E1. A group of students play the following game: they are counting one by one from 00 to 99 taking turns, but instead of every number they only say one of its digits. (The numbers in order are 00, 01, 02, ..., meaning that one-digit numbers are regarded as two-digit numbers with a first digit 0.) One way of starting the counting could be for example 0, 1, 2, 0, 4, 0, 6, 7, 8, 9, 1, 1, 2, 1, 1, 5, 6, 1, 8, 1, 0, 2 etc. When they reach 99, the counting restarts from 00. At some point Csongor enters the room and after listening to the counting for a while, he discovers that he is able to tell what number the counting is at. How many digits has Csongor heard at least?

Solution: Csongor had to hear at least three digits to figure out where the counting is at. First we show that two digits are not enough. Let the two digits he heard be a and b. If $a = b$, they could be anywhere between $a+1$ and $a+9$. If $b = a+1$ (including $9+1 = 0$), then $a, a+1$ and $10+a, 10+(a+1)$ could be a pair of numbers. If $a \neq b$ and $a \neq b-1$, then a could be the digit in the tens place of the first number and b in the ones place of the second number, or vice versa (a is in the ones place of the first, b is in the tens place of the second). These pairs of numbers exist, we only need to show they are different. This is true, because in the first pair of numbers the first number has a in the tens place, but in the second pair of numbers the first number can only have $b-1$ or b in the tens place, because the next number has b in the tens place.

If he heard three digits, it could be enough, for example, in the case of 1, 5, 3, he knows from the first two digits that just the number pairs 14, 15 or 51, 52 are possible and knowing the third digit, he will know that the counting is on 51, 52, 53.

E2. In a Greek village of 100 inhabitants in the beginning there lived 12 Olympians and 88 humans, they were the first generation. The Olympians are 100% gods while humans are 0% gods. In each generation they formed 50 couples and each couple had 2 children, who form the next generation (also consisting of 100 members). From the second generation onwards, every person's percentage of godness is the average of the percentages of his/her parents' percentages. (For example the children of 25% and 12.5% gods are 18.75% gods.)
a) Which is the earliest generation in which it is possible that there are equally many 100% gods as 0% gods?
b) At most how many members of the fifth generation are at least 25% gods?

Solution: I'll call 100% gods gods, 0% gods humans, everyone else partly god.

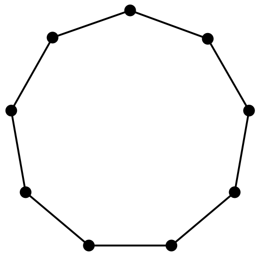
a) As you go from one generation to the next, the difference between the number of humans and the number of gods always decreases by at most the number of partly gods. If two parents are the same (human, partly god, god), their children will be the same, it won't change the difference of humans and gods in the next generation. If a god and a human are the two parents, then the difference will also not change, as they will have two partly god children. If a partly god and a human or a partly god and a god are the parents, they will have two partly god children, in which case the difference between the number of humans and gods will decrease, or increase, by 1. Therefore, for the whole generation, the difference between the number of gods and humans can be reduced by at most the number of partly gods.

In the first generation there are no partly gods, in the second generation the at most 24 children of the 12 gods can be partly gods, in the third generation the at most 48 grandchildren of the god 12 can be partly gods. Originally the difference between the number of gods and humans is $88 - 12 = 76$, this can be reduced in the next three generations by at most the number of partly gods, in total $0 + 24 + 48 = 72$, so in the fourth generation the difference between the number of gods and humans is at least $76 - 72 = 4$. That means that the earliest generation where the number of gods and humans can be the same is the fifth. This is possible, for example, if in the first generation there are 12 god - human pairs and 38 human - human pairs, in the second generation there are 24 demigod - human pairs and 26 human - human pairs, in the third generation there are 48 quarter-god - human pairs and 2 human - human pairs, in the fifth generation there are 4 eighth-god - human pairs and 46 eighth-god - eighth-god pairs. Thus, in the fifth generation there will be 92 eighth-gods and 8 sixteenth-gods, there will be the same number (0) of humans and gods in this generation.

