E+1. Show that for every positive real number r, the perimeter of a rectangle of size $1 \times r$ can be covered by pairwise non-intersecting circles of radius 1.
The circles can be tangent to each other.

Solution: Consider the rectangle on a grid with vertices at points $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, r)$, $(\frac{1}{2}, r)$.

Let the centres of the unit circles be the points of the form $(-\frac{1}{2}, (2k-\frac{1}{2})\sqrt{3})$, $(\frac{1}{2}, (2k-\frac{1}{2})\sqrt{3})$ and $(0, (2k+\frac{1}{2})\sqrt{3})$ where k is an integer.

These circles cover the side of length r since they are tangent to each other at points of the form $(-\frac{1}{2}, k\sqrt{3})$, $(\frac{1}{2}, k\sqrt{3})$, $(2k\sqrt{3}, 0)$ where (k is an integer). This means that the circles cover the lines $y = -\frac{1}{2}$ and $y = \frac{1}{2}$ and the longer sides of the rectangle are on this line.

Now we only have to cover the sides of length 1. The $(-\frac{1}{2}, 0)(\frac{1}{2}, 0)$ side can be covered by the circle with centre $(0, \frac{1}{2}\sqrt{3})$. Let us call a positive number z good if there exists an integer k such that $|z - (2k+\frac{1}{2})\sqrt{3}| \leq \frac{\sqrt{3}}{2}$. This is equivalent to the circles covering the segment $(-\frac{1}{2}, z)(\frac{1}{2}, z)$. The good numbers are a union of closed intervals of length $\sqrt{3}$ and the non good numbers are a union of open intervals of length $\sqrt{3}$.

If r is good, then we are done, but if r is not good, then there exists a real number $0 < l < \sqrt{3}$ for which $|(r+l) - (2k+\frac{1}{2})\sqrt{3}| \leq \frac{\sqrt{3}}{2}$ because of the lengths of the intervals. If now we translate all of the cirlces by l parallel to the y axis in the positive direction, then we obtain a desired covering of the perimeter of the rectangle.

E+2. **a)** Find all solutions to the equation $a^2 + b^2 + c^2 = abc$, where a, b and c are positive primes.
b) Prove that for every positive integer N there exist integers $a, b, c \geq N$ that satisfy the equation $a^2 + b^2 + c^2 = abc$.

Solution: a) Consider the equation modulo 3. If $p \neq 3$ prime, then $p^2 \equiv 1 \pmod 3$. Therefore if none of the primes are 3, the left hand side is divisible by 3, it is a contradiction. So at least one of the primes is equal to 3, and the right hand side is divisible by 3. By the above, the left hand side can only be divisible by 3 if $a = b = c = 3$. These numbers satisfy the equation.

b) Suppose that (a, b, c) is a solution, with $a \leq b \leq c$. Then

$$f(x) = x^2 - bcx + b^2 + c^2$$

is a monic quadratic polynomial in $\mathbb{Z}[x]$ with an integer root. Then the other root a' is integer as well as the sum of the two roots is bc by the Viéte formulas. Hence (a', b, c) is a solution to the original equation as well. We also have $a' = \frac{b^2+c^2}{a} > a$, again by Viéte formulas, so we found another solution where we managed to increase the minimal value of (a, b, c) and leave the other two values the same. Starting from the solution $(3, 3, 3)$, and repeating the above $3N - 9$ times we get a solution with $a, b, c \geq N$.

The method used to solve part b) is called Viéte jumping.

E+3. At the end of the first quarter of the Greece-Egypt basketball game, the score was 26-25. During the first quarter, Áron wrote down the total number of points of the Greeks after every Greek basket, while Benedek wrote down the total number of points of the Egyptians after every Egyptian basket. In the break they noticed that there is no number that both of them wrote down. In how many ways could they have written down the numbers, if there were 21 baskets, and every basket was a 2-pointer or a 3-pointer?
Two options are different if at least one of them wrote down different numbers.

Solution: Write down the numbers from 1 to 26 and colour red the numbers, that Áron has written down and colour blue the numbers, that Benedek has written down. We know that there is no number, that was written down by both Áron and Benedek, so every number was coloured at most once. Then erase the numbers that haven't been coloured. We know that there were 21 buckets, so we have 5 erased numbers.

