E+1. Find all positive integers n such that $\lfloor\sqrt{n}\rfloor + \left\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\right\rfloor > 2\sqrt{n}$.

If k is a real number, then $\lfloor k\rfloor$ means the floor of k, this is the greatest integer less than or equal to k.

Solution:

Answer:

Numbers of the form $x(x+1)$ and $x(x+2)$, where x is an arbitrary positive integer.

Solution:

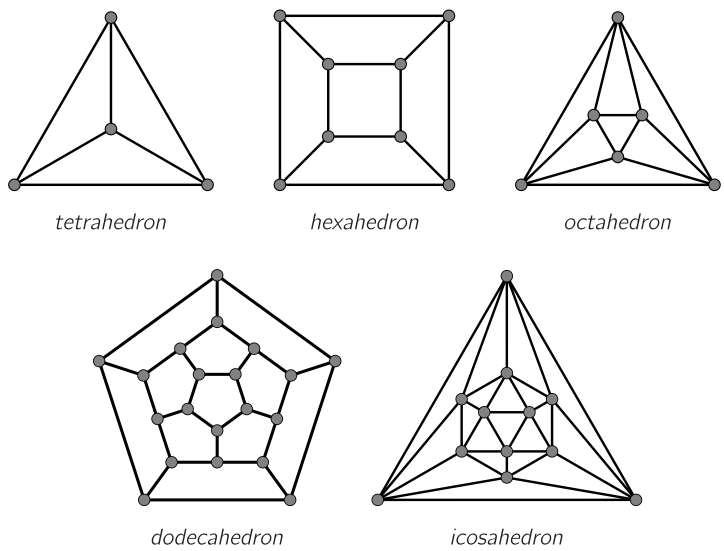
Let $\lfloor\sqrt{n}\rfloor = s$, where s is a positive integer. Then $s \le \sqrt{n} < s+1$, so $s^2 \le n < (s+1) = s^2+2s+1$. Since n and s are integers, $n \le s^2+2s = s(s+2)$, so $s^2 \le n \le s(s+2)$, this means $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{n}{s}\rfloor$. We look for n positive integers for which $s + \lfloor\frac{n}{s}\rfloor > 2\sqrt{n}$. There are two cases depending on whether or not s divides n.

If s divides n, $s^2 \le n \le s(s+2)$ so $n = s^2$, $n = s(s+1)$ or $n = s(s+2)$. For the first of these $\sqrt{n} = s = \lfloor\sqrt{n}\rfloor = \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor$, so that $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = 2\sqrt{n}$, there is equality. If $n = s(s+1)$, then $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{s(s+1)}{s}\rfloor = s+1$. Thus $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = s+(s+1) = 2s+1 = \sqrt{4s^2+4s+1}$ and $2\sqrt{n} = \sqrt{4s^2+4s}$, so $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor > 2\sqrt{n}$ for the numbers of the form $s(s+1)$, where s is a positive integer. If $n = s(s+2)$, then $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{s(s+2)}{s}\rfloor = s+2$. Thus $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = s+(s+2) = 2s+2 = \sqrt{4s^2+8s+4}$ and $2\sqrt{n} = \sqrt{4s^2+8s}$, so $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor > 2\sqrt{n}$ for the numbers $s(s+2)$, where s is a positive integer. (In all cases we use that out of two positive real number, the larger has larger square.)

If s does not divide n, from $s^2 \le n \le s(s+2)$ we get that $s^2 < n < s(s+1)$ or $s(s+1) < n < s(s+2)$. In the first case $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{n}{s}\rfloor = s$, so $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = s+s = 2s$, while $s^2 < n$. Then $s < \sqrt{n}$, so $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = 2s < 2\sqrt{n}$. In the second case $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{n}{s}\rfloor = s+1$, so $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = s+(s+1) = 2s+1$, while from $s^2+s < n$, $(s+0,5)^2 = s^2+s+0,25 < n$ because s^2+s and n are integers. Thus $s+0,5 < \sqrt{n}$, therefore $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = 2s+1 < 2\sqrt{n}$. So if s does not divide n, the right-hand side, $2\sqrt{n}$ is the greater.

In summary for a positive integer n, $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor > 2\sqrt{n}$ is satisfied exactly if $n = x(x+1)$ for some positive integer x or $n = y(y+2)$ for some positive integer y.

