E+ category

XVI. DÜRER COMPETITION

First round: 18^{th} November 2022.

MATHEMATICS
PROBLEMS

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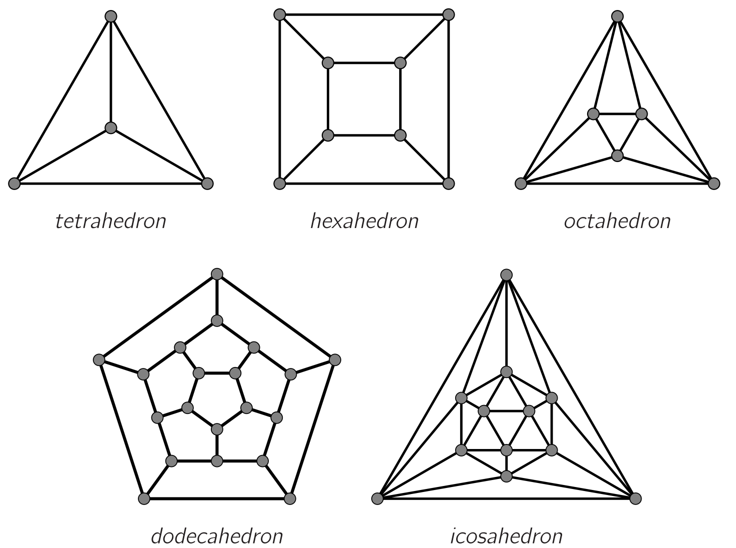
1. Find all positive integers n such that $\lfloor\sqrt{n}\rfloor + \left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rfloor > 2\sqrt{n}$.

If k is a real number, then $\lfloor k \rfloor$ means the floor of k, this is the greatest integer less than or equal to k.

2. We say that a graph G is *divisive*, if we can write a positive integer on each of its vertices such that all the integers are distinct, and any two of these integers divide each other if and only if there is an edge running between them in G. Which Platonic solids form a divisive graph?



3. Let $n \geq 3$ be an integer and A be a subset of the real numbers of size n. Denote by B the set of real numbers that are of the form $x \cdot y$, where $x, y \in A$ and $x \neq y$. At most how many distinct positive primes could B contain (depending on n)?

4. We are given an angle $0° < \varphi \leq 180°$ and a circular disc. An ant begins its journey from an interior point of the disc, travelling in a straight line in a certain direction. When it reaches the edge of the disc, it does the following: it turns clockwise by the angle φ, and if its new direction does not point towards the interior of the disc, it turns by the angle φ again, and repeats this until it faces the interior. Then it continues its journey in this new direction and turns as before every time when it reaches the edge. For what values of φ is it true that for any starting point and initial direction the ant eventually returns to its starting position?

5. Consider an acute triangle ABC. Let D, E and F be the feet of the altitudes through vertices A, B and C. Denote by A', B', C' the projections of A, B, C onto lines EF, FD, DE, respectively. Further, let H_D, H_E, H_F be the orthocenters of triangles $DB'C'$, $EC'A'$, $FA'B'$. Show that

$$H_DB^2 + H_EC^2 + H_FA^2 = H_DC^2 + H_EA^2 + H_FB^2.$$

Please write all the solutions on separate sheets. Make sure to write the name of your team and the category on every paper.

Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVI. Dürer Competition