of the data tells us that Kishegyes has a larger army than Bácsfeketehegy, while Sóvé cannot send a message to Kishegyes. Only Bácshegyes is located between the latter two, so the army of Sóvé is smaller than that of Bácshegyes. Thus, in decreasing order according to army size, the towns are: Topolya, Petrőc, Kishegyes, Bácsfeketehegy, Sóvé.



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2. For the solution, see Category C Problem 3.

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3. For the solution, see Category C Problem 4.

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4. For the solution, see Category C Problem 5.

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5. The numbers can be divided into 5 groups as follows: the numbers divisible by 2; of the rest, the numbers divisible by 3; of the rest, the numbers divisible by 5; of the rest, the numbers divisible by 7, and the remaining numbers.

In the first four groups, the greatest common divisors are always at least 2, 3, 5, or 7. In the fifth group, the numbers are pairwise coprime, for the following reason. If n is a composite number, its smallest prime divisor must be 2, 3, 5, or 7, as the smallest composite number without this property is $11 \cdot 11 > 100$. So n must be in one of the first four groups. The numbers in the fifth group are therefore all prime or 1, so they are pairwise coprime.

Next, we will show that the numbers cannot be divided into a smaller number of groups. Proceeding by contradiction, suppose 4 groups can be realised. Then some two of the numbers 2, 2^2, 2^3, 2^4, 2^5 are in the same group, by the pigeonhole principle. In that group, the numbers pairwise have a common divisor greater than 1. As the powers of 2 only have 2 in their factorization, all members of this group shall be even. We may move all the even numbers to this group. 3 groups remain for the odd numbers.

Similarly, some two of 3, 3^2, 3^3, 3^4 are in the same group of the remaining three. That group has common divisors greater than 1, and all members are multiples of 3. We may assume all the multiples of 3 are in this group. Two groups remain.

Of the numbers $5, 7, 5^2, 7^2, 5 \cdot 7$, some three fall in the same group, by pigeonhole. As no three of these are pairwise coprime, the members of this group are not pairwise coprime. If $7, 7^2, 7 \cdot 5$ are in one group, then $5, 5^2$ are in the other, as they are coprime with 7. Similarly, if $5, 5^2, 7 \cdot 5$ are in one group, then $7, 7^2$ are in the other. In both cases, it is true that both groups have pairwise taken common divisors greater than 1, but then 1 itself cannot be in any of the groups, which is a contradiction. Therefore, 5 groups are indeed necessary.

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4.2.3 Category E

1. For the solution, see Category C Problem 2.

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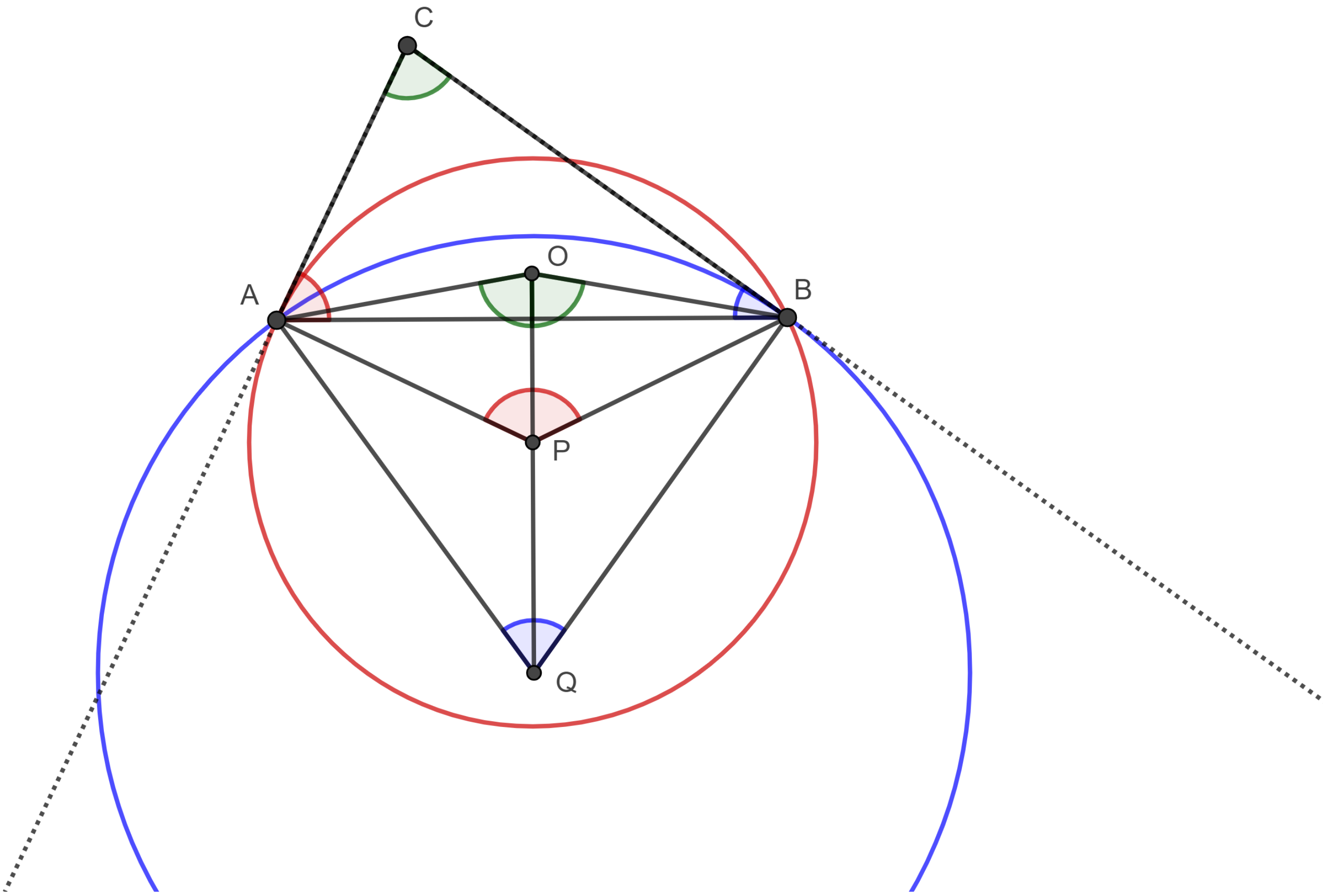
2. Let the centre of the circumcircle be O, denote the angles of the triangle by α, β, γ and let k_1 and k_2 be the circles defined in the problem, tangent to sides BC and AC.

Since points O, P, Q all lie on the perpendicular bisector of segment AB, they are collinear.



We will show that triangles AQO and PAO are similar. In this case because of the similarity, $\frac{OQ}{AO} = \frac{AO}{OP}$ holds and since O is the centre of the circumcircle, $AO = R$ from which we get the desired equality.

Now we will show that both of the triangles AQO and PAO are similar to triangle ABC. For this we will check that all of them have the same angles.

Using the inscribed angle theorem for circle k_2 and angle $\angle BAC$ we get that the inscribed angle belonging to chord BA is equal to α, therefore $\angle BPA = 2\alpha$. Similarly we get that in circle k_1, $\angle BQA = 2\beta$. Now by the inscribed angle theorem $\angle BOA = 2\gamma$.

Since line OQ is the perpendicular bisector of segment AB, it bisects the angles mentioned, meaning that $\angle OPA = \alpha$, $\angle OQA = \beta$ and $\angle QOA = \gamma$. Now in triangle PAO we know that two of the angles are α and γ, this means that the third angle is β. Similarly in triangle AQO the third angle is α, thus we have proven that triangles AQO and PAO are similar.

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3. **a)** It can't reach all points of the first quadrant, for example, it can't reach the point $(1, 2)$. Assume by contradiction that the flea can reach this point. Then for all of its jumps, $p \leq 1$ has to hold. This means $p^2 \leq p$, so for all the jumps the y coordinate can increase by at most as much as the x coordinate. Because of this, using only jumps with $p \leq 1$, the flea can only reach points for which $x \geq y$. However, this is not true for the point $(1, 2)$, hence it is unreachable.

b) Assume that the flea reaches the point $(100, 1)$ in n steps, using the numbers $p_1, p_2, \ldots, p_n$. Then we have

$$\sum_{i=1}^{n} p_i = 100 \qquad \text{and} \qquad \sum_{i=1}^{n} p_i^2 = 1.$$

Using the AM-QM inequality for p_i and squaring both sides, we get

$$\frac{\sum_{i=1}^{n} p_i^2}{n} \geq \left(\frac{\sum_{i=1}^{n} p_i}{n}\right)^2.$$

Therefore,

$$\frac{1}{n} = \frac{\sum_{i=1}^{n} p_i^2}{n} \geq \left(\frac{\sum_{i=1}^{n} p_i}{n}\right)^2 = \frac{10000}{n^2},$$

thus $n \geq 10000$. But this lower bound is also achievable: if $n = 10000$, and $p_i = \frac{1}{100}$ for all i, then the sum of p_i is indeed 100, and the sum of their squares is $10000 \cdot \frac{1}{10000} = 1$.

Hence the flea needs at least 10000 jumps to get to the point $(100, 1)$.

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4. We can partition the numbers into 5 groups the following way: let us put into the first group all the numbers that are divisible by 2 and into the second group the numbers that are divisible by 3 and which are not in the first group. The third group will contain all the numbers that are divisible by 5, but are contained neither in the first nor in the second group. We put into the fourth group the numbers that are divisible by 7 and are not contained in any of the previous groups. And finally the last group will contain the rest of the numbers.

For the first, second, third and fourth group the condition holds, since any two numbers from the same group have a common divisor of $2, 3, 5$ or 7. We will prove that the fifth group contains only primes and the number 1, and therefore any two of them are coprime. This is true because the smallest prime divisor of any composite number $n \leq 100$ must be either $2, 3, 5$ or 7, since $11^2 > 100$. Therefore any composite number is contained in one of the first four groups.

Now we prove, that we cannot partition the numbers into fewer groups. Assume by contradiction, that we have partitioned them into 4 groups. Consider the numbers $2, 2^2, 2^3, 2^4, 2^5$. Using the pigeonhole principle, we get that there is a group which contains at least two of these integers, which means that in this group any two numbers must have a common divisor greater than 1. And since the only prime divisor of the numbers above is 2, every other number from that group must be divisible by 2. Hence there are 3 groups left.

Consider the numbers $3, 3^2, 3^3, 3^4$. In the same way as before one can show that there is a second group, in which every number is divisible by 3. So there are two more groups left.

Now take the numbers $5, 7, 5^2, 7^2, 5 \cdot 7$. Using pigeonhole principle we get that there is a group which contains at least 3 of the integers above. Examining all possible cases we get that there are only two possible ways to partition these numbers into two groups: $5, 5^2$ and $7, 7^2, 7 \cdot 5$

or $7, 7^2$ and $5, 5^2, 7 \cdot 5$. In both cases it is true that any two numbers from the same group must have a common divisor greater than 1, which leads to a contradiction, since for example we cannot put 1 into any of these groups. Therefore at least five groups are needed.

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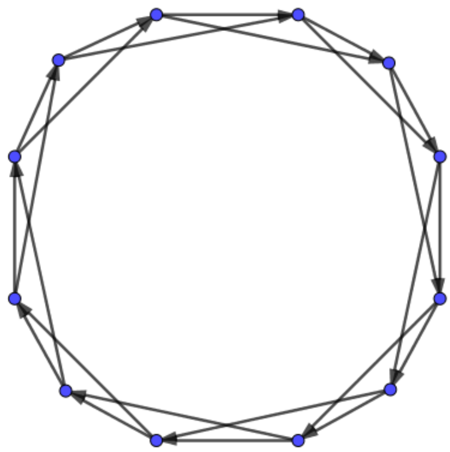
5. a) Yes, he can achieve this, for example if he imagines that the players are arranged on a circle, such that the players from team A and team B alternate, and he gives to each player the names of the next two players on their right. This method works since if two neighbouring players were teammates, then the player immediately to the left of them would have received the names of two players in the same team, so it would not be possible that one of them is his teammate and the other is not. This means that along the circle, the players from the two teams must alternate. (In the figure, the points correspond to the players, and there is an arrow pointing from player A to player B if A received B's name.)