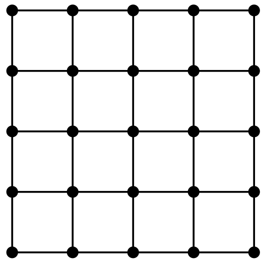
b) The sum of the parents' percentages of godness is equal to the sum of the percentages of godness of their children, since a couple has two children and the percentages of both are the average of the parents' percentages. In all 50 families this is true, so the children's generation always has the same god percentage as their parents'. So in every generation this sum is the same. In the first generation this sum is $12 \cdot 100 + 88 \cdot 0 =$1200, that is the sum in all generations. So in the fifth generation there can't be 49 or more at least 25% gods, because the sum would be $49 \cdot 25 = 1225 > 1200$, which is not possible. Thus, at most 48 members of the fifth generation can be at least 25% gods, and this is possible: for example, in the first two generations there are 6 god - god pairs and 44 human - human pairs, in the third there are 12 god - human pairs and 38 human - human pairs, in the fourth there are 24 demigod - human pairs and 26 human - human pairs, then in the 5th generation there will be 48 quarter-gods and 52 humans.

E3. Pitagoras drew some points in the plane and connected some of these with segments. Now Tortillagoras wants to write a positive integer next to every point, such that one number divides another number if and only if these numbers are written next to points that Pitagoras has connected. Can Tortillagoras do this for the following drawings?

a)



b)



In part b), vertices in the same row or column but not adjacent are not connected.

Solution:

Let $k(x)$ be the number on the point x.

a) Answer: No.

Suppose indirectly that we can write numbers on the points so that for any two numbers on adjecent points, one divides the other. We can orient the segments so that if $k(a)$ divides $k(b)$, then the segment between them is oriented towards b.

We have a polygon with 9 sides. But in this case we must have two adjecent segments, which are oriented in the same way. This means, we have a, b, c numbers, such that $k(a) \mid k(b)$ and $k(b) \mid k(c)$. This also means, that $k(a) \mid k(c)$. So we got that any segments between these three points must be part of the drawing, but there is no triangle on the picture.

b) Answer: Yes.

Let us denote points as coordinates, such that if a point has coordinates (x, y), it means that the point lies in the xth coloumn from the left, and the yth row from the bottom.

First we write different primes on the points, let $p(x)$ be the prime on the point x. Then we take a point with even coordinate sum, and multiple this number with the numbers on the adjecent points, and write this number on the point instead. We do that for every point with even coordinate sum. This is a good constrution, so we need to prove if $k(a) \mid k(b)$ then a and b are connected. Firstly note that if two points are connected, their coordinate sums have different parities.

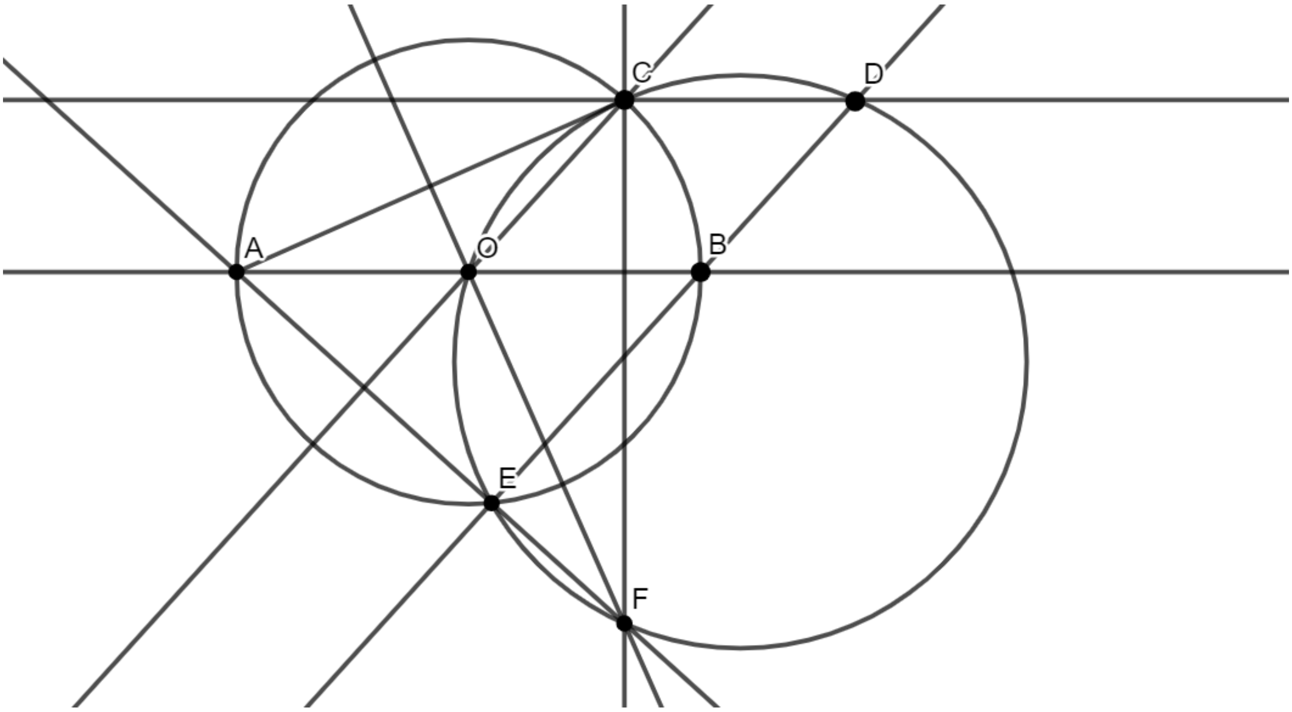
If the coordinate sum for both a and b is odd, they cannot be connected and $k(a)$ and $k(b)$ are different primes, so none of them divides the other.