It is easy to see that the remaining numbers are coloured red and blue by alternating pattern. Let's take a look at the erased numbers. We know that 1 is erased, therefore 2 and 3 must be coloured, and also the numbers 25 and 26 are coloured. Two erased numbers must have a difference of at least 3, because otherwise there would be two adjacent numbers with the same colour with a difference of at least 4.

If we know the erased numbers we can reconstruct what numbers they have written down. We must colour the numbers alternating in such way, that the last number, 26 is red. If the difference of any two erased numbers is at least 3, then this will be possible.

Therefore, we need to select 4 numbers from 4 to 24, such that every two numbers has a difference at least 3. This is equivalent to the problem that we must select 4 numbers from 4 to 18, because if we selected the numbers $a < b < c < d$ for the second problem, then the $a, b+2, c+4, d+6$ selection will be good for the first problem, and vice versa. Hence the answer is $\binom{15}{4}$.

E+4. For an integer $n \geq 2$, the n-level pyramid consists of $1^2 + 2^2 + 3^2 + \cdots + n^2$ cubes of size $1 \text{ m} \times 1 \text{ m} \times 1 \text{ m}$, and each cube is made of marble or sandstone. On the kth level, the cubes are arranged in a square grid of size of $(n+1-k) \times (n+1-k)$, and the centers of these grids fall on the same vertical line for all $1 \leq k \leq n$. In addition, the cube faces are parallel, hence each cube of the pyramid is either on the ground or stands on 4 other cubes. The top cube is made of marble, and to ensure the stability of the building, it is true for every marble cube that it is either on the ground or at least 3 out of the 4 cubes on which it stands are marble. What is the least possible number of marble cubes in the pyramid?

Solution: Answer: There are at least n^2 marble cubes.

Construction for n^2 marble cubes: On all levels put marble cubes on the diagonal starting from the southwest corner and going to the northeast corner, also put marbles on the diagonal on the east side of this diagonal. Then we have n^2 marble cubes in total and all of the properties are satisfied.

Proof that we can't have less than n^2 marble cubes: We will prove that on the level with $k \times k$ cubes, we have at least $2k-1$ marble stones. We prove 2 propositions first:

Proposition 1: In an optimal construction, on each level, the set of marble cubes forms a connected set, where we consider 2 cubes adjacent if they share a side. We prove this by induction going from top to bottom. It is clear that the top level is connected. Using the fact that each marble stone has at least 3 marble cubes below it, we clearly get the induction step. Note that if we had a marble cube with no marble cube on top, we could replace it with sandstone and get another allowed construction with less marble.

Proposition 2: For the set of $k \times k$ cubes, there is at least 1 marble stone on each side. We prove this by induction again. For the top layer, this is clear. If we have a cube on the side from the $k-1$-th layer, then there are 2 cubes below it in the next level, being on the same side. At least one is marble, so we are done by induction.

Combining the two propositions, we know that on the level of $k \times k$ cubes, there is a path from south to north and a path from east to west using only marble stones. These both have at least $k-1$ south-north and $k-1$ east-west steps. We count the number of sides of the marble stones on this level that is not horizontal. From these steps, we get $2 \cdot 2(k-1)$ of them. Looking at the level from the four directions, we see k sides from each direction, each of them is disjoint from each other. Hence there are at least $4k + 2 \cdot 2(k-1)$ not horizontal sides on this level, corresponding to $2k-1$ marble cubes.

E+5. Let ABC be an acute triangle and let O be its circumcentre. Let O_A, O_B and O_C be the circumcentres of triangles BCO, CAO and ABO respectively. Prove that lines AO_A, BO_B and CO_C are concurrent.

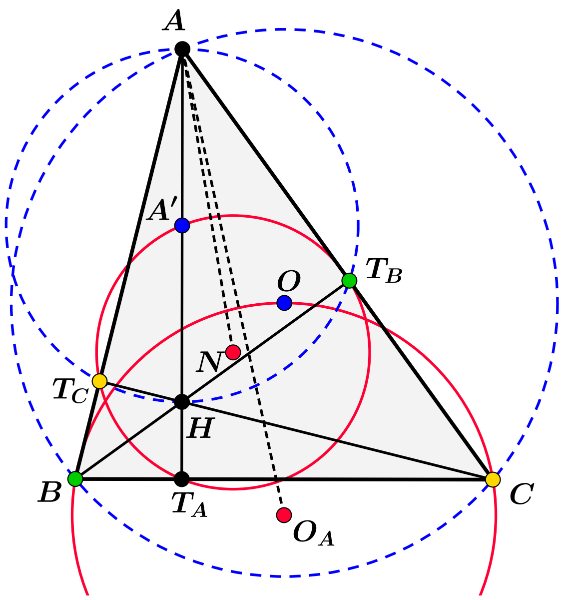
First solution:

Denote the feet of altitudes of triangle ABC by T_A, T_B and T_C, respectively. Let H be the orthocentre, A' be the midpoint of AH and N be the centre of the nine-point circle of $\triangle ABC$. Define f as the composition of the homothety with centre A and ratio $\frac{AT_B}{AB}$ and the reflection over the angle bisector of $\angle BAC$.

It is easy to see that f takes B to T_B. Since $\triangle ABC \sim \triangle AT_CT_B$ (as they have equal angles) and since f is a similarity, f takes C to T_C. Observe that T_C and T_B lie on the circle with diameter AH, thus the circumcentre of AT_CT_B triangle is A'. However, f is a similarity, so it takes the circumcentre of triangle ABC, O, to the circumcentre of triangle AT_CT_B, which is A'. So f takes O to A'.

By the above, f takes triangle BCO and its circumcircle to triangle T_BT_CA' and its circumcircle. Knowing that A' is on the nine-point circle of triangle ABC, circle BCO is sent to the nine-point circle of ABC by f. So f takes O_A to N.

As f takes A to itself, the image of line AO_A is AN. Since a homothety with center A maps AO_A to itself, we have that lines AO_A, AN are symmetric w.r.t. the angle bisector of $\angle BAC$. Thus, line AO_A passes through the isogonal conjugate of N. The argument above holds for the other two lines, BO_B and CO_C, hence they all pass through the isogonal conjugate of the nine-point circle.



Note: The point of concurrency has a name too, Kosnita point. (In the Encyclopedia of Triangle Centers it is denoted by X_{54}.) As we have seen from the proof above, it is the isogonal conjugate of the nine-point centre. The statement of the problem (also known as Kosnita's theorem) is due to the Romanian mathematician, Cezar Coşniţă.

Second solution:

First of all, note that the segment bisector of AO passes through points O_B, O_C as they are circumcentres of two circles having segment AO as a side. Thus, $AO \perp O_BO_C$. Similarly, $BO \perp O_CO_A$ and $CO \perp O_AO_B$. This means that $\angle O_CO_AB = 90° - \angle O_ABO$. However, OBO_AC is a kite (because $OB = OC$ and $O_AB = O_AC$ as they are radii), so $90° - \angle O_ABO = 90° - \angle O_ACO = \angle O_BO_AC$. Similarly, $\angle O_AO_BC = \angle O_CO_BA$ and $\angle O_BO_CA = \angle O_AO_CB$

Let $A' = AO_A \cap O_BO_C$, $B' = BO_B \cap O_CO_A$ and $C' = CO_C \cap O_AO_B$. Further, denote by α, β, γ the equal angles proven previously around points O_A, O_B, O_C, respectively (in the figure they are shown by different colours).

Using the sine rule first on triangles AO_BA', AO_CA', then on triangles AO_AO_B, AO_AO_C, we get

$$\frac{O_BA'}{O_CA'} = \frac{\frac{\sin\angle A'AO_B}{\sin\beta}\cdot AA'}{\frac{\sin\angle A'AO_C}{\sin\gamma}\cdot AA'} = \frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin\angle A'AO_B}{\sin\angle A'AO_C} = \frac{\sin\gamma}{\sin\beta}\cdot\frac{\frac{O_AO_B}{AO_A}\cdot\sin\angle AO_BO_A}{\frac{O_AO_C}{AO_A}\cdot\sin\angle AO_CO_A} = \frac{\sin\gamma}{\sin\beta}\cdot\frac{O_AO_B\cdot\sin\angle AO_BO_A}{O_AO_C\cdot\sin\angle AO_CO_A}.$$