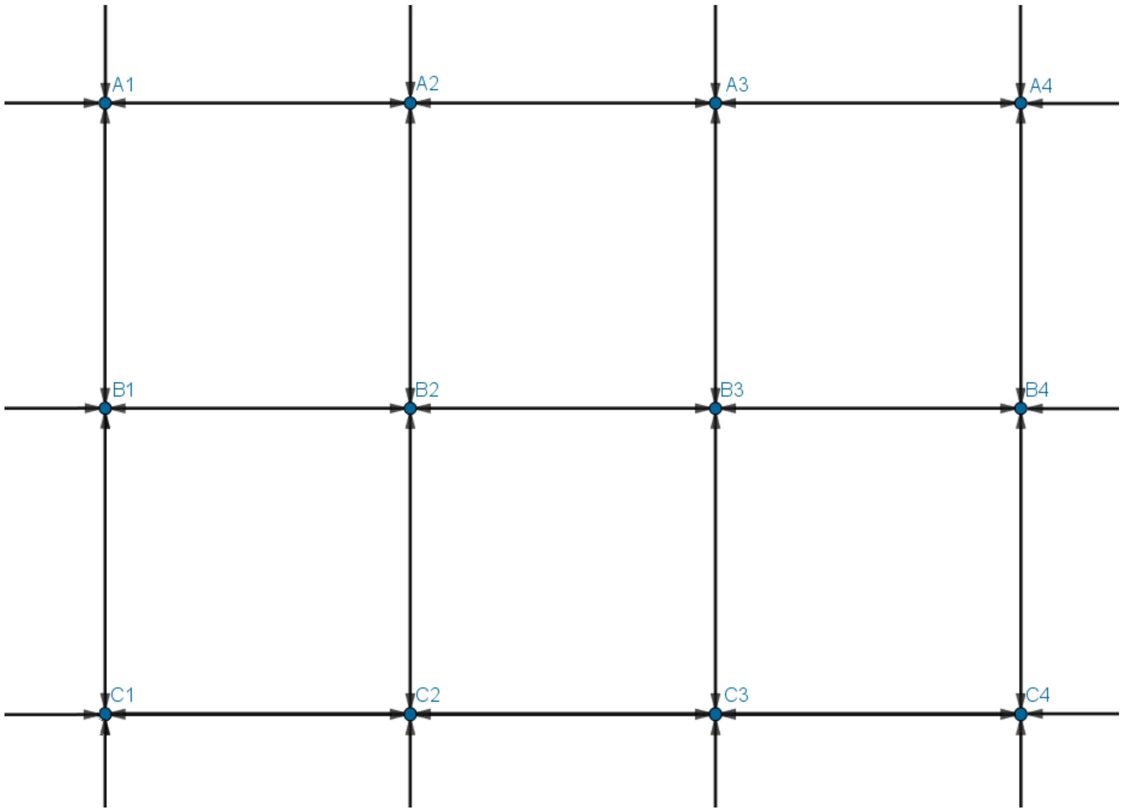


b) Yes, he can achieve this. Let the players be A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3 and C_4. Then the game master gives a card to each player containing the names of those players who either:

- have the same number, or
- have the same letter and a number differing by 1 or 3.

For example, C_3 receives the names of A_3, B_3, C_2 and C_4; and player A_1 receives the names of B_1, C_1, A_2 and A_4. Then the two teams can be as follows: one team contains those players whose number is 1 or 2, and the other contains those whose number is 3 or 4. But alternatively, one team could contain those players whose number is 1 or 4, and the other could contain those with 2 or 3.

(In the following figure, arrows leaving the figure on the left come back on the right and vice versa, and arrows leaving on the top come back on the bottom, and vice versa.)



c) No, he cannot achieve this. We know that each player is a teammate of themselves, so if everyone added his own name to his card, then among the 6 players on his card, 5 would be teammates, and 1 would be an opponent. So everybody would have one teammate not appearing on his card. If two players are in the same team, then after they have added their own names, their cards contain at least 4 names in common (since in their team, there is 1 player not appearing on one of the cards, and 1 not appearing on the other card. If these two non-appearing names are different then the two cards contain at least 4 names in common, and if they are the same then they contain at least 5). However if two players are not on the same team, then their cards can only contain at most 2 names in common, since each of them has 1 player who is not his teammate (and this player can appear on the other player's card as the other player's teammate). So if any two players compare their cards (after adding their own name) then depending on the number of common names, they can determine if they are teammates. So the players (working all together) can determine the two teams.

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4.2.4 Category E$^+$

1. For the solution, see Category E Problem 4.

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2. If ABC is an isosceles triangle, then the altitude corresponding to the the base of the triangle splits the triangle into two congruent right triangles.

If ABC is not isosceles, then we claim there is no such cut. Assume by contradiction that there is a suitable cut, consisting of $P_1, P_2, \dots, P_n$. If neither of P_1, P_n coincides with a vertex of the triangle, then one of the resulting polygons would have $n+1$ sides, while the other would have $n+2$ sides, which is a contradiction. Because of this we can assume $P_1 = A$ from now on.

The perimeter of the two resulting polygons is equal, and the segments AP_2, P_2P_3, ..., $P_{n-1}P_n$ are sides of both, thus $AB + BP_n = AC + CP_n$ has to hold. Furthermore, because of the congruence, the multiset of the side lengths of the polygons has to be same for both. They have $n-1$ coinciding sides, therefore this implies that $\{AB, BP_n\} = \{AC, CP_n\}$. However, we assumed that the triangle is not isosceles, so $AB \neq AC$, thus $AB = CP_n$ and $AC = BP_n$. From this we get the equation $AB + AC = BP_n + CP_n = BC$, which contradicts the triangle inequality.

Therefore there is a suitable cut if and only if the triangle is isosceles.

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3. a) Yes, he can achieve this. The game master imagines that the 4 teams are sitting around two circular tables with 20 seats each. At the first table, the players in the first two teams sit alternatingly, and at the second table the players of the other two teasm sit alternatingly. The game master gives to each player the names of the two players immediately to their right at the same table. It is clear that everybody received the name of one teammate and one opposing player. Take any possible assignment of the 4 teams that satisfy the cards just described. If there are two players at the same table who are teammates and sit next to each other, then the player sitting immediately to their left would have received the names of two players on the same team, which is impossible. So nobody is in the same team as their neighbour to the right. So everybody must be in the same team as the second player to their right. This implies that a team must contain every other player from the same table. So the composition of the four teams is indeed unique.

b) No. Make everybody add their own name to their card, so their cards will contain 8 teammates and 2 opposing players. If two players are in the same team then their cards must contain at least 6 names in common, as both of them only have 2 teammates not appearing on their card, so their team has only at most 4 players not appearing on either of the two cards. However if two players are in different teams then there can be at most 4 players appearing on both of their cards, as if someone appears on both then he is not a teammate of at least one of the two players, and the two cards contain 4 such players in total. So any two players can determine (by comparing their cards) whether they are in the same team or not, so the players (all working together) can determine the teams.

c) Yes. Divide the players into 8 groups $A_1, A_2, \dots, A_8$ each containing 5 players. Let the real teams be the union of groups A_{2i-1} and A_{2i} for $1 \leq i \leq 4$. For every i, let each person in group A_i get the names of everybody in the same group, and two people from group A_{i+1} and two from A_{i-1} (indices are considered modulo 8). It is clear that according to the real teams, everybody received the names of 6 teammates and 2 opposing players, however if the teams were the unions of groups A_{2i} and A_{2i+1} for $1 \leq i \leq 4$, then the same cards would also be appropriate. So the players cannot determine what the actual teams are.

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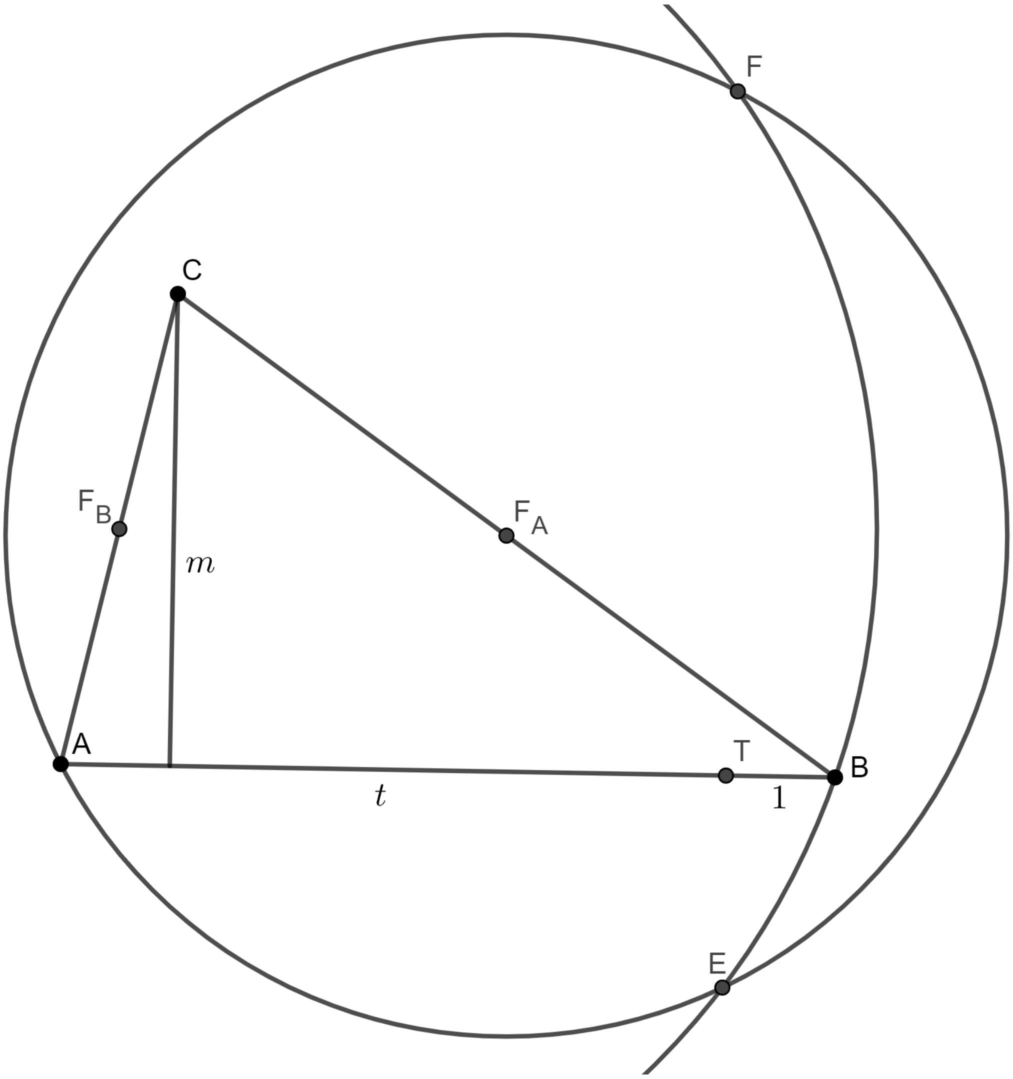
4. Firstly we will show that line FE is perpendicular to line AB and that the reflection of point C by line AB also lies on this line.

Segment F_AF_B is a midsegment in triangle ABC therefore it is parallel to AB. Since E and F are reflections of each other by line F_AF_B, it means that FE is perpendicular this line, therefore also to AB.

Denote the circle with centre F_A passing through A by k_A, the circle with centre F_B passing through B by k_B. Now since line FE is the power line of circles k_A and k_B, it is enough to show that the power of the reflection of the foot of the height from C in triangle ABC is the same in both circles.

Without loss of generality we can assume that BT is of length 1. Then if $AT = t$ and the height at C is m then we would like to prove that $TF_A^2 - AF_A^2 = TF_B^2 - BF_B^2$.



By the length of the medians (with the usual notation):

$$AF_A^2 - BF_B^2 = \frac{2b^2 + 2c^2 - a^2}{4} - \frac{2a^2 + 2c^2 - b^2}{4} = \frac{3}{4}(b^2 - a^2).$$

Using the Pythagorean theorem:

$$TF_A^2 - TF_B^2 = m^2/4 + (t/2-1)^2 - m^2/4 - (t-1/2)^2 = \frac{3}{4}(1-t^2),$$

$$\frac{3}{4}(b^2-a^2) = \frac{3}{4}(1+m^2-m^2-t^2).$$

Thus indeed EF is the line perpendicular to AB passing through the reflection of C to the perpendicular bisector of AB.

Using this result now in triangle CEF we get that the points of intersections mentioned in the problem will lie on the line perpendicular to EF, containing the reflection of the feet of the height from C by line EF. Clearly, this is line AB therefore we have finished the proof.

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5. Let's take the $n \times n$ matrix M whose rows are numbered from 0 to $n-1$ and its k-th row is $(a_1^{2k}, a_2^{2k}, \ldots, a_n^{2k})$.

