E+ category

XVI. DÜRER COMPETITION

Final round
10-12 February, 2023

MATHEMATICS
TRADITIONAL ROUND PROBLEMS

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1. Show that for every positive real number r, the perimeter of a rectangle of size $1 \times r$ can be covered by pairwise non-intersecting circles of radius 1.
The circles can be tangent to each other.

2. a) Find all solutions to the equation $a^2 + b^2 + c^2 = abc$, where a, b and c are positive primes.
b) Prove that for every positive integer N there exist integers $a, b, c \geq N$ that satisfy the equation $a^2 + b^2 + c^2 = abc$.

3. At the end of the first quarter of the Greece-Egypt basketball game, the score was 26-25. During the first quarter, Áron wrote down the total number of points of the Greeks after every Greek basket, while Benedek wrote down the total number of points of the Egyptians after every Egyptian basket. In the break they noticed that there is no number that both of them wrote down. In how many ways could they have written down the numbers, if there were 21 baskets, and every basket was a 2-pointer or a 3-pointer?
Two options are different if at least one of them wrote down different numbers.

4. For an integer $n \geq 2$, the n-level pyramid consists of $1^2 + 2^2 + 3^2 + \cdots + n^2$ cubes of size $1\text{ m} \times 1\text{ m} \times 1\text{ m}$, and each cube is made of marble or sandstone. On the kth level, the cubes are arranged in a square grid of size of $(n+1-k) \times (n+1-k)$, and the centers of these grids fall on the same vertical line for all $1 \leq k \leq n$. In addition, the cube faces are parallel, hence each cube of the pyramid is either on the ground or stands on 4 other cubes. The top cube is made of marble, and to ensure the stability of the building, it is true for every marble cube that it is either on the ground or at least 3 out of the 4 cubes on which it stands are marble. What is the least possible number of marble cubes in the pyramid?

5. Let ABC be an acute triangle and let O be its circumcentre. Let O_A, O_B and O_C be the circumcentres of triangles BCO, CAO and ABO respectively. Prove that lines AO_A, BO_B and CO_C are concurrent.

6. Game: There are four piles of discs given, numbered from 1 to 4. Every turn the current player chooses integers m and n that satisfy $1 \leq m < n \leq 4$ and takes m discs from pile number n and distributes them into the piles $n-1, n-2, \ldots, n-m$ by adding one disc to every pile. The player that has no available moves loses.
Beat the organisers in this game twice in a row! Based on the number of discs in the piles you can decide if you would like to be the first or the second player.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. For a substantially different second solution or generalization, up to 2 extra points per problem might be awarded. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVI. Dürer Competition