E+2. We say that a graph G is *divisive*, if we can write a positive integer on each of its vertices such that all the integers are distinct, and any two of these integers divide each other if and only if there is an edge running between them in G. Which Platonic solids form a divisive graph?



Solution: Direct the edges of our graph in a way such that each edge points towards the number which is divisible by the number on the start of the edge. A finite graph is divisive if and only if the edges can be directed in a way that the directed graph exactly forms the relationships of a partially ordered set. Our first observation is that a divisive graph can not contain an odd cycle of length at least 5 as an induced subgraph. This is because two adjacent edges would be directed in the same direction hence creating a new edge (a diagonal of the cycle) and so the cycle wouldn't be an induced subgraph.

Now we will answer the cases one by one:

- **Tetrahedron:** We can get a construction by writing different powers of 2 on all nodes.
- **Hexahedron:** This is a bipartite graph, so let us form two classes of nodes such that there is no edge running inside any of the classes. We write different prime numbers on the nodes of the first class, and we write the product of the three adjacent primes to the nodes of the second class.
- **Octahedron:** If the octahedron is put on one of its vertices, there is a single top and bottom vertex and there is a square in the middle layer. The four nodes of the square shall be filled with numbers $2, 12, 3, 18$ respectively. The bottom vertex shall be given number $36 \cdot 5 = 180$ and we assign $36 \cdot 7 = 252$ to the top vertex. This way, all nodes in the middle layer will divide the top and bottom vertices, but these two won't divide each other.
- **Dodecahedron:** The graph of the dodecahedron is not divise, beacause it contains a pentagon as an induced subgraph (any 5 points lying on the same face of the hexagon form a pentagon).
- **Icosahedron:** The graph of the icosahedron is not divise, beacause it contains a pentagon as an induced subgraph (the neighbours of any vertex form a pentagon in this case).

E+3. Let $n \geq 3$ be an integer and A be a subset of the real numbers of size n. Denote by B the set of real numbers that are of the form $x \cdot y$, where $x, y \in A$ and $x \neq y$. At most how many distinct positive primes could B contain (depending on n)?

Solution: Let us regard the prime elements of set B, for all $p \in B$ pick a pair (x, y) from set A for which $xy = p$. Then we create the following graph G: the vertices are the elements of set A, and two vertices are connected by an edge if and only if (x, y) is a pair that we picked earlier. It is clear that the number of edges in G is exactly the number of prime elements in B.

Now we claim that in G there is no closed walk of even length with an edge that has been only visited once. For the contrary suppose that such a walk exists and let its vertices be $c_0, c_1, c_2 \ldots c_{2k}$, where $c_0 = c_{2k}$. Let also $x_0, x_1, x_2 \ldots x_{2k}$ be the numbers from A corresponding to vertices $c_0, c_1, c_2 \ldots c_{2k}$ (meaning that $x_0 = x_{2k}$). Since $c_i c_{i+1}$ is an edge in G, $x_i \cdot x_{i+1}$ is prime for all $0 \leq i < 2k$. Now let us consider the product $\prod_{i=1}^{2k} x_i$, we will write it two different ways:

$$\prod_{i=1}^{2k} x_i = \prod_{i=1}^{k} x_{2i-1} \cdot x_{2i} = \prod_{i=0}^{k-1} x_{2i} \cdot x_{2i+1}.$$

Since we know that the product of x_i and x_{i+1} is always prime, thus both $\prod_{i=1}^{k} x_{2i-1} \cdot x_{2i}$ and $\prod_{i=0}^{n-1} x_{2i} \cdot x_{2i+1}$ are the products of k (not necessarily different) primes. But since we assumed that there exists an edge of the walk that we only passed once, this prime would divide one of the products but not the other, which is a contradiction by the fundamental theorem of number theory.

Now we will show that G has at most n edges. For this it is enough to show that otherwise it contains a walk of even length with an edge that is visited only once. Suppose indirectly that G has at least $n+1$ edges. Then pick a component of G that has more edges than vertices. Since we can always find such a component by the pigeonhole principle, we can assume that G is connected.