If the coordinate sum for both a and b is even, they are again not connected, then lets look at the primes we first written to the poins, lets sign them $p(a)$ and $p(b)$. These points are not adjecent, because of the coordinates, so $p(a) \nmid k(b)$ and $p(b) \nmid k(a)$, which means $k(a) \nmid k(b)$ and vice versa.

The last case is when the coordinate sum of a is even, and the coordinate sum of b is odd. This means, that $k(b)$ is a prime, but $k(a)$ has more than one prime divisor, so $k(a) \nmid k(b)$. But it can happen that $k(b) \mid k(a)$, but this means that $p(b) \mid k(a)$, which is happens exactly when a and b are adjecent. And this is exactly what we wanted.

E4. Let k be a circle with diameter AB and centre O. Let C be an arbitrary point on the circle different from A and B. Let D be the point for which O, B, D and C (in this order) are the four vertices of a parallelogram. Let E be the intersection of the line BD and the circle k, and let F be the orthocenter of the triangle OAC. Prove that the points O, D, E, C, F lie on a circle.

Solution:



From the reverse of Thales's theorem, it is enough to prove that $FOD\angle = FED\angle = FCD\angle = 90°$, as all the five points would be on the circle with diameter FD.

By definition AF is an altitude in triangle AOC, so AF is perpendicular to CO. Also, as $OCDB$ is a parallelolgram, we know that CO and DB are parallel, so DB is perpendicular to AF. But as DBE is collinear, from the reverse of Thales's theorem, we know that AEF is also collinear. This implies that $DEF\angle = 90°$.

By definition CF is an altitude in triangle AOC, so CF is perpendicular to AO. Also, as $OCDB$ is a parallelogram, we know that AO and CD are parallel, so CD is perpendicular to CF. This implies that $DCF\angle = 90°$.

As OB and OC have the length of the radius of k, they have equal length. This implies that $OCDB$ is a parallelogram, with adjacent sides with equal lengths. This implies that $OCDB$ is a rhombus: its diagonals are perpendicular to each other, so BC is perpendicular to OD. As AB is a diameter in k, we know that BC is perpendicular to AC, so AC and OD are parallel. Also FO is an altitude in triangle AOC, so FO is perpendicular to AC. Then as AC and OD are paralalel, we know that FO is also perpendicular to OD, so $DOF\angle = 90°$. So we proved all the three desired right angles, which completes the proof.

E5. Let $n \geq 3$ be an integer. Timi thought of n different real numbers and then wrote down the numbers which she could produce as the product of two different numbers she had in mind. At most how many different positive prime numbers did she write down (depending on n)?

Solution: Let the set of Timi's numbers be A and the numbers she had written down be B. The following solution uses graph theoretic notation but the only property of graphs it uses is that if a graph on m vertices has at least m edges, then it must contain a cycle (which you can prove for yourselves as an exercise). So don't get intimidated by the notation, just follow each of the logical steps carefully.