We multiply this matrix M by the vector $(a_1, a_2, \ldots, a_n)$. The resulting vector's elements are $(a_1 + a_2 + \ldots + a_n)$, $(a_1^3 + a_2^3 + \ldots + a_n^3)$, and so on, up until $(a_1^{2n-1} + a_2^{2n-1} + \ldots + a_n^{2n-1})$

We know that all these values are 0, from the assumptions in the problem. We can assume that the vector $(a_1, a_2, \ldots, a_k)$ is not the zero vector, as otherwise the statement of the problem is trivially true. Hence there is a non-zero vector v, for which $M \cdot v = \mathbf{0}$, so M's rows are linearly dependent.

Therefore there is an integer $0 < m \leq n-1$, for which the m-th row of M is a linear combination of the rows $0, 1, \ldots, m-1$.

Let's prove that for all integers $k \geq 0$, $\sum\limits_{j=1}^{n} a_j^{2k+1} = 0$ holds by induction.

According to the problem statement the induction hypothesis holds for $0 \leq k < n$. Let's assume that for some $r \geq n$ we already know that for all $0 \leq k < r$, $\sum\limits_{j=1}^{n} a_j^{2k+1} = 0$ is true, and we want to prove that $\sum\limits_{j=1}^{n} a_j^{2r+1} = 0$.

Let's take the vector $(a_1^{2(r-m)+1}, a_2^{2(r-m)+1}, \ldots, a_n^{2(r-m)+1})$. If we multiply this vector by the zeroth, first, ..., $(m-1)$-th row of matrix M, we get the values $\sum\limits_{j=1}^{n} a_j^{2(r-m)+1}$, $\sum\limits_{j=1}^{n} a_j^{2(r-m+1)+1}$, ... and $\sum\limits_{j=1}^{n} a_j^{2(r-1)+1}$ respectively. According to our inductive hypothesis all of these values are 0.

The m-th row of matrix M can be written as a linear combination of the first m. Hence if we multiply $(a_1^{2(r-m)+1}, a_2^{2(r-m)+1}, ..., a_n^{2(r-m)+1})$ by the m-th row of the matrix, then the result is a linear combination of the previous products, thus this is 0 as well.

Therefore $\sum\limits_{j=1}^{n} a_j^{2r+1} = 0$, and we have proven the inductive step.

Let $T = \max(|a_j|)$, and if not all a_j have absolute value T, let the second largest absolute value be R.

As $\frac{R}{T} > 1$, there exists a large enough k, so that $(\frac{R}{T})^{2k+1} < \frac{1}{2n}$.

Among the a_j-s with absolute value T, let there be s with the value T and h with the value $-T$. Let's assume $s \neq h$. Then $0 = \sum\limits_{j=1}^{n} a_j^{2k+1} = (s-h)T^{2k+1} + \sum\limits_{|a_j| \leq R} a_j^{2k+1}$.

On the other hand, for our sufficiently large k, $\left|\sum\limits_{|a_j|\leq R} a_j^{2k+1}\right| \leq nT^{2k+1}(\frac{R}{T})^{2k+1} \leq \frac{1}{2}T^{2k+1}$.

Using $|s-h| \geq 1$, we have

$$\left|\sum_{j=1}^{n} a_j^{2k+1}\right| = \left|(s-h)T^{2k+1} + \sum_{|a_j|\leq R} a_j^{2k+1}\right| \geq \frac{T^{2k+1}}{2},$$

which is a contradiction, as this value should be 0.

Hence $s = h$, thus for the ordering $a_1 \leq a_2 \leq \ldots \leq a_n$, we have $a_1 = a_2 = \ldots = a_s = -a_{n-s+1} = -a_{n-s+2} = \ldots = -a_n$, so for these elements $a_i = -a_{n-i+1}$ holds. Removing these elements, the condition $\sum\limits_{j=s+1}^{n-s} a_j^{2k+1} = 0$ will still hold for all k, as the removed terms had a sum of 0.

This way we have less elements in the sum, and by induction on n, we can prove that $a_i = -a_{n-i+1}$ holds for all i.

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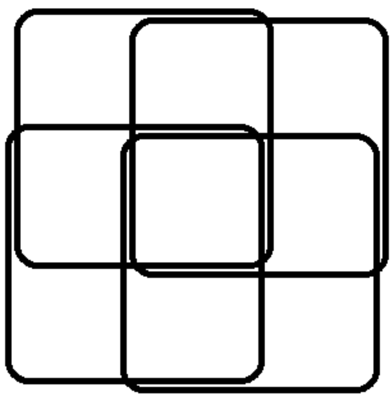
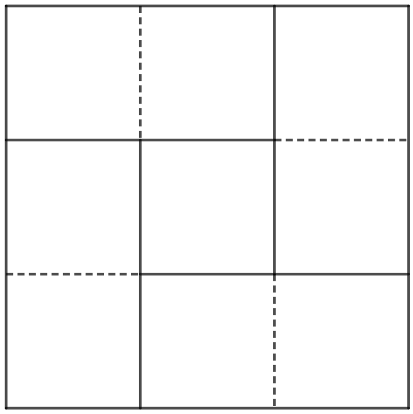
4.3 Final round – day 1

4.3.1 Category C

1. With drawing four 2×2 squares Csongor could create the desired grid. (see the diagram below)



At the same time, we can realize that less than four squares will not be enough. Out of the four dashed lines no two could be covered with such a square that does not go outside of the figure. This way, Csongor needs at least four squares for drawing the grid.



Second solution: We will show another solution for proving that less than four squares will not be enough. Let us indirectly suppose that 3 squares will suffice. It is clear that we can only use squares whose dimensions are 1×1, 2×2 or 3×3. We would like to cover all the four corners, therefore there will have to be such a square that covers at least two corners. This is only possible if we use a 3×3 square which covers the whole perimeter of the figure. We can observe that the grid consists of 24 short segments, 12 of which is already covered by the 3×3 square. A 1×1 square and a 2×2 square can both cover at most 4 new short segments since they overlap with the 3×3 square in at least 4 short segments. Therefore if we have 3 squares, one will have to have dimensions of 3×3 and in total the squares can cover maximum $12 + 4 + 4 = 20$ short segments which would not cover the whole grid. Thus, Csongor will need at least one more square.

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2. a) Every positive integer can be written in the form $4k + m$, where k is a non-negative integer and $1 \leq m \leq 4$. Hence, because of the move $x + 4$, it is sufficient to show that the numbers 1, 2, 3, and 4 arise using the permitted moves, starting from 11. We explain a way to produce each of them.

- **Producing 2:** Let's take the digit sum of the initial value (11), which equals 2.
- **Producing 1:** Starting from 2, which we have just produced, let's perform twice the move $x + 4$, to arrive at 10. Next, let's take the digit sum, which equals 1.
- **Producing 4:** Starting from 1, let's perform the move $2(x + 1) + 1$, making 5, and then perform this move once more, making 13. Finally, let's take the digit sum.
- **Producing 3:** Starting from 4, let's perform the $x + 4$ move twice, then take the digit sum.

Thus, we have shown that every positive integer can be produced from the given initial value using the permissible set of moves.

b) To verify the claim, we will show that the two permissible moves map a non-multiple of 3 to a non-multiple. It is sufficient to prove this, as the initial value is not a multiple of 3, yet 2022 is.

This, however, is easy to see with both moves. Summing the digits, the mod 3 residue of the initial value is the same as the mod 3 residue of the resulting value, by the well-known divisibility rule. With the $2(x+1)+1=2x+3$ move, if x is not divisible by 3, then neither is $2x$, nor $2x+3$.

This completes the proof.

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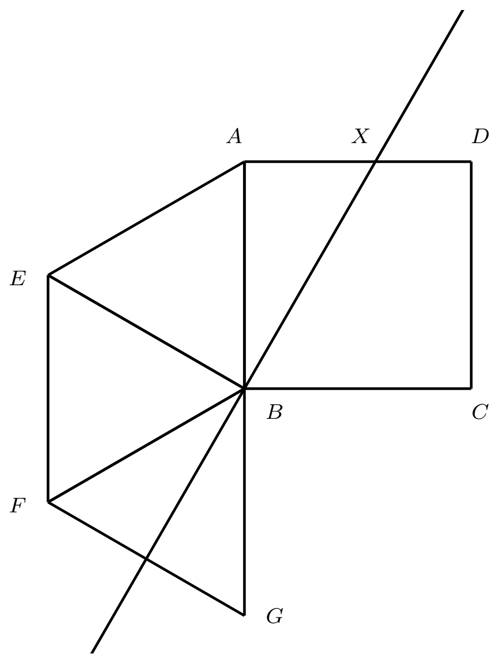
3. As reflection is a congruence transformation, and F is by definition the reflection of point A over line BE, triangle BEF is congruent to triangle ABE, so triangle BEF is regular. Moreover, by definition of G, triangle FBG is the reflection of BEF over line BF, so they are congruent, whence triangle FBG is also regular.

By regularity of triangle FBG, the perpendicular bisector of segment FG is the same as the angle bisector of angle $FBG\sphericalangle$, which subtends an angle of 30° with line BG.

However, $ABG\sphericalangle = ABE\sphericalangle + EBF\sphericalangle + FBG\sphericalangle = 60° + 60° + 60° = 180°$, so line AB coincides with line BG. Therefore, the perpendicular bisector of segment FG subtends an angle 30° with line AB.

As X lies on line AD, which is perpendicular to AB, the distance of X from line AB is equal to the length of segment XA. Hence, the tangency of the circle centered at X with radius XA to line BF is equivalent to the fact that X has distance XA from BF, i.e., X is of equal distance from lines AB and BF.

This is true, because the set of points of equal distance from two lines is the union of their two angle bisectors. As X is by definition on the perpendicular bisector of FG, which bisects $FBG\sphericalangle$, we have proved the claim.



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4. **a)** Answer: No. Let us indirectly suppose that there exist 12 such consecutive numbers. Let n denote the smallest of the 12 numbers. With this notation, the sum of the 12 numbers will be: $n + (n+1) + \ldots + (n+11) = 6 \cdot (2n+11) = 12n + 66$. This number is even, but it is not divisible by four, which means that it cannot be a square number.

b) Answer: Yes. Let n denote the smallest of the 11 numbers. Their sum will be $(2n+10) \cdot 5 + n + 5 = 11n + 55 = 11 \cdot (n+5)$. If we set $n+5$ to be 11, meaning that n is set as 6, the sum will be $11 \cdot 11 = 121$, a square number.

c) Answer: Yes. Let n denote the smallest of the 10 numbers. With this notation, the sum of the 10 numbers will be $10n + 45 = 5 \cdot (2n+9)$. $(2n+9)$ cannot be 5 because that way n would be negative. Let n therefore be $5 \cdot 3^2 = 45$. This way, $n = 18$ and the sum of the numbers is $(5 \cdot 3)^2 = 225$, which is a square number.

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5. Donna Duck could have layed a maximum of 18 eggs so far. The solution has two parts, first we will show an exaple for 18, then we will prove that 19 or more is impossible.

Grey eggs are represented by the letter G, brown by B, Orange by O. A suitable exapmle for 18 eggs.

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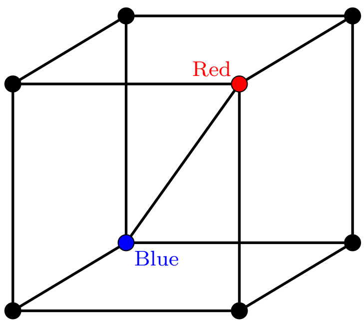
6. The key idea is that if the two ends of the space diagonal are colored, then the first player wins. When its two colors are red and blue, say, then the neighbors of the red vertex can always be painted blue, and the neighbors of the blue vertex can always be painted red. Thus, all vertices will receive a color.



Hence, the starting player has a winning strategy. In her first move, she paints one end of the space diagonal, and in the next move, if it isn't painted yet, she can paint the other end of the space diagonal, as a color is still available for it. After this, all vertices attain a color as discussed previously, so she will win the game.

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4.3.2 Category D

1. We'll call the $a + b$ number of years an interval. We know that in 2005 it was a wild duck, in 2010 it was a domestic duck, then in 2011 a wild duck, while in 2015 a domestic duck that was the delegate. Therefore the domestic ducks could become delegates earliest in 2006 for b years, then the wild ducks for a years, latest till 2014. Because of this the lenght of one period is $a + b \leq 2015 - 2006 = 9$ years.

Based on the rules, with an interval difference the same type of ducs will be delegates, therefore if the difference is the multiple of the interval, still the same type of ducks will be delegates. Thus, if in two distinct years the type of delegates was different, the interval can not be the divisor of the difference.

Since in 2022 the Duckdelegate was domestic and in 2004 a wild duck, the length of a period can not be the divisor of $2022 - 2004 = 18$, so it can not be 1, 2, 3, 6 or 9

In 2015 the delegate was domestic, in 1999 wild, therefore an interval must not be the divisor of $2015 - 1999 = 16$, that rules 4 and 8 out.