It is clear that G cannot contain a cycle of even length, since then this would form an undersired walk. But since G has more than $n-1$ edges, it must contain a cycle. Let the vertices of this cycle be $c_0, c_1, c_2 \ldots c_l$ where $c_0 = c_l$. Now erase the edge $c_0 c_1$ from the graph. Since the graph will still contain at least $n-1$ edges, it will stay connected and we can find another cycle. Let the vertices of this cycle be $b_0, b_1 \ldots b_k$ where $b_0 = b_k$. Since G remained connected after ereasing the edge $c_0 c_1$, can find a path between c_0 and b_0, let its vertices be $c_0 = a_0, a_1, \ldots a_m = b_0$. Since we know that G does not contain a cycle of even length, both n and k must be odd. Now consider the following closed path:

$$c_0, c_1 \ldots c_l = c_0 = a_0, a_1 \ldots a_m = b_0, b_1 \ldots b_k = b_0 = a_m, a_{m-1}, \ldots a_0 = c_0.$$

This is clearly a closed path and it has exactly $l + m + k + m$ vertices, which is even. We also know that the path passes through $c_0 c_1$ only once since the second cycle was constructed without this edge in the graph. Therefore we have obtained a path of even length with an edge that has only been visited once, which is a contradiction.

Thus we have shown that G can have at most n edges, therefore B cannot have more than n prime elements. Now we just need to construct a set A for which B has exactly n prime elements.

Let $p_1, p_2 \ldots p_n$ the first n primes and let the elements of A be as follows: $x_1 = \frac{\sqrt{p_3 p_2}}{\sqrt{p_1}}, x_2 = \frac{\sqrt{p_1 p_3}}{\sqrt{p_2}}$ and $x_3 = \frac{\sqrt{p_1 p_2}}{\sqrt{p_3}}$. From this on define the elements of A recursively: let for every $n-1 \geq k \geq 3$ be $x_{k+1} = \frac{p_{k+1}}{x_k}$. (Since none of the x_k are 0, this recursive formula is valid). Then the products $x_2 \cdot x_3, x_3 \cdot x_1, x_1 \cdot x_2$ and $x_3 \cdot x_4, x_4 \cdot x_5 \ldots x_{n-1} \cdot x_n$ equal to primes $p_1, p_2 \ldots p_n$, therefore we have shown that there could be at most n distinct positive primes in B.

E+4. We are given an angle $0° < \varphi \leq 180°$ and a circular disc. An ant begins its journey from an interior point of the disc, travelling in a straight line in a certain direction. When it reaches the edge of the disc, it does the following: it turns clockwise by the angle φ, and if its new direction does not point towards the interior of the disc, it turns by the angle φ again, and repeats this until it faces the interior. Then it continues its journey in this new direction and turns as before every time when it reaches the edge. For what values of φ is it true that for any starting point and initial direction the ant eventually returns to its starting position?

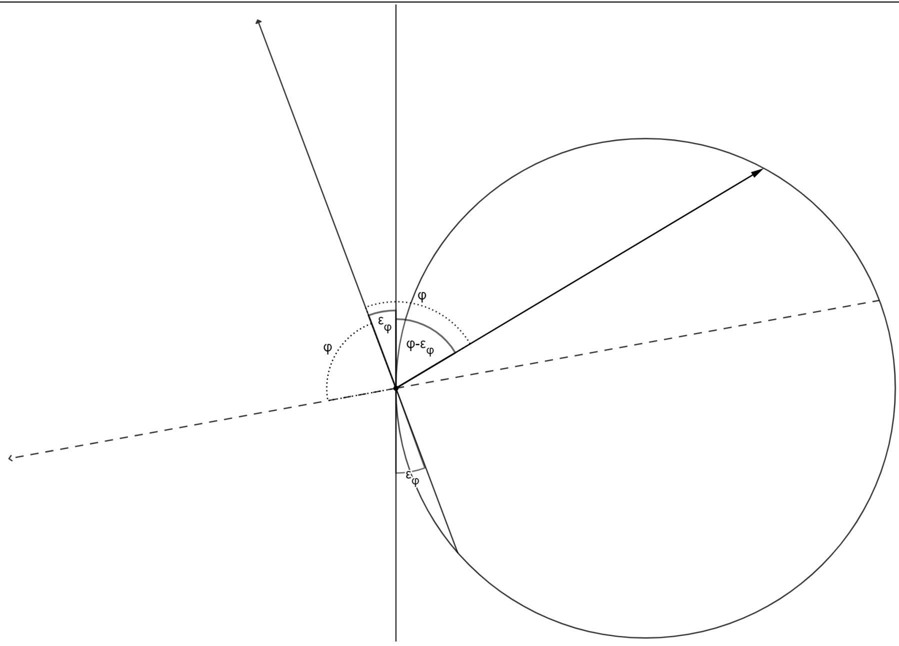
Solution: The answer is the ant only returns if $\varphi = 90°$ or $\varphi = 180°$. In the second case, after it reaches the edge of the disc, it goes back on the same chord, and will return to its initial position, regardless of its starting position and initial direction. If $\varphi = 90°$, we have two cases depending on whether it turns once or twice the first time it reaches the edge. It cannot turn more, as after two turns, the direction is pointing back to just where it came from. Clearly, if it turns twice, it is the same case as when $\varphi = 180°$.