Let us regard the prime elements of set B, for all $p \in B$ pick a pair (x, y) from set A for which $xy = p$. Then we create the following graph G: the vertices are the elements of set A, and two vertices are connected by an edge if and only if (x, y) is a pair that we picked earlier. It is clear that the number of edges in G is exactly the number of prime elements in B.

Now we claim that in G there is no closed walk of even length with an edge that has been only visited once. For the contrary suppose that such a walk exists and let its vertices be $c_0, c_1, c_2 \dots c_{2k}$, where $c_0 = c_{2k}$. Let also $x_0, x_1, x_2 \dots x_{2k}$ be the numbers from A corresponding to vertices $c_0, c_1, c_2 \dots c_{2k}$ (meaning that $x_0 = x_{2k}$). Since $c_i c_{i+1}$ is an edge in G, $x_i \cdot x_{i+1}$ is prime for all $0 \leq i < 2k$. Now let us consider the product $\prod_{i=1}^{2k} x_i$, we will write it two different ways:

$$\prod_{i=1}^{2k} x_i = \prod_{i=1}^{k} x_{2i-1} \cdot x_{2i} = \prod_{i=0}^{k-1} x_{2i} \cdot x_{2i+1}.$$

Since we know that the product of x_i and x_{i+1} is always prime, thus both $\prod_{i=1}^{k} x_{2i-1} \cdot x_{2i}$ and $\prod_{i=0}^{n-1} x_{2i} \cdot x_{2i+1}$ are the products of k (not necessarily different) primes. But since we assumed that there exists an edge of the walk that we only passed once, this prime would divide one of the products but not the other, which is a contradiction by the fundamental theorem of number theory.

Now we will show that G has at most n edges. For this it is enough to show that otherwise it contains a walk of even length with an edge that is visited only once. Suppose indirectly that G has at least $n+1$ edges. Then pick a component of G that has more edges than vertices. Since we can always find such a component by the pigeonhole principle, we can assume that G is connected.

It is clear that G cannot contain a cycle of even length, since then this would form an undersired walk. But since G has more than $n-1$ edges, it must contain a cycle. Let the vertices of this cycle be $c_0, c_1, c_2 \dots c_l$ where $c_0 = c_l$. Now erase the edge c_0c_1 from the graph. Since the graph will still contain at least $n-1$ edges, it will stay connected and we can find another cycle. Let the vertices of this cycle be $b_0, b_1 \dots b_k$ where $b_0 = b_k$. Since G remained connected after ereasing the edge c_0c_1, can find a path between c_0 and b_0, let its vertices be $c_0 = a_0, a_1, \dots a_m = b_0$. Since we know that G does not contain a cycle of even length, both n and k must be odd. Now consider the following closed path:

$$c_0, c_1 \dots c_l = c_0 = a_0, a_1 \dots a_m = b_0, b_1 \dots b_k = b_0 = a_m, a_{m-1}, \dots a_0 = c_0.$$

This is clearly a closed path and it has exactly $l + m + k + m$ vertices, which is even. We also know that the path passes through c_0c_1 only once since the second cycle was constructed without this edge in the graph. Therefore we have obtained a path of even length with an edge that has only been visited once, which is a contradiction.

Thus we have shown that G can have at most n edges, therefore B cannot have more than n prime elements. Now we just need to construct a set A for which B has exactly n prime elements.

Let $p_1, p_2 \dots p_n$ the first n primes and let the elements of A be as follows: $x_1 = \frac{\sqrt{p_3 p_2}}{\sqrt{p_1}}, x_2 = \frac{\sqrt{p_1 p_3}}{\sqrt{p_2}}$ and $x_3 = \frac{\sqrt{p_1 p_2}}{\sqrt{p_3}}$. From this on define the elements of A recursively: let for every $n-1 \geq k \geq 3$ be $x_{k+1} = \frac{p_{k+1}}{x_k}$. (Since none of the x_k are 0, this recursive formula is valid). Then the products

$x_2 \cdot x_3, x_3 \cdot x_1, x_1 \cdot x_2$ and $x_3 \cdot x_4, x_4 \cdot x_5 \dots x_{n-1} \cdot x_n$ equal to primes $p_1, p_2 \dots p_n$, therefore we have shown that there could be at most n distinct positive primes in B.