The Duckdelegate of 2010 being a domestic and in 2005 being a wild rules the divisors of $2010 - 2005 = 5$, so 5 out.

The only possible solution is $a + b = 7$ years. Since there was a change of electives after 2010 (from domestic to wild) , an interval starts in 2011, so there will be one starting in 2018 and 2025, thus the wild ducks must wait 3 more painfully long years

Sidenote We have solved the problem without specifying the exact values of a and b. For the completeness of the solution we will show the there are a and b values that satisfy the problem.

The intervals started in 2018, 2011, 2004, 1997. Since in 1999 a wild duck was the delegate, $a \geq 3$, and since in 2014 the wild duck was not reeelected, $a \leq 4$.

If $a = 4$ and $b = 3$ the constuction is: 1997-2000 wild ducks, 2001-2003 domestic ducks, 2004-2007 wild, 2008-2010 domestic, 2011-2014 wild, 2015-2017 domestic, 2018-2021 wild, 2022-2024 domestic, 2025-2028 wild. This is indeed correct.

If $a = 3$ and $b = 4$, the construction is as follows: 1997-1999 wild, 2000-2003 domestic, 2004-2006 wild, 2007-2010 domestic, 2011-2013 wild, 2014-2017 domestic, 2018-2020 wild, 2021-2024 domestic, 2025-2027 wild. This chechks out as well. So we do indeed have two different solution, but in any case the wild ducks would have to wait 3 more years to be reelected.

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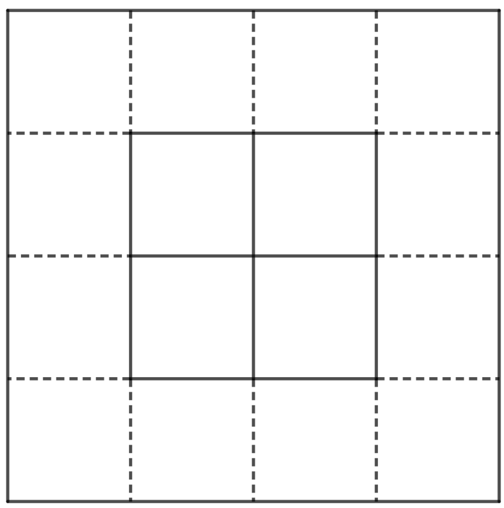
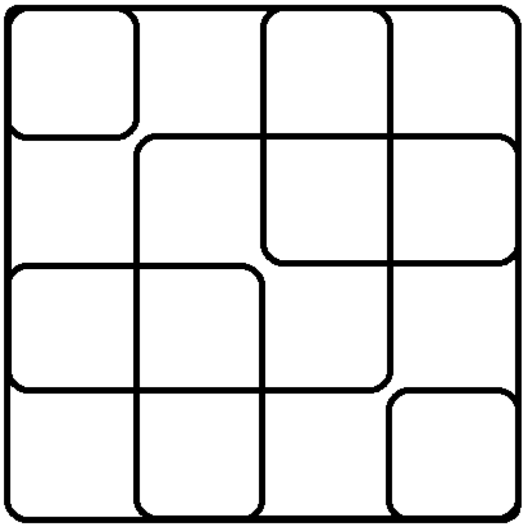
2. If we look at the figure below, we can see that at least 6 squares are needed to cover the whole grid.



We can spot that a square can cover maximum two dashed line segments. Because there are 12 dashed segments on the grid, we would need at least 6 squares to cover the whole thing.

Six squares are indeed enough, as can be seen below.



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3. a) Answer: No. Let us indirectly suppose that there exist 20 such consecutive numbers. Let n denote the smallest of these numbers. With this notation, the sum of the 20 numbers will be: $n + (n + 1) + \ldots + (n + 19) = 10 \cdot (2n + 19) = 20n + 190$. This number is even, but it is not divisible by four, which means that it cannot be a perfect square.

b) Answer: Yes. Let n denote the smallest of the 21 numbers. Their sum will be: $21n+210 = 21 \cdot (n + 10)$. If we set $n + 10$ to be 21, meaning that n is 11, the sum will be $21 \cdot 21$, a perfect square.

c) Answer: Yes. Let n denote the smallest of the 2022 numbers. With this notation, the sum of the these numbers will be $1011 \cdot (2n + 2021)$. We cannot set $(2n + 2021)$ to be 1011 because then n would be negative. Let n therefore be $1011 \cdot 3^2$. This way, $n = 3539$ and the sum of the numbers is $(1011 \cdot 3)^2$, which is a perfect square.

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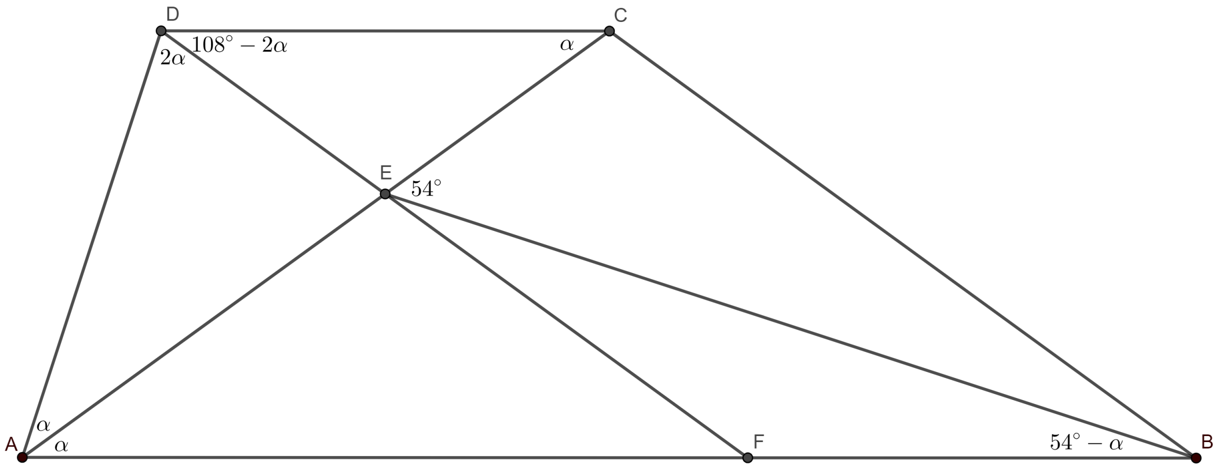
4. Let $BAC\sphericalangle = CAD\sphericalangle = \alpha$. Side AB is parallel with side CD, so $DCA\sphericalangle = BAC\sphericalangle = \alpha$. Thus, triangle ADC is isosceles, and using the condition of the task, we obtain $FB = AD = DC$. Quadrilateral $FBCD$ is a parallelogram, as its sides FB and CD are equal and parallel. Comparing this with the condition of the task, $AF = BC = FD$, so triangle AFD is isosceles, $ADF\sphericalangle = 2\alpha$.

From here on, let's calculate angles. $BEC\sphericalangle$ is an external angle of triangle AEB of measure $54°$, so $EBA\sphericalangle = 54° - BAE\sphericalangle = 54° - \alpha$. Line BE is an angle bisector and $FBCD$ is a parallelogram, whence

$$FDC\sphericalangle = CBF\sphericalangle = 2 \cdot EBF\sphericalangle = 108° - 2\alpha.$$

$ABCD$ is a trapezoid, so $BAD\sphericalangle + ADC\sphericalangle = 180°$, meaning $108° + 2\alpha = 180°$, $\alpha = 36°$. Now, our task becomes easy:
$BAD\sphericalangle = 2\alpha = 72°$.
$ADC\sphericalangle = 180° - BAD\sphericalangle = 108°$.
$CBA\sphericalangle = 108° - 2\alpha = 36°$.
$DCB\sphericalangle = 180° - CBA\sphericalangle = 144°$.

Comment: There does indeed exist such a quadrilateral, i.e., if the angles of $ABCD$ are as calculated, then the conditions of the task are fulfilled.



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5. For the solution, see Category C Problem 5.

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6. For the solution, see Category C Problem 6.

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4.3.3 Category E

1. For the solution, see Category C Problem 3.

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2. a) Regardless of the order of the X-es, Anett will get points for each row and column exactly 5 times. For a row she can get maximum $1+2+3+4+5$ points. (Since for example for putting the third X in a row, Anett can get maximum 3 points.) This is true for the columns as well, therefore she can get at most $(5+5)\cdot(1+2+3+4+5) = 150$ in total.

150 points can be collected: first she should put an X in the middle field. Then, the second X should be put in a field that has an adjacent side with the middle one. In the upcoming

rounds she should always put the new X-es in those fields that are empty and whose sides are adjacent to the previously marked X-es. This way, moving from the middle to the sides, she would gradually fill in the grid. During the process, X-es will be put in consecutive fields, therefore in any ith row (or column), i row-points (or column-points) will be given. Altogether this means $(5+5)\cdot(1+2+3+4+5)=150$ points.

		X		

		X		
	X	X	X	
		X		

		X		
	X	X	X	
X	X	X	X	X
	X	X	X	
		X		

	X	X	X	
X	X	X	X	X
X	X	X	X	X
X	X	X	X	X
	X	X	X	

X	X	X	X	X
X	X	X	X	X
X	X	X	X	X
X	X	X	X	X
X	X	X	X	X

b) Similarly to part a) let us check the minimum number of points that Anett can get for one row or column. For the 5th X in a row she will surely get 5 row-points and for the first X she will surely get 1. She can collect 11 points in two ways: if first she puts an X in the middle then in the far left and right fields, she will get $1+1+1+3+5=11$ points. On the other hand, if she fills in the 2-2 far left and right fields and leaves the middle one for the last, she will also get $1+2+1+2+5=11$ points.

She will not be able to get any less points because if the last X goes in the middle, she will get 11 points for sure. Let us check what happens if the last X goes in the field two away from the edges (in the second or the fourth fields from left to right). In this case, on one of the sides 3 fields will remain. The last X put drawn in these fields will be worth 3 row-points, whereas the rest will be worth at least 1. This means that again she will get $1+5+1+3+1=11$ points. If she draws the last X in the far right or left fields then from the remaining 4 fields she will get 4 points for the last X put in. This way she will get at least $5+4+1+1+1=12$ points. It is on the other hand possible to get 11 points for each row and column:

Let's colour the grid like a chessboard in a way that the middle field is black. Then, let's draw an X in all of the black fields. Afterwards, let's go from the outside to the middle: in a random order let's put X-es in the outer fields, then let's fill in the remaining ones in an arbitrary order. This way, all rows and columns will be filled in with one of the above mentioned methods that are worth 11 points, meaning that this is indeed an optimal solution. The minimal number of points that can be collected is therefore $(5+5)\cdot 11=110$.

X		X		X
	X		X	
X		X		X
	X		X	
X		X		X

X	X	X	X	X
X	X		X	X
X		X		X
X	X		X	X
X	X	X	X	X

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3. If $n=1$, then 0 claps are needed for the distribution.

If $n=2$, then with the first clap, the first person divides the heap in two, and then with the second, they give the sheet to the second one. 2 claps are needed.

Now, let's investigate the general case $n>2$. In this case, we claim that $n+1$ claps are needed.

Let the strategy of the first person be the following. As long as there are at least 3 sheets before them, they make the following two moves in turn:

- Taking two sheets from the pile.
- Giving the two-sheet pile to the second person.

If there were an even number of sheets originally, then after $n-1$ moves, only 2 sheets remain with them. Then, in the n-th move, they separate the two sheets, and in the $(n+1)$-th move, they give one sheet to the second person. If there were an odd number of sheets originally, then after n moves, only 1 sheet remains with them, and they do nothing in the $(n+1)$-th move.

The strategy of everyone else:

- If a heap of sheets is received at move $k < n-1$, then they pass it on in the $(k+1)$-th move.
- If during the $k = n-1$-th move, they receive a heap (two sheets), then in the next two moves, they divide it in half, and then pass on one of these. (For this reason, over the $k = n$-th move, nobody can receive a sheet, as in that move, everyone divides a heap in half.)
- When they receive a sheet in the $k = n+1$-th move, they keep it.

It is easy to see that with this method, everyone has exactly 1 sheet after $n+1$ claps.

Now we further show that if $n \geq 3$, then n moves are not sufficient for distributing the sheets. For the n-th person to receive a sheet, that sheet must be passed on $n-1$ times, so that sheet can be separated at most once. In the first move, the first person cannot pass on the entire heap, as they would remain without a sheet. So the first two moves of the first person must be to prepare a sheet and pass it on. Then, however, person $n-1$ coould get their own sheet in at least $2+1+(n-2)$ moves, so it's impossible to distribute the sheets in n moves.