In case it turns only once, its path will be the inscribed rectangle which has initial chord (on which the ant starts moving) as a side. This is because initially it is true, and whenever it reaches the edge, it can only continue its journey on the same line as it came from or on the other side of the rectangle at that point. After reaching the edge the first time it will travel in the negative direction on the perimeter of the rectangle.

Now we prove that if φ is not 90° or 180°, then there is a starting point and direction so that the ant never returns there. For a contradiction assume that for a certain such φ the ant returns to its starting point regardless of the starting point and direction.

Observe that two non-identical chords intersect at most at one point, that there are uncountably many points on a chord, and that the ant's journey consists of countably many chords. If we could choose a starting chord and direction along it so that the ant would never go on that chord, then we could choose a starting point on that chord that the other chords in the journey don't contain, contradicting our assumption. Therefore for this φ, for any starting point and initial direction, the ant has to return to the chord during its journey.

Since it must return an infinite numer of times because of this, there will be two instances when it goes in the same direction through the chord. Without loss of generality, we may assume that it is the n^{th} chord when it first travels through the initial chord again in the initial direction. It is easy to see that the $(n+k)^{\text{th}}$ chord will be the same as the $(1+k)^{\text{th}}$ chord, since the chord and direction determine the next chord and direction. Because of this, whenever the ant returns to the first chord in the initial direction it must come from the same chord and direction. So if we can show a second chord, which would be the second chord for two different initial chords, we get a contradiction, as only one of those two chords can be in the cycle we get when we start the ant from the second chord. Starting from the other initial chord, the ant gets into the cycle which doesn't include that initial chord. Contradiction.



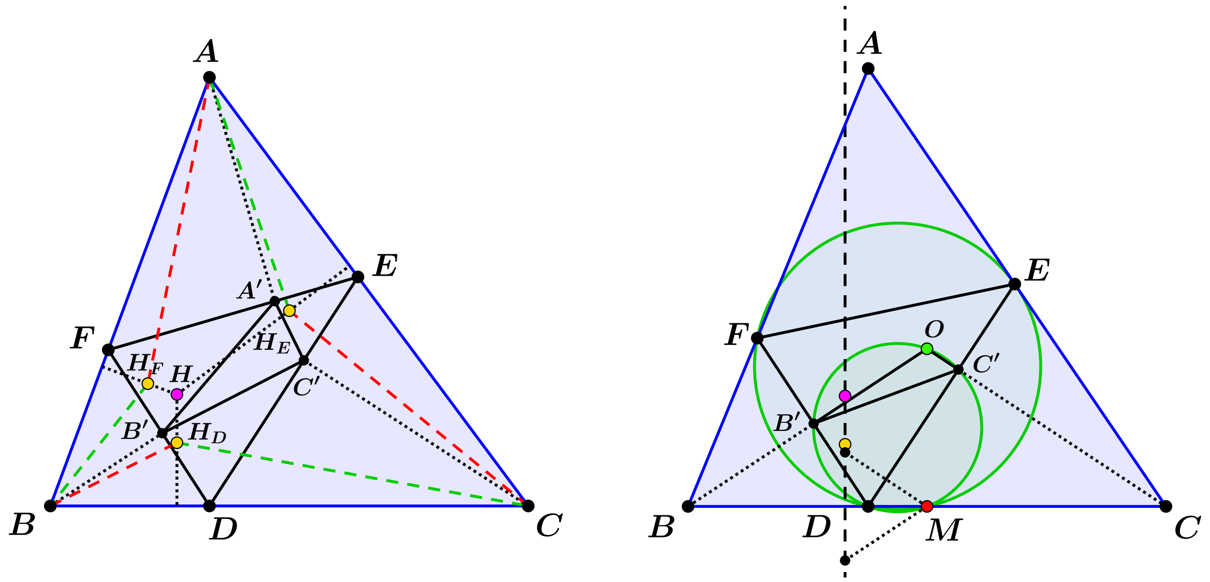
Draw the tangent to the disc from an arbitrary edge point and measure the angles as in the above figure, where ε_φ is small enough. The figure will be similar to the above (except for that the dashed and the vector chords may coincide or can switch order) when $\varphi - \varepsilon_\varphi$, ε_φ and φ are positive and $\varepsilon_\varphi + \varphi < 180°$. For example $\varepsilon_\varphi = \min\left(\frac{\varphi}{2}, \frac{180° - \varphi}{2}\right)$ works. In this case, we can see that using the dashed or solid chord as the initial chord and directing the initial direction towards the point results in the vector chord as the second chord. Contradiction.