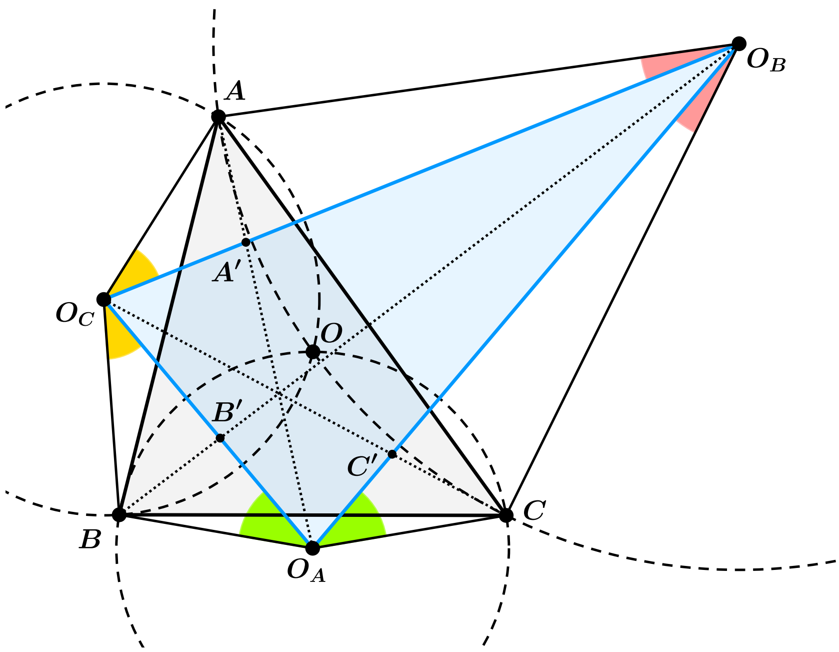
Similarly, we have

$$\frac{O_CB'}{O_AB'} = \frac{\sin\alpha}{\sin\gamma}\cdot\frac{O_BO_C\cdot\sin\angle BO_CO_B}{O_BO_A\cdot\sin\angle BO_AO_B} \quad\text{and}\quad \frac{O_AC'}{O_BC'} = \frac{\sin\beta}{\sin\alpha}\cdot\frac{O_CO_A\cdot\sin\angle CO_AO_C}{O_CO_B\cdot\sin\angle CO_BO_C}.$$

However, due to the equal angles we get $\angle AO_BO_A = \angle CO_BO_C$, $\angle BO_CO_B = \angle AO_CO_A$ and $\angle CO_AO_C = \angle BO_AO_B$, so multiplying the three equations from before yields

$$\frac{O_BA'}{O_CA'}\cdot\frac{O_CB'}{O_AB'}\cdot\frac{O_AC'}{O_BC'} = 1.$$

Therefore, due to the converse of Ceva's theorem we have proved the desired concurrency.



Note: The proof above basically shows Jakobi's theorem.

E+6. **Game:** There are four piles of discs given, numbered from 1 to 4. Every turn the current player chooses integers m and n that satisfy $1 \leq m < n \leq 4$ and takes m discs from pile number n and distributes them into the piles $n-1, n-2, \ldots, n-m$ by adding one disc to every pile. The player that has no available moves loses.
Beat the organisers in this game twice in a row! Based on the number of discs in the piles you can decide if you would like to be the first or the second player.

Solution: The second player has a winning strategy if and only if the number of discs in pile number 2 is even and in pile number 4 is congruent to 0 or 2 modulo 5 or if there are odd in pile number 2, and congruent to 1 modulo 5 in pile number 4. We only need to see that from a winning state we cannot step into a winning one but can always from a losing one.

If we are not in a winning state, then we can always get to a winning state as if the number of discs in pile 4 is congruent to 3 or 4 modulo 5, then by taking away 2 or 3 discs from this pile then the parity of the number of discs will change in pile 2, and we can choose which of $5k$ and $5k+1$ or $5k+1$ and $5k+2$ fits our strategy. If in pile 4 the number of discs is congruent to 0 or 2, then there is an odd number in pile 2, therefore by taking away one from there, we get to a winning state. And lastly if there is 1 modulo 5 in pile 4, then by taking away one disc from pile 4 we again get to a winning state as there has to be an even number of discs in pile 2.

From a winning state we cannot get to a winning state as if they take away from piles 2 or 3 then pile 4 does not change but the parity changes in pile 2. If they take away from pile 4 not resulting in 3 or 4 modulo 5, then if there was 0 modulo 5, then 3 has to be taken away, and that changes the parity of pile 2, and if there was 1 or 2 modulo 5, then by taking away only one disc, the parity of pile 2 does not change, and by taking away 2 the parity of pile 2 changes, so it is easy to see that we always get to a losing state.

The game is clearly finite, so we are done.