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4. One direction is clear, as if the product in both groups of k, then the product of divisors is k^2, a square number. Let's show that this condition is sufficient.

Case 1: n is not a square number. Let's denote the divisors of n with $d(n)$. If n is not a square number, then $d(n)$ is an even number, of the form $2l$, where l is a positive integer. The divisors can be divided into divisor pairs in which the product is n. The number of pairs is l, so the product of all divisors is n^l. As n is not a square number, n^l will be one if and only if l is even, of the form $2m$. If this holds, then we may place m divisor pairs in each group, making the product of divisors n^m. This proves the first case.

Case 2: n is a square number. Then $d(n)$ is an odd number. Excepting $\sqrt{n}$, the divisors of n can be collected into divisor pairs, the product of which is n, a square number. Therefore, the product of divisors is a square number if and only if $\sqrt{n}$ is a square number, i.e., n is the square of a square, a fourth power.

Let $n = x^4$, where x is a positive integer. Then excepting x^2, its divisors can be grouped into divisor pairs. Two such pairs are 1 and x^4, x and x^3. The plan is to put 1, x, x^4 in one group and x^2, x^3 in another group, as well as including an equal number of divisor pairs into both groups. This is possible exactly when the number of divisors is of the form $4m+1$.

But if the prime factorization of n is $p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, then its number of divisors is $d(n) = (a_1+1)(a_2+1)\dots(a_r+1)$. If $n = x^4$ is a fourth power, then all a_i are multiples of 4, so $d(n)$

is a product of numbers of the form $4m + 1$, hence of this form. Therefore, the divisors can be grouped into equal products in this case as well. This concludes the proof.

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5. $n + k - 1$.

Let's notice that given the area of three of four rectangles determined by two rows and two columns are known, then the fourth is also determined.

To show this, let the heights of the two rows be a and b, the width of the selected columns be c and d. The values of ac, ad, bc are known, and bd is determined via the equality $bd \cdot ac = ad \cdot bc$.

Upper estimate: Let's give the area values of all rectangles in the first row and first column. This is indeed $n + k - 1$ small rectangles, and the area of the p-th row q-th column is determined from the rectangles in the 1-st row q-th column, p-th row 1-st column, and the 1-st row 1-st column, as above.

Lower estimate: We show an approach for the lower estimate. Consider the bipartite graph whose vertices are the columns and the rows, where the l-th row and m-th column are connected with an edge if the area of the corresponding rectangle is known. Suppose that fewer than $n+k-1$ rectangles are sufficient. This means fewer than $n+k-1$ edges in the graph, so it is not connected. In particular, the rows can be partitioned into parts A and B, the columns into C and D, such that only rectangles in $A \times C$ and in $B \times D$ have a given area. Now, let's multiply the height of the rows in A by 2 and the columns in C by $\frac{1}{2}$. None of the given areas changes, but a rectangle determined by a row in A and a column in D has area doubled. Therefore, Benny cannot uniquely determine the rest of the areas.

Comment: There is a simple yet advanced alternative to the latter proof. The idea is that up to scaling the rows and columns, there are $n+k-1$ unknown row heights and column widths to be determined. These form a system of linear equations, after taking logarithms. Therefore, at least $n + k - 1$ equations are needed to determine all the variables.

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6. Let's analyse the game for small n-s first. For $n = 1$ the far left and far right cells would be the same, so the game would not make any sense. For $n = 2$ the blue player immediately wins with the starting position. If $n = 3$ or 4 the red player can easily win if he moves his original disc and colours the disc of the blue player.

When $n = 5$ and a player either decides to replace his disc instead of passing or places a new disc on the board, the other player easily wins by replacing his own disc. Therefore, the first player should always pass and so should do the second one until he wins the game with just passing.

From now on we focus on $n \geq 6$ in general. Let's name the middle cell (or if n is odd then the right one from the two middle cells) 'critical' cell. First, we can realize that the red disc that is furthest to the right will still always be to the left from the blue disc that is furthest to the left. If this was not so then there would be such a move during the game that creates this position for the first time. This however could not be obtained by any of the possible moves, which is a contradiction.

Let's name those positions critical when a player has his discs on cells that are 2 and 3 cells away from the critical cell whereas the other player does not have discs these distances away (or closer) from the critical cell. We claim that the player who can create such a critical position and can guarantee that he will be the first one to have his discs 2 and 3 cells away from critical cell, has a winning strategy. Let's name this person X.

Because of the symmetry of the board for any odd numbered n-s, the red player will be X. For even n-s there are two cases: if $n = 4k$ then the red player will need at least k moves to have discs 2 and 3 away from the critical cell, while the blue player only needs $k-1$, so the blue player will be X. If $n = 4k+2$ then the blue player also needs k moves, so the red player will be X. For odd n-s this works the following way: for $n = 4k+1$ and for $n = 4k+3$ both contestants need k moves to reach the position above.

The winning strategy of X is the following: From the starting position he should reach a critical position with the least possible moves and then continue the game with the strategy below:

1. case: If his opponent places a disc on the critical field: X places a new disc of his own, 1 away from the critical cell that is adjacent to his discs. This way the only disc of his opponent will change its colour and X will easily win. The opponent can therefore have only one disc since we supposed that X will reach the critical position within the least possible moves, which is either k or $k-1$, based on the remainder of n when divided by 4.

2. case: If his opponent does not occupy the critical cell: This time X will replace his disc that was 2 away from the critical cell to the critical cell itself. If within the next moves his opponent coloured the disc on the critical cell then X could move his disc that was 3 away from the critical cell to the adjacent one and this way the critical cell would have X's colour again. If his opponent did not colour the disc on the cricital cell, X if he can, places a disc on a cell that is next to his own, is the furthest away from his starting cell and is closer to the starting position of his opponent. If this is not possible because the cell already contains a disc then he fills up the empty cells between the critical cell and his own starting cell. If this is done in the proper order (not starting with filling up the cell next to the critical one) X can easily win the game.

Following the above detailed strategy the red player has a winning strategy for $n \geq 6$ if $4 \mid n$, otherwise it is the blue player who has a winning strategy. The strategy excludes the possibility that a game would last longer than 200 moves if $n \geq 6$, so either the red player will have more than $\frac{n}{2}$ discs on the board or the blue one will have at least $\frac{n}{2}$.

(Back to problems)

4.3.4 Category E$^+$

1. Let us suppose that c is divisible by such an odd prime number p for which there exist an odd prime number q that divides $p-1$. This way $p|a_1 = c$ and for all $n \geq 2$, $a_n = c\varphi(a_{n-1})$ is divisible by c. We know that for all $n \geq 2$, $p|a_{n-1}$ therefore $p-1|a_n$, so $q|a_n$. Let v_n denote the exponent of 2 in a_n. This way for $n \geq 2$, $a_n = 2^{v_n}p^s q^t r$, where $s, t \geq 1$ and r are coprimes to 2, p and q. So $\varphi(a_n) = 2^{v_n-1}(p-1)p^{s-1}(q-1)q^{t-1}\varphi(r)$ which is divisible by 2^{v_n+1} because $p-1$ and $q-1$ are both divisible by 2. This means that $v_{n+1} \geq v_n + 1$. Consequently, the exponent of 2 is continuously increasing and the sequence cannot be bounded.

If c is divisible by such an odd prime number for which $p-1$ is divisible by 4 then because for all $n \geq 1$, $p|a_n$ we know that $a_n = 2^{v_n}p^s r$ where $s \geq 1$ and r are coprimes to 2 and p. This way $\varphi(a_n) = 2^{v_n-1}(p-1)p^{s-1}\varphi(r)$ which is divisible by 2^{v_n+1} since $4|p-1$. The sequence cannot be bounded in this case either.

If $4|c$ then $a_n = 2^{v_n}r$ where r is odd. In this case $\varphi(a_n) = 2^{v_n-1}\varphi(r)$ and $a_{n+1} = c\varphi(a_n)$ are divisible by 2^{v_n+1} (since $4|c$). The sequence cannot be bounded this time either.

If $2|c$ and p is an odd prime number that is a factor of c then $a_n = 2^{v_n}p^s r$ and $a_{n+1} = c2^{v_n-1}(p-1)p^{s-1}\varphi(r)$, which is divisible by 2^{v_n+1} since $2|p-1$ and $2|c$.

So we are left with three options: $2|c$, but c is not divisible by 4 and it is not divisible by any of the odd prime numbers either, meaning that $c = 2$. Or c is not divisible by 2 and it is only divisible by such an odd prime number p for which $p-1$ does not have any odd prime factors and for which $p-1$ is not divisible by 4. This is only possible if $p-1 = 2$, so $p = 3$.

If $9|c$ then $a_n = 3^{w_n}r$ where r is coprime to 3 and $a_{n+1} = c\varphi(a_n) = c \cdot 2 \cdot 3^{w_n-1}\varphi(r)$, which is divisible by 3^{w_n+1} (since $9|c$). This way the exponent of 3 is continuously increasing therefore it cannot be bounded.

So the only options left are $c = 2$ or $c = 3$. These are indeed good solutions: For $c = 2$, all $a_n = 2$ since $2\varphi(2) = 2$ whereas for $c = 3$, $a_1 = 3$ and for all $n \geq 2$, $a_n = 6$ since $3\varphi(3) = 6$ and $3\varphi(6) = 6$.

Second solution: The cases where $c = 2, 3$ are solved similarly to the previous one. Let $c \geq 4$. First we conclude that the only prime factor of a_n is at most c. We prove this by induction. The statement is trivial for $n = 1$.

It is known that $\varphi(x) = x \prod\limits_{p_i|x} \frac{p_i-1}{p_i}$ from which we can conclude that the prime factors of $\varphi(x)$ cannot be bigger than the prime factors of x. Let us suppose that the statement is true for n, meaning that the prime factors of a_n cannot be bigger than c. This way $a_{n+1} = c \cdot \varphi(a_n)$. We also know that all the prime factors of $\varphi(a_n)$ are at most the prime factors of a_n which, due to the induction cannot be bigger than the prime factors of c. Therefore the prime factors of a_{n+1} are also at most the prime factors of c. Let's look at the following expression:

$$\frac{a_{n+1}}{a_n} = \frac{c \cdot \varphi(a_n)}{a_n} \geq c \cdot \prod_{\substack{q \text{ prím} \\ q \leq c}} \frac{q-1}{q} \geq c \cdot \frac{4}{3} \prod_{2 \leq k \leq c} \frac{k-1}{k} = \frac{4}{3},$$

where the last inequality is true because $c \geq 4$, so in the product on the right we have $\frac{3}{4}$ whereas on the left we do not. From here with induction $a_{n+1} \geq c \cdot \left(\frac{4}{3}\right)^n$. Thus, it is not bounded.

$$\frac{a_{n+1}}{a_n} = \frac{c \cdot \varphi(a_n)}{a_n} \geq c \cdot \prod_{\substack{q \text{ prím} \\ q \leq c}} \frac{q-1}{q} \geq c \cdot \frac{4}{3} \prod_{2 \leq k \leq c} \frac{k-1}{k} = \frac{4}{3},$$

Third solution: With using the formula $\varphi(x) = x \prod\limits_{p_i|x} \frac{p_i-1}{p_i}$, it is easy to see that if $a|b$ for some a, b positive integers then $\varphi(a)|\varphi(b)$. Now let a_n and b_n be the two sequences that we obtain by the method described in the task with using the constants $c = a_1$ and $c = b_1$ where $a_1|b_1$. With the help of induction now we can conclude that $a_n|b_n$ for all n. The statement is true for $n = 1$. Let us suppose that $a_n|b_n$. This way $\varphi(a_n)|\varphi(b_n)$, meaning that $a_{n+1} = a_1 \cdot \varphi(a_n)|b_1 \cdot \varphi(b_n) = b_{n+1}$. So it is enough to confirm the statement for primes $p \geq 5$

and for $p = 4, 6, 9$ because if the sequence is not bounded for p then (based on the previous conclusions) it is not bounded for $p \cdot k$ either and some p from above will divide all $n > 3$ numbers. Therefore, it is enough to prove the following lemma:

Lemma: For all primes $p \geq 5$ and for $p = 4, 6, 9$ the sequence is not bounded.