Note: The reason why the above proof doesn't work for 90° is that that is the case when the dashed and the vector lines coincide, or equivalently when the initial chord is also the second chord. In this case, this is the chord from which it never returns *directly* to the second chord, but it agrees with it. This is only true when $2\varphi = 180°$, so no other cases are affected.

E+5. Consider an acute triangle ABC. Let D, E and F be the feet of the altitudes through vertices A, B and C. Denote by A', B', C' the projections of A, B, C onto lines EF, FD, DE, respectively. Further, let H_D, H_E, H_F be the orthocenters of triangles $DB'C'$, $EC'A'$, $FA'B'$. Show that

$$H_DB^2 + H_EC^2 + H_FA^2 = H_DC^2 + H_EA^2 + H_FB^2.$$

Solution: First of all, let's create a diagram. Denote by H the orthocenter of triangle DEF. We are going to prove that $HH_D \perp BC$, $HH_E \perp CA$ and $HH_F \perp AB$.



By symmetry, it is enough to prove that $HH_D \perp BC$. Let M be the midpoint of side BC. We will show that the Miquel point of quadrilateral $B'C'EF$ (that is the second intersection of circles $(DB'C'), (DEF)$) is in fact, point M. It is well-known that the nine-point circle of a triangle passes through the feet of the altitudes and the midpoints of the sides. Thus, circle (DEF) goes through M. Denote by O the circumcenter of triangle ABC. We will first show that lines BB', CC' intersect at O. Again by symmetry, it suffices to show that B, B', O are collinear or its equivalent form: $\angle DBB' = \angle CBO$. Since quadrilateral $ACDF$ is cyclic, applying the inscribed angles theorem we get $\angle DBB' = 90^\circ - \angle B'DB = 90^\circ - \angle FDB = 90^\circ - \angle BAC = \dfrac{180^\circ - 2\angle BAC}{2} = \dfrac{180^\circ - \angle BOC}{2} = \angle CBO$ because $BO = CO$. (One can show the previous collinearity by noticing that in the cyclic quadrilateral $ACDF$ lines AC, DF are antiparallel and lines BO, BH' are isogonal - where H' denotes the orthocenter of triangle ABC - so $BH' \perp AC \implies BO \perp DF$). Therefore, both B' and C' lies on the circle with diameter DO, which passes through M as O lies on the bisector of segment BC.

It is well-known that in a triangle with orthocenter K, the Simson-line of a point P on the circumcircle halves segment PK. Since M lies on circles $(DB'C'), (DEF)$, the reflections of M onto lines DE, DF lie on HH_D. Also, it is known that in triangle DEF the inner angle bisectors are precisely the altitudes, while the external angle bisectors are precisely the sidelines of triangle ABC. Thus, M lies on the external angle bisector of $\angle EDF$, implying that the line connecting the reflections of M onto lines DE, DF is perpendicular to the bisector (since if we reflect any of the reflections on the bisector, we must get the other reflection). Indeed, $HH_D \perp BC$.

To finish the proof, we will use a well-know lemma stating that $PQ \perp AB \iff PA^2 - PB^2 =$

$QA^2 - QB^2$ (the proof of which is an easy application of the Pythagorean theorem). This means that

$$\begin{aligned}
HH_D \perp BC &\implies HB^2 - HC^2 = H_DB^2 - H_DB^2 \\
HH_E \perp CA &\implies HC^2 - HA^2 = H_EC^2 - H_EA^2 \\
HH_F \perp AB &\implies HA^2 - HB^2 = H_FA^2 - H_FB^2 \qquad +
\end{aligned}$$

$$0 = H_DB^2 - H_DC^2 + H_EC^2 - H_EA^2 + H_FA^2 - H_FB^2.$$

Therefore, $H_DB^2 + H_EC^2 + H_FA^2 = H_DC^2 + H_EA^2 + H_FB^2$.