16th Dürer Competition

Problems, Solutions

C D E E+

Contents

Introduction – About the Dürer Competition

Hungary has a rich tradition of hosting mathematics contests catering to participants of all ages and qualifications, from primary schoolers to university students. While most contests are individual-based, the Dürer Competition stands out as a unique experience, where teams of three work together to solve problems, fostering the advantages of cooperative thinking. Notably, students tend to be happier and more at ease during this collaborative setting.

One of the fundamental goals of the Dürer Competition is to present intriguing problems that showcase the elegance of mathematics and the joy of thinking to a wide array of students. The organizers strive to include as many original problems as possible, usually introducing around 150 problems each year. Although some problems may not be entirely original, the majority of the more challenging ones are thoughtfully crafted by the team.

Mythology was the theme of the 16^{th} competition.

An essential aspect worth mentioning is the youthful and dynamic composition of the organizing team. Traditionally comprising young individuals, mostly university students with a passion for mathematics, many of them are former competitors who now contribute as organizers. This community, ranging from 30 to 70 people, plays a pivotal role in the competition's success. Some have been devoted organizers for as long as 16 years, actively participating even while balancing full-time jobs, while others step into significant responsibilities as first-year undergraduates. Remarkably, several organizers have also served as coordinators at prestigious international events such as EGMO 2022 and IMO 2022.

Maintaining this spirit, the Dürer Competition has been successfully organized for 16 years, consistently drawing a growing number of students and schools each year. In the academic year 2022-23, over 1000 Hungarian high school students participated in the mathematics categories, along with more than 650 students from primary schools, demonstrating the competition's increasing popularity and impact on the mathematics community.

This was the fourth year that we opened our two hardest categories for international competitors. The contest was held online with more than 150 students from 15 countries taking part, with new countries Israel, Ukraine, UK and Moldova. For the first time this year we were able to invite international teams to the finals in Miskolc, the Israeli and Ukrainian team wrote the competition in English and were able to join the other activities over the weekend.

Primary school students can take part in our competition in the following two categories:

- **Category A** is open to 5^{th} and 6^{th} grade students.
- **Category B** is open to 7^{th} and 8^{th} grade students.

In these two categories the contest is regional: the first round is organised in 6 cities in north-eastern Hungary, but is open to anyone provided that they travel to one of the locations.

Four categories are available for high school students:

- **Category C** is open to 9^{th} and 10^{th} graders who have never previously qualified for the final of any national math contest.
- **Category D** is open to 9^{th} to 12^{th} graders who are a bit more experienced, but do not come from a school that is outstanding in handling mathematical talents.
- **Category E** is open to 9^{th} to 12^{th} graders who already have good results from other contests, or come from a school outstanding in maths.
- **Category E$^+$** is designed for competitors who actively take part in olympiad training. In this category, most teams include some student who has taken part at an international olympiad (IMO, MEMO, EGMO, RMM, IMSC), or is about to qualify for one in the same academic year.

We also organise the contest in physics (*category F* and *F$^+$*) and chemistry (*categories K, K$^+$* and *L*), but these are omitted from this booklet.

High school students participating in categories C, D, and E face an initial **online relay round** comprising 9 problems. Each question requires an integer answer between 0 and 9999. Teams start with the first question and have three attempts to submit an answer. A correct response grants them a set number of points, allowing them to advance to the next question. However, each incorrect attempt reduces the potential score by 1. If a team fails to answer a question correctly after three attempts, they must move on to the next question without scoring. Additionally, an **online game** is included in this round.

In the second round for categories E and E$^+$, a traditional **olympiad-style contest** takes place, where teams must provide detailed proofs for 5 problems within a 3-hour timeframe. For categories C and D, the second round involves four short-answer and three olympiad-style questions, also lasting 3 hours. Notably, proof is not required for the short-answer questions, making the round more accessible to students less accustomed to competing. This round is organized in approximately 25 locations across the country.

The final round is hosted in Miskolc. For high schoolers (categories C, D, E, E$^+$, F, F$^+$ K, K$^+$, L), it spans a weekend in early February, from Thursday to Sunday. Friday serves as the first competition day, where students tackle five **olympiad-style problems** and participate in a **game**. If a team believes they have discovered a winning strategy for the game, they can challenge the organizers. By defeating the organizers twice in a row, the team earns the maximum score for the problem. However, if they lose, they still have two more attempts to challenge for a partial score. Saturday holds a **relay round** comprising 16 questions, following similar rules to the online round. The rankings are determined based on a combined score from both competition days.

Throughout the final weekend, students and teachers have the opportunity to engage in various educational and recreational activities, including lectures, games, and discussions about universities.

The primary school student competition follows a separate but similar format. Their first round consists of a **relay round** with 15 questions, and the final round mirrors the structure of the high school students' final round.

The competition is expanded year after year which contributes to the establishment of the Dürer Competition as a well-known and renowned contest which is one of our main objectives.

A subset of the organizers of the 16^{th} Dürer Competition

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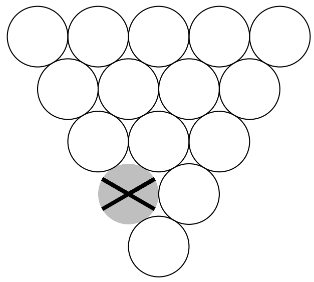
1 Problems

1.1 Online round

1.1.1 Category C

1. In Dionysus' garden, one of the grapes in a bunch of grapes got moldy (it is marked grey in the figure). Every day, the grapes that are in contact with at least one already moldy grape, also get moldy. In how many days will the entire bunch become moldy? (The figure shows the entire bunch.)