Proof: We will not go into details on checking the statement for $p = 4, 6, 9$, it is easy to do so. Let the given prime be $p \geq 5$ and let's suppose that the statement has been confirmed for all values that are smaller than the one in the lemma. We can realize that we have shown that the sequence is not bounded either for compound numbers that are smaller than p. We can conclude that if a_n denotes the sequence that is obtained by $p = c$ and if b_n denotes the sequence where $p - 1 = c$ then $a_n = p \cdot b_{n-1}$ for all integers $n \geq 2$. We prove this by induction:

$$a_2 = p\varphi(p) = p \cdot (p-1) = p \cdot b_1$$

$$a_2 = p\varphi(p) = p \cdot (p-1) = p \cdot b_1$$

Next, we can conclude that all prime factors of b_n are smaller than p for all n. With the help of the formula $\varphi(x) = x \prod\limits_{p_i|x} \frac{p_i-1}{p_i}$ it is easy to confirm that all the prime factors of $\varphi(x)$ are not bigger than the biggest prime factor of x. So if we suppose that all the prime factors of b_{n-1} are smaller than p then, because of the statement above, all the prime factors of $\varphi(b_{n-1})$ will be smaller than p. This means that all prime factors of $b_n = (p-1) \cdot \varphi(b_{n-1})$ are smaller than p, too. We go on with proving the lemma. Let us suppose that $a_n = p \cdot b_{n-1}$. This way $a_{n+1} = p \cdot \varphi(p \cdot b_{n-1})$, but previously we have concluded that p and b_{n-1} are coprimes so with using the multiplicity of the function φ, $\varphi(p \cdot b_{n-1}) = \varphi(p)\varphi(b_{n-1}) = (p-1)\varphi(b_{n-1}) = b_n$. So $a_n = p \cdot b_{n-1}$ for all n. Since in the condition we have already confirmed the statement for $p-1$, the sequence b_n will not be bounded. Consequently, a_n will not be bounded either, which proves both the lemma and the statement of the task.

(Back to problems)

2. For the solution, see Category E Problem 5.

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3. Let the sides of a triangle be $a = y + z$, $b = z + x$ and $c = x + y$. Then, according to the law of cosines

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{2x^2 + 2xy + 2xz - 2yz}{2(x+y)(x+z)} = \frac{x(x+y+z) - yz}{x(x+y+z) + yz} = \frac{x - yz}{x + yz}.$$

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{2x^2 + 2xy + 2xz - 2yz}{2(x+y)(x+z)} = \frac{x(x+y+z) - yz}{x(x+y+z) + yz} = \frac{x - yz}{x + yz}.$$

Similarly, we will get the two other fractions appearing in the inequality as the cosine of the two other angles of the triangle.

The triangle generated is an acute triangle since

$$z > xy \implies 2xz + 2yz + 2z^2 = 2z(x+y+z) > 2xy \implies a^2 + b^2 > c^2,$$

Similarly $a^2 + c^2 > b^2$ and $b^2 + c^2 > a^2$.

The inequality for acute triangles

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) < 1$$

can be used to prove the statement.

To complete the proof we will show why the inequality above stands true.

$$\cos\gamma = -\cos(\alpha+\beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos^2\gamma = \sin^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta - 2\sin\alpha\sin\beta\cos\alpha\cos\beta =$$
$$= 1 - \cos^2\alpha - \cos^2\beta + 2\cos^2\alpha\cos^2\beta - 2\sin\alpha\sin\beta\cos\alpha\cos\beta$$

Rearranging the equation we get that the statement to be proven is

$$0 < 2\sin\alpha\sin\beta - \cos\alpha\cos\beta = -2\cos(\alpha+\beta),$$

which is true since the triangle is acute.

Second solution: (Sketch of the proof) Using the notations of the first solution, the inequality can be expressed with letters a, b, c. Let $A = a^2$, $B = b^2$ és $C = c^2$. This way, the inequality to be proven is

$$\sum_{\text{cyc}} \frac{(B+C-A)^2}{4BC} < 1.$$

After multiplying by the denominator, rearranging everything on one side, simplifying and factoring out we will get

$$0 < (A+B-C)(A+C-B)(B+C-A).$$

This is true since the triangle inequality stays true for numbers $A = a^2$, $B = b^2$, $C = c^2$, just as we have seen in the previous solution.

Third solution: After modifying the left side we will obtain the inequality:

$$1 - \frac{4xyz}{(x+yz)^2} + 1 - \frac{4xyz}{(y+zx)^2} + 1 - \frac{4xyz}{(z+xy)^2} < 1.$$

After rearraging the inequality and dividing by 2 we will get an inequality that is equivalent to the one in the task.

$$1 < 2xyz\left(\frac{1}{(x+yz)^2} + \frac{1}{(y+zx)^2} + \frac{1}{(z+xy)^2}\right).$$

$x + y + z = 1$, so $x + yz = x(x+y+z) + yz = (x+y)(x+z) = (1-z)(1-y)$. Similarly $y + zx = (1-x)(1-z)$ and $z + xy = (1-y)(1-x)$. Therefore

$$\left(\frac{1}{(x+yz)^2} + \frac{1}{(y+zx)^2} + \frac{1}{(z+xy)^2}\right) = \left(\frac{1}{(1-y)^2(1-z)^2} + \frac{1}{(1-z)^2(1-x)^2} + \frac{1}{(1-x)^2(1-y)^2}\right) =$$

$$= \frac{(1-x)^2+(1-y)^2+(1-z)^2}{(1-x)^2(1-y)^2(1-z)^2} = \frac{1+x^2+y^2+z^2}{(1-x)^2(1-y)^2(1-z)^2},$$

in the latter we again used the condition that $x+y+z=1$. So

$$2xyz\left(\frac{1}{(x+yz)^2}+\frac{1}{(y+zx)^2}+\frac{1}{(z+xy)^2}\right) = 2xyz\left(\frac{1+x^2+y^2+z^2}{(1-x)^2(1-y)^2(1-z)^2}\right).$$

Therefore the statement of the task is equivalent to the inequality below:

$$1 < 2xyz\left(\frac{1+x^2+y^2+z^2}{(1-x)^2(1-y)^2(1-z)^2}\right).$$

This stays true if and only if $\Big((1-x)(1-y)(1-z)\Big)^2 < 2xyz(1+x^2+y^2+z^2)$.

$$((1-x)(1-y)(1-z))^2 = (1-x-y-z+xy+yz+zx-xyz)^2 = (xy+yz+zx-xyz)^2 =$$

$$= x^2y^2+y^2z^2+z^2x^2+x^2y^2z^2+2x^2yz+2xy^2z+2xyz^2-2x^2y^2z-2x^2yz^2-2xy^2z^2$$

using the condition that $x+y+z=1$. Therefore the statement of the task is only true if

$$x^2y^2+y^2z^2+z^2x^2+x^2y^2z^2+2x^2yz+2xy^2z+2xyz^2-2x^2y^2z-2x^2yz^2-2xy^2z^2 < 2xyz(1+x^2+y^2+z^2),$$

rearranged $x^2y^2+y^2z^2+z^2x^2+x^2y^2z^2+2xyz(x+y+z) < 2xyz(1+x^2+y^2+z^2+xy+yz+zx)$.

$$2xyz(1+x^2+y^2+z^2+xy+yz+zx) = xyz(2+x^2+y^2+z^2+(x+y+z)^2) =$$

$$= xyz(2+x^2+y^2+z^2+1^2) = xyz(3+x^2+y^2+z^2).$$

Consequently, the statement of the task is equivalent to the inequality

$$x^2y^2+y^2z^2+z^2x^2+x^2y^2z^2+2xyz(x+y+z) < xyz(3+x^2+y^2+z^2)$$

and

$$0 < 3xyz+xyz(x^2+y^2+z^2)-2xyz(x+y+z)-x^2y^2-y^2z^2-z^2x^2-x^2y^2z^2$$

. After rearranging the inequality and using the condition $x+y+z=1$:

$$3xyz+xyz(x^2+y^2+z^2)-2xyz(x+y+z)-x^2y^2-y^2z^2-z^2x^2-x^2y^2z^2 =$$

$$= xyz(3-2(x+y+z))+xyz(x^2+y^2+z^2)-x^2y^2-y^2z^2-z^2x^2-x^2y^2z^2 =$$

$$xyz+xyz(x^2+y^2+z^2)-x^2y^2-y^2z^2-z^2x^2-x^2y^2z^2 = (x-yz)(y-zx)(z-xy),$$

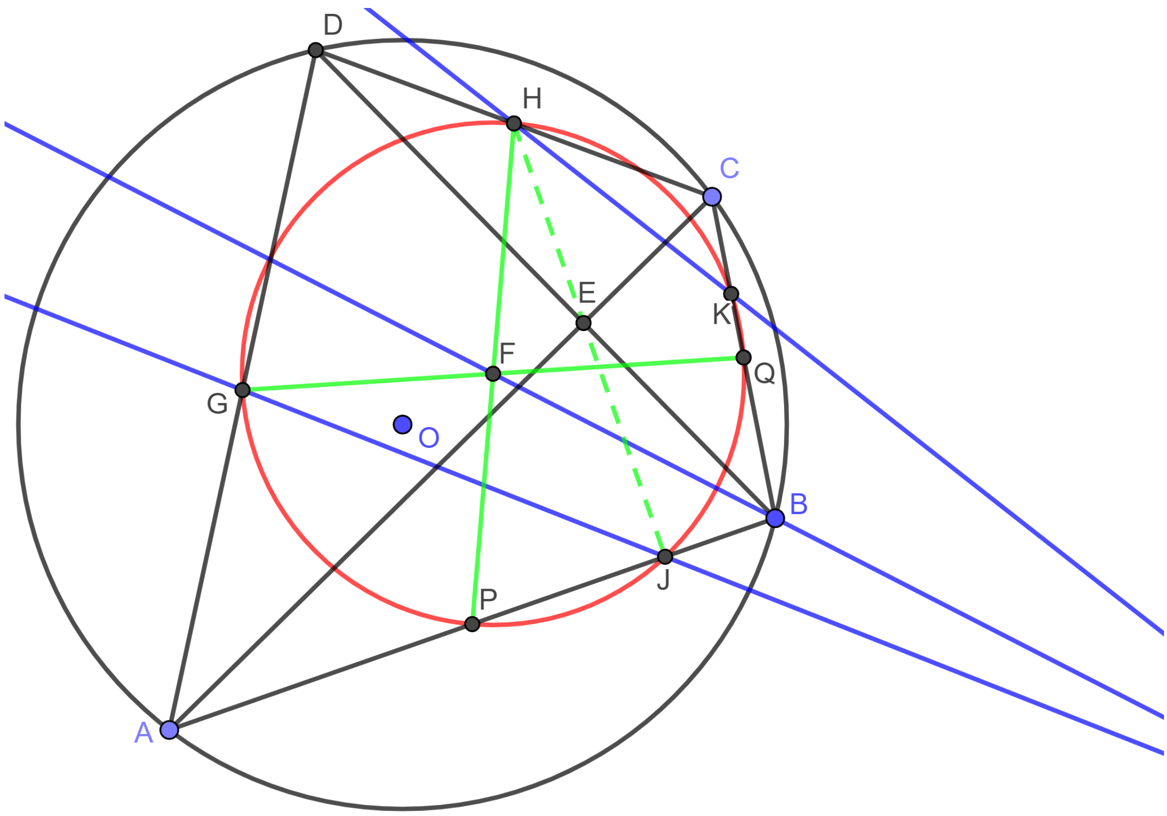
which is greater than 0 due to the conditions of the task $x > yz$, $y > zx$, $z > xy$. This is equivalent to the statement of the task, therefore we have proven the task's statement.

(Back to problems)

4. Let P and Q denote the midpoints of the AB and BC sides.

$$HEC\sphericalangle = HCE\sphericalangle = DBA\sphericalangle = AEJ\sphericalangle,$$

so H, E and J are collinear. Similarly, G, E and K are also collinear. HG and QP are parallel to AC, whereas QH and GP are parallel to BD. Therefore the vertices of the rectangle $GPQH$ lie on a circle, k. In addition, because H, E and J are collinear $HJP\sphericalangle$ is a right angle. Consequently, J also lies on circle k. Similarly, point K lies on circle k as well.



Based on the statement above HE is perpendicular to AB and OP is also perpendicular to AB. Similarly, PE is perpendicular to CD and OH is perpendicular to CD, meaning that $OHEP$ is a parallelogram. Similarly, $OGEQ$ is also a parallelogram. Therefore, F is the midpoint of both the HP and GQ segments.

Let us apply Pascal's theorem for points G, J, P, H, K, Q in order to obtain that X, B and F are collinear, thus proving the statement.

(Back to problems)

5. We will show that k is appropiate if and only if it is a prime power. Let the initial numbers be $a_1, a_2, \dots, a_n$. All equalities are viewed modulo n. By induction, the number on the paper of the first person after k minutes will be

$$A_k = \sum_{0 \le i \le k} (-1)^i \binom{k}{i} a_{i+1}.$$

Lemma: Given that p is a prime, $\binom{p^l}{a} \equiv 0 \pmod{p}$ if $0 < a < p^l$.

Proof: $\binom{p^l}{a} = \frac{p^l}{a} \cdot \frac{(p^l-1)(p^l-2)\dots(p^l-a+1)}{1\cdot 2\cdot\dots\cdot(a-1)}$. Observe that $p^l - i$ and i are divisible by exactly the same powers of p, so all factors p are cancelled in their quotient. Also, a is not divisible by p^l. Thus, when reducing this fraction, we arrive at an integer divisible by p. (For a more general fact, look up the Lucas lemma.)

Let us return to the problem at hand by looking at the different cases:

- If $n = 1$, $k = 1$ works.
- If $n = p^l$, where l is a positive integer, then by the Lemma, $A_n \equiv a_1 + (-1)^{p^l} a_1 \equiv 0 \pmod{p}$, as if p is odd, $(-1)^{p^l} = -1$, and if $p = 2$, then $1 + (-1)^{p^l} = 2 \equiv 0 \pmod{2}$. Therefore, after n minutes, there is a number divisible by p before the first player, as well as before the rest. From here, by induction, it is clear that after tn moves, everyone has a number divisible by p^t, so $k = l \cdot n$ is a suitable choice.
- If n has two distinct prime divisors, p and q, then let's see the numbers on the sheets of paper after p^l moves, where l is a positive integer. From the Lemma, we have that

  $$A_{p^l} \equiv a_1 + (-1)^{p^l} a_{p^l+1} \pmod{p}.$$

  Suppose that the starting numbers are $a_1 = 1$ and all others 0. As n is not a prime power, n does not divide p^l, so $a_{p^l+1} = 0$, meaning $A_{p^l} \equiv 1 \pmod{p}$, so n does not divide A_{p^l}. This is true for any positive integer l, so there will be no suitable k. This completes the proof.

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6. For the solution, see Category E Problem 6.

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4.4 Final round – day 2

4.4.1 Tables

#	ANS	Problem	P
C-1	2022	Thirty-three minutes, thirty-three seconds,	3p
C-2	7	A prime number is called *middle aged*	3p
C-3	15	A regular 12-gon is visible on the diagram,	3p
C-4	3	Csaba stands in the middle of a 15 m × 15 m room	3p
C-5	21	In duck language, only letters **q**, **a**, and **k** are used.	4p
C-6	23	Momma duck laid 50 eggs this years,	4p
C-7	180	At the beginning of each year, Kartal writes a sentence	4p
C-8	13	At the water bird olimpics,	4p
C-9	72	In Dürer's duck school, there are two rows of doors,	5p
C-10	8	Benedek draws circles with the same center in the following way.	5p
C-11	525	Three palaces, each rotating on a duck leg,	5p
C-12	42	One angle of a triangle equals the sum of the other two,	5p
C-13	4	6 teams took part in a soccer contest,	6p
C-14	46	Write some positive integers in the following table such that	6p
C-15	24	Csongi taught Benedek how to fold a duck in 8 steps	6p
C-16	1018	The product of Albrecht's three favorite numbers is 2022,	6p

#	ANS	Problem	P
D-1	15	A regular 12-gon is visible on the diagram,	3p
D-2	7	A prime number is called *middle aged*	3p
D-3	3	Csaba stands in the middle of a 15 m × 15 m room	3p
D-4	12	How many 10-digit sequences are there,	3p
D-5	13	At the water bird olimpics,	4p
D-6	525	Three palaces, each rotating on a duck leg,	4p
D-7	8	Benedek draws circles with the same center in the following way.	4p
D-8	4	6 teams took part in a soccer contest,	4p
D-9	1490	The *fragments* of a positive integer are the numbers	5p
D-10	864	In Dürer's duck school, there are three rows of doors,	5p
D-11	113	In Kacs Aladár street,	5p
D-12	24	Csongi taught Benedek how to fold a duck in 8 steps	5p
D-13	46	Write some positive integers in the following table such that	6p
D-14	252	Every side of a right triangle is an integer when measured in cm,	6p
D-15	6912	The pair of positive integers (a, b) is such that a does not divide b,	6p
D-16	41	Doofy duck buy tangerines in the store.	6p

#	ANS	Problem	P
E-1	21	In duck language, only letters **q**, **a**, and **k** are used.	3p
E-2	3	Csaba stands in the middle of a 15 m × 15 m room	3p
E-3	525	Three palaces, each rotating on a duck leg,	3p
E-4	5	At least how many regular triangles are needed	3p
E-5	8	Benedek draws circles with the same center in the following way.	4p
E-6	113	In Kacs Aladár street,	4p
E-7	1490	The *fragments* of a positive integer are the numbers	4p
E-8	1018	The product of Albrecht's three favorite numbers is 2022,	4p
E-9	252	Every side of a right triangle is an integer when measured in cm,	5p
E-10	6912	The pair of positive integers (a, b) is such that a does not divide b,	5p
E-11	7	In rectangle $ABCD$,	5p
E-12	24	Csongi taught Benedek how to fold a duck in 8 steps	5p
E-13	46	Write some positive integers in the following table such that	6p
E-14	2992	In Dürer's duck school, there are six rows of doors,	6p
E-15	41	Doofy duck buy tangerines in the store.	6p
E-16	5500	The number 60 is written on a blackboard.	6p

#	ANS	Problem	P
E^+-1	12	How many 10-digit sequences are there,	3p
E^+-2	5	At least how many regular triangles are needed	3p
E^+-3	525	Three palaces, each rotating on a duck leg,	3p
E^+-4	113	In Kacs Aladár street,	3p
E^+-5	33	On a circle k, we marked four points (A, B, C, D)	4p
E^+-6	252	Every side of a right triangle is an integer when measured in cm,	4p
E^+-7	6912	The pair of positive integers (a, b) is such that a does not divide b,	4p
E^+-8	7	In rectangle $ABCD$,	4p
E^+-9	24	Csongi taught Benedek how to fold a duck in 8 steps	5p
E^+-10	46	Write some positive integers in the following table such that	5p
E^+-11	2992	In Dürer's duck school, there are six rows of doors,	5p
E^+-12	41	Doofy duck buy tangerines in the store.	5p
E^+-13	42	Circle k_1 has radius 10,	6p
E^+-14	5	Benedek scripted a program which calculated the following sum:	6p
E^+-15	35	An ant crawls along the grid lines of an infinite quadrille notebook.	6p
E^+-16	5500	The number 60 is written on a blackboard.	6p

4.4.2 Category C

1. $33 \cdot 60 + 33 + 3 \cdot 3 = 1980 + 33 + 9 = 2022$.

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2. The numbers $n-4$, n, and $n+4$ give pairwise different remainders when divided by 3, so they can only be all prime if one of them is 3. Only $n-4=3$ is possible, as otherwise $n-4$ is negative. $n=7$ really is middle aged, so the solution is 7.

(Back to problems)

3. Let's number the vertices of the quadrangle such that the required angle is $A_1A_4A_2\sphericalangle$. An internal angle of the regular 12-gon is $\frac{10\cdot 180}{12} = 150$ degrees. $A_2A_3A_4$ is isosceles, while $A_1A_2A_3A_4$ is a trapezoid, so $A_3A_4A_2\sphericalangle = 15°$, while $A_3A_4A_1\sphericalangle = 30°$, so the answer is 15.

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4. If 3 people stand in a regular triangle of side length 2.9 centered at Csaba, then it is clear that he cannot reach the wall, albeit social distancing is upheld. If there are only two people other than Csaba (C), A and B, then let D be the projection of Csaba on line AB. Walking away from D along line CD, the distance of Csaba from A and B increases, so the rule is followed, and Csaba reaches the wall eventually. The answer is 3.

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5. One word is **aaaa**. If there is a consonant, that could be in four places, with two possible letters each, so 8 options. If there are two consonants, they could be in three possible places: first and third, first and fourth, second and fourth. The consonants could be in $2 \cdot 2$ ways, so this is 12 options. Three or four consonants are impossible, because then there would be two adjacently. So there are a total of $1 + 8 + 12 = 21$ words.

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6. Let Pappa's boy and girl guesses be b and g. Then $b + g = 50$ and $\frac{90}{100}g + \frac{115}{100}b = 50$. Comparing, $\frac{1}{10}g = \frac{3}{20}b$, so $2g = 3b$, and so $b = 20$. Therefore, 23 boy ducklings hatched.

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7. There cannot be two digits which occur once, nor can there be two digits which occur twoce, so the only possibility is that there is one digit appearing once and another digit appearing three times. Therefore, after 2022, the first possibility is 2111, but this doesn't check out. The next possibility, 2202, does work.

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8. The question is, how many circles contain at least three points. Of the six outer birds, there are $\frac{6\cdot 5}{2} = 15$ ways to choose two. The opposite pairs are on a line with the centre, but the other pairs with the centre yield a circle, and it is easy to see that all these circles are distinct, a total of 12 circles. The six outer points are on a circle, as well. This counts all circles, containing the central point or not, so there are 13 circles in all.

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9. Essentially, we have two different options: middle, school, middle, street, middle, school OR middle, street, middle, school, middle, school. In either cases, we may choose the use of doors in 3! ways at both levels, so we have a total of $2 \cdot 6 \cdot 6 = 72$ options.

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10. By the area of a circle, the first circle has area π, so the area of the n-th circle is $\frac{n(n+1)}{2}\pi$. The radius will be an integer when $\frac{n(n+1)}{2}$ is a square number. It is simple to check that this occurs first when $n = 8$.

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11. If they face south after n days, they will face north after $2n$ days. It is clear that their least common multiple 1050 tells us how many days later they face north. For this reason, it is after 525 days at the soonest that they could all face south. Checking that they do indeed face south at this time, the answer is established.

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12. The triangle is right-angled, so $a^2 + b^2 = c^2$. Further, we know that $c^2 + 168 = (a+b)^2 = a^2 + b^2 + 2ab$, so $ab = 84$. The area is $T = \frac{ab}{2} = 42$.

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13. The 0 point team lost five times. This means the 5 point team won at least once, so it had 1 win and 2 draws. The 8 points could only be 2 wins, 2 draws and 1 loss, and the 11 points could only be 3 wins and 2 draws. The 13 point team had 4 wins and 1 draw. Thus, the five given teams had 10 wins, 7 draws and 8 losses in total. The sixth team had two more losses than wins, and at least one win versus the 0 point team, so it must be 1 win, 3 losses, and 1 draw. In total, the sixth team gained 4 points.

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14. Logical steps lead one to fill in the diagram uniquely as follows.

					1
1		2	**3**	2	3
2	3	**4**	2		**2**
	2	3		1	3
		2			**2**
1	2	3	1		1

The total sum is 46.

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15. Let's observe that in the first step, the paper is folded in half, and from then on, every fold appears on the upper and bottom half, in red on one half and in blue on the other. Therefore, everything falls out from the second fold on, only the contribution from the first fold remains. Thus, the solution is 24. (If you don't believe it, try folding it!)

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16. Let Albrecht's three favorite numbers be a, b, and c. Then $abc = 2022$, $(a+1)(b+1)(c+1) = 1514$ and $a + b + c = 0$.

$$1514 = (a + 1)(b + 1)(c + 1) = abc + ab + ac + bc + a + b + c + 1 = 2023 + ab + ac + bc,$$

whence $ab + ac + bc = -509$. Thus,

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2ab - 2ac - 2bc = 0 + 2 \cdot 509 = 1018.$$

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4.4.3 Category D

1. For the solution, see Category C Problem 3.

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2. For the solution, see Category C Problem 2.

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3. For the solution, see Category C Problem 4.

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4. Let's place the four ones first. Next, let's place the twos, so the sequence is 1212121. Then, the threes can be placed in four ways, on either side of the respective 1. Let's see where the 4 can be placed in each case. Clearly, we have 4, 3, 3, 2 possibilities in the respective cases, so the number of possibilities is $4 + 3 + 3 + 2 = 12$.

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5. For the solution, see Category C Problem 8.

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6. For the solution, see Category C Problem 11.

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7. For the solution, see Category C Problem 10.

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8. For the solution, see Category C Problem 12.

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9. The four-digit number $\overline{abcd}$ has 10 fragments, which sum to $\overline{abcd} + \overline{abc} + \overline{bcd} + \overline{ab} + \overline{bc} + \overline{cd} + a + b + c + d = 1111a + 222b + 33c + 4d$.

The fragment sum of 2022 is 2296, so we seek the solutions of $1111a+222b+33c+4d = 2296$, where a, b, c, d are digits.

If $a \geq 3$, the sum is too large.

If $a = 2$ and $b \geq 1$, it is still too much. If $a = 2$ és $b = 0$, then $33c + 4d = 74$. Here, $c \geq 3$ make the sum too large, and there is no solution with $c = 0, 1$. If $c = 2$, we find $\overline{abcd} = 2022$.

If $a = 1$ and $b \geq 6$, the sum is too large (at least 2443). If $b = 5$, then $33c + 4d = 75$. Here, $c \leq 2$, but none of $c = 0, 1, 2$ yield a solution.

If $a = 1$ and $b = 4$, then $33c+4d = 297$. Here, $c = 9$ and $d = 0$ yield a good solution. If $c = 8$, then $4d = 33$, for which there is no solution; if $c \leq 7$, then $33c + 4d \leq 7 \cdot 33 + 9 \cdot 4 = 267 < 297$.

If $a = 1$ and $b \leq 3$, then the sum is at most $1111 + 3 \cdot 222 + 9 \cdot 33 + 9 \cdot 4 = 2110 < 2296$, so we don't obtain a good solution.

Therefore, the only good solution other than 2022 is 1490.

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10. The number of essentially different possibilities is 4. These are the following (denoting the layers from the street to the school as A, B, C):

$AAABBBCCC$, $AAABCCBBC$, $ABBAABCCC$, $ABCCBAABC$.

In every single possibility, the doors A can be visited in $3! = 6$ ways, as well as the doors B and the doors C. Therefore, the number of possibilities is $4 \cdot 6 \cdot 6 \cdot 6 = 864$ in total.

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11. Denote the central house number x, then the number of allotments is exactly x. If all three numbers on Scrooge's villa were put up, the sum of numbers would be x^2. However, the numbers x and $x+2$ are missing. This means $x^2 - x - (x+2) = 3133$, so $x^2 - 2x = x(x-2) = 3135$. From the factorisation $3135 = 3 \cdot 5 \cdot 11 \cdot 19$, one can guess that $x = 3 \cdot 19 = 57$ és $x - 2 = 5 \cdot 11 = 55$. The greatest house number is therefore $2x - 1 = 113$.

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12. For the solution, see Category C Problem 15.

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13. For the solution, see Category C Problem 14.

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14. Let the sides of the triangle be a, b, and $a+75$. From the Pythagorean theorem, $a^2+b^2=(a+75)^2$. Rearranging, we obtain $b^2=150a+75^2=75\cdot(2a+75)=5^2\cdot 3\cdot(2a+75)$, so $2a+75$ is three times a square number. Therefore, we seek an odd square number whose triple is greater than 75. The smallest such square number is 49, meaning $b=105$, $a=36$. The perimeter equals 252.

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15. Let $d=(a,b)$, then the number of common divisors of a and b equals the number of divisors of d. If $a=kd$ and $b=ld$, then $k,l\neq 1$ (as a and b do not divide one another), so at least one of k and l is at least 3. This shows $d\leq 33$. The most divisors of numbers between 1 and 33 are had by 24 and 30, 8 in both cases. We may suppose $a<b$, and the largest value ab is attained in the case $d=24$ with the pair $(72,96)$, and in the case $d=30$, with the pair $(60,90)$. Of these, the former is larger, and the answer is 6912.

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16. A whole tangerine can be assembled in precisely the following ways:

9	10	11	12	13
9	0	0	0	0
0	10	0	0	0
0	0	11	0	0
0	0	0	12	0
0	0	0	0	13
0	5	0	6	0
3	5	0	2	0
3	0	0	8	0
6	0	0	4	0

Let's see what the most number of tangerines is where a whole cannot be assembled. At most 10 of the 11, at most 12 or the 13 can be chosen. Of the 9, 10, 12 tangerines, the most that can be purchased is 8 of the 9, 9 of the 10, 1 of the 12. This means that $8+9+10+1+12+1=41$ tangerines are needed for a whole tangerine to be assembled.

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4.4.4 Category E

1. For the solution, see Category C Problem 5.

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2. For the solution, see Category C Problem 4.

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3. For the solution, see Category C Problem 11.

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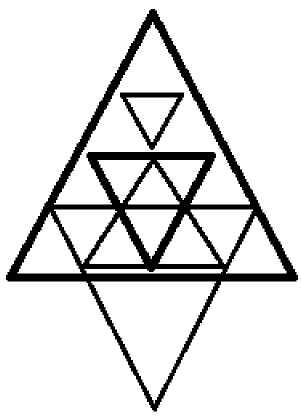
4. The diagram can be covered with 5 triangles:



Four triangles are not sufficient for the covering. Every triangle covers at most 1 horizontal line, so to cover the 4 horizontal lines, each triangle must completely cover one such line. This is true in all three directions. Now the length 4 segments can only be covered as on the diagram. The length 3 segments are completely covered by another three different triangles. As the length 2 segments are not covered this way, 4 triangles are not sufficient.

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5. For the solution, see Category C Problem 10.

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6. For the solution, see Category D Problem 11.

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7. For the solution, see Category D Problem 9.

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8. For the solution, see Category C Problem 16.

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9. For the solution, see Category D Problem 14.

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10. For the solution, see Category D Problem 15.

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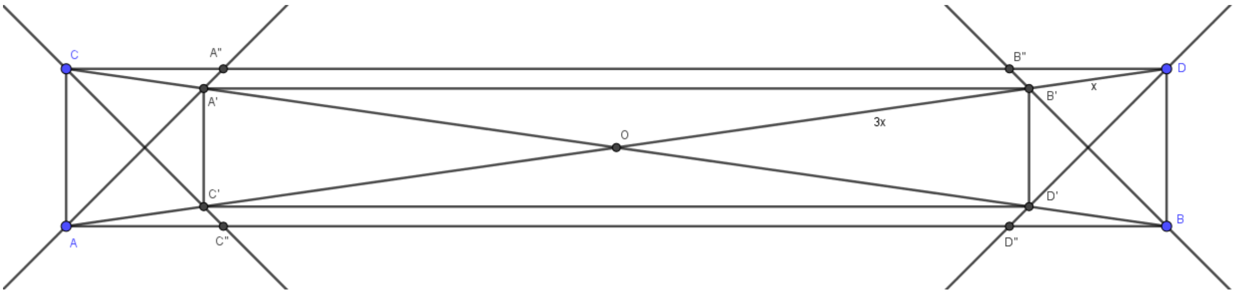
11. Let the intersection of line AA' and side CD be A'', and similarly (according to the diagram) let's define the points B'', C'', and D''. Now if $x = DB'$, then because of the area ratio, $B'O = 3x$ (O is the centre), and so $DB' : B'A = 1 : 7$. Now triangles $DB''B'$ and ABB' are similar, and the similarity ratio is $1 : 7$. But this equals the ratio $B''D : AB = DB : AB$ aránnyal. Thus, the answer is 7.



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12. For the solution, see Category C Problem 15.

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13. For the solution, see Category C Problem 14.

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14. When there are k lines of doors (3 doors per row), let the number of ways to pass between the lines be t_k. In other words, we do not distinguish between doors in the same row.

If we pass i doors before first turning back, Dodo must pass through i doors backwards and then i doors forwards before passing to the problem with $k - i$ rows. Here, i may vary from 1 to k, so $t_k = t_{k-1} + t_{k-2} + \ldots + t_1 + t_0$. Solving this recursion, $t_0 = t_1 = 1$ and for $k \geq 2$, $t_k = 2^{k-1}$.

For each row of doors, we can pass through the doors in 6 ways, so the answer for k rows is $6^k \cdot 2^{k-1}$. If $k = 6$, this is 1492992.

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15. For the solution, see Category D Problem 16.

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16. Let's say that sequence $d_0, d_1, d_2, \ldots, d_{10}$ is a chain of divisors if $d_0 = 60$, $d_{10} = 1$, and for all i, $d_{i+1} \mid d_i$. The question in the task is how many chains of divisors there are. Note that the sequence of exponents of 2, 3, 5 are decreasing, where $60 = 2^2 \cdot 3^1 \cdot 5^1$. There are $10 + \binom{10}{2}$ ways to choose the two places where the exponent 2 decreases down to 0, whether in one step or in two. There are 10 ways to choose the place where the exponent 1 decreases to 0. In total, the number of possible divisor chains is $10 \cdot 10 \cdot 55 = 5500$.

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4.4.5 Category E$^+$

1. For the solution, see Category C Problem 4.

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2. For the solution, see Category E Problem 4.

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3. For the solution, see Category C Problem 11.

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4. For the solution, see Category D Problem 11.

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5. Notice on the diagram that $\alpha_1 = \beta_2$, $\alpha_2 = \delta_1$, $\beta_1 = \gamma_2$, and $\gamma_1 = \delta_2$. Therefore, $\alpha_1\beta_1\gamma_1\delta_1 = \alpha_2\beta_2\gamma_2\delta_2$, so $\frac{\alpha_1}{\alpha_2} \cdot \frac{\beta_1}{\beta_2} \cdot \frac{\gamma_1}{\gamma_2} \cdot \frac{\delta_1}{\delta_2} = 1$. From these, $\frac{\delta_1}{\delta_2} = \frac{5 \cdot 11 \cdot 3}{2 \cdot 7 \cdot 10} = \frac{33}{28}$. Thus, the solution is 33.

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6. For the solution, see Category D Problem 14.

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7. For the solution, see Category D Problem 15.

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8. For the solution, see Category E Problem 15.

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9. For the solution, see Category C Problem 15.

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10. For the solution, see Category C Problem 14.

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11. For the solution, see Category E Problem 14.

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12. For the solution, see Category D Problem 16.

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13. Let the radius of k_3 be R, let the distance from the tangency point T of k_3 and e from k_1 be t, and let the radius of k_4 be r.

If the centre of k_n is O_n $(n = 1, 2, 3, 4)$, then by writing the Pythagorean theorem for triangles TO_3O_1, TO_3O_2, and TO_3O_4, we obtain three equations for three unknowns (R, t, r). These are readily solved by rearranging and applying the quadratic formula. The answer is $r = 42$.

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14. By the Chinese Remainder Theorem, it is sufficient to calculate the residue of the sum mod 5 and mod 7.

Modulo 5, $1^4 \equiv 2^4 \equiv 3^4 \equiv 4^4 \equiv 1$. Therefore, $1^1 + 6^6 + 11^{11} + 16^{16} \equiv 1 + 1 + 1 + 1 \equiv 4$, $2^2+7^7+12^{12}+17^{17} \equiv 2^2+2^3+2^0+2^1 \equiv 4+3+1+2 \equiv 0$, $3^3+8^8+13^{13}+18^{18} \equiv 3^3+3^0+3^1+3^2 \equiv 2 + 1 + 3 + 4 \equiv 0$, $4^4 + 9^9 + 14^{14} + 19^{19} \equiv (-1)^0 + (-1)^1 + (-1)^0 + (-1)^1 \equiv 0$.

In total, the sum of the first 20 terms is 4 mod 5.

For all x coprime to 5, $(x + 20)^{x+20} \equiv (x + 20)^x \cdot 1 \equiv x^x \pmod 5$, so in each block of 20, the sum is 4 mod 5. There are $2020/20 = 101$ such blocks, each with sum -1 mod 5. Thus, the complete sum is congruent to $-1 + 2021^{2021} \equiv 0$ mod 5.

Due to similar reasons, the sum in every block of 42 is $6 \equiv -1$ mod 7, and $2021 = 48 \cdot 42 + 5$, so the complete sum is $48 \cdot (-1) + 2017^{2017} + 2018^{2018} + 2019^{2019} + 2020^{2020} + 2021^{2021} \equiv 1 + 1^1 + 2^2 + 3^3 + 4^4 + 5^5 \equiv 1 + 1 + 4 + 6 + 4 + 3 \equiv 5$ mod 7.

The residue class mod 35 which is 0 mod 5 and 5 mod 7 is given by the numbers of the form $35k + 5$. Therefore, the required residue is 5.

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15. We may assume the ant makes its first move upwards. Its path must be a rectangle, where the first turn is to the left or to the right. If it turns to the right, when it moves a times up and b times to the right. The probability of the ant completing this rectangle is $\left(\frac{1}{3}\right)^{2a+2b-1}$.

We calculate $\sum\limits_{a=1}^{\infty}\sum\limits_{b=1}^{\infty}\left(\frac{1}{3}\right)^{2a+2b} = \left(\sum\limits_{a=1}^{\infty}\left(\frac{1}{9}\right)^{a}\right)^2 = \left(\frac{\frac{1}{9}}{1-\frac{1}{9}}\right)^2 = \frac{1}{64}$.

Therefore, the probability of a rectangle when the ant turns right is $\sum\limits_{a=1}^{\infty}\sum\limits_{b=1}^{\infty}\left(\frac{1}{3}\right)^{2a+2b-1} = \frac{3}{64}$. The probability when it turns left is the same, so the answer is $\frac{3}{32}$.

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16. For the solution, see Category E Problem 16.

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