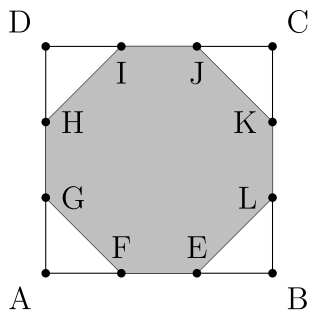


(Solution)

2. As it is commonly known, the Greeks got some of their soldiers into Troy in a huge wooden horse, disguised as a gift. The wooden horse had four legs but the Troyans let 25 times as many legs into their city as they thought they did by accepting the gift. How many Greek soldiers hid in the Troyan horse?

(Solution)

3. Morpheus, the god of dreams, has a square-shaped garden. Each side of the garden is 30 m long (the garden is the $ABCD$ square on the figure). He wants to plant poppy in the octagon defined by the trisecting points (this is the $EFGHIJKL$ octagon on the figure). What is the area of this octagon in square meters?



(Solution)

4. Zeus draws a large regular polygon on a huge piece of paper for the mortals. The polygon is too big for them to walk around it but they could measure it that its angles are 160°. How many sides does the polygon have?

(Solution)

5. A flea is sitting on the 0 point of the number line. In the first step, the flea jumps one unit either right or left, in the second step, it jumps two units either right or left, and so on, in any nth step it jumps n units either left or right. Which is the smallest possible natural n, for which the flea can reach point 7 after the nth step?

(Solution)

6. Sasha, the cat starts a special diet because he became allergic to pork. He will eat three different meals from now on: chicken, beef, and fish. According to his diet, he cannot eat the same meal on two consecutive days and if he ate fish then he cannot eat it in the following two days. In how many different ways can he plan the first week of his diet if he wants to eat fish on the third day and in the two days before the diet, he only ate pork?

(Solution)

7. We'd like to fill a 4×4 table with 0s and 2s, such that every cell has exactly one number written in it. How many different ways are there to fill it, in which in every row and column there's exactly one 0, and there's both at least one row and at least one column, where the numbers give 2022 when read from left to right, and from top to bottom?

(Solution)

8. Ares has centaurs, humans, and pegasusses fighting in his army. Everyone has 1 head, centaurs and pegasusses have 4 legs, while humans have 2, and centaurs and humans have 2 arms, while pegasusses have 0. Upon inspecting his army Ares sees 193 heads, 666 legs, and 244 arms. How many centaurs are there in Ares' army?

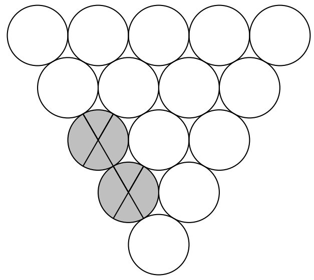
(Solution)

9. The gate of the underworld is guarded by Hades' three-headed dog, Cerberus. Orpheus can only enter if he can put all three of Cerberus' heads to sleep at the same time, so he plays his harp until all three heads fall asleep. After he started playing, the first head immediately fell asleep, then it takes turns in sleeping for 2 minutes and then being awake for 9 minutes. The second head fell asleep after 2 minutes, then it takes turns in sleeping for 2 minutes and then being awake for 4 minutes. The third head fell asleep after 4 minutes, then it takes turns in sleeping for 2 minutes and then being awake for 6 minutes. At least how many minutes does Orpheus need to play for in order to be able to enter the underworld, if it takes 10 seconds for him to walk past Cerberus?

(Solution)

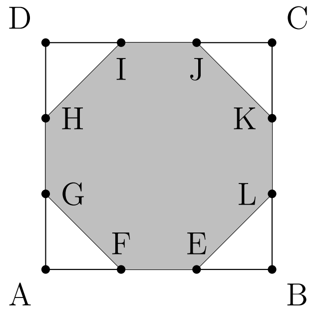
1.1.2 Category D

1. In Dionysus' garden, two of the grapes in a bunch of grapes got moldy (they are marked grey in the figure). Every day, the grapes that are in contact with at least two already moldy grapes, also get moldy. In how many days will the entire bunch become moldy? (The figure shows the entire bunch.)



(Solution)

2. Morpheus, the god of dreams, has a square-shaped garden. Each side of the garden is 30 m long (the garden is the $ABCD$ square on the figure). He wants to plant poppy in the octagon defined by the trisecting points (this is the $EFGHIJKL$ octagon on the figure). What is the area of this octagon in square meters?



(Solution)

3. Jóska, Miki, and Viktor went to the Hungry Trojan Horse Restaurant. Jóska ordered 2 plates of gyros and 3 baklavas for 3660 forints while Miki paid 5330 forints for 3 plates of gyros and 4 baklavas. How much did Viktor pay if he ordered 5 plates of gyros and 6 baklavas?

(Solution)

4. A flea sits at the point 0 on the number line. In the first step, the flea jumps one unit to the left or to the right, in the second step it jumps two to the left or to the right, and so on, in every step jumping as much as the number of the given step. What's the first step after which it can land on the point 12?

(Solution)

5. Sasha, the cat starts a special diet because he became allergic to pork. He will eat three different meals from now on: chicken, beef, and fish. According to his diet, he cannot eat the same meal on two consecutive days and if he ate fish then he cannot eat it in the following two days. In how many different ways can he plan the first week of his diet if he wants to eat fish on the third day and in the two days before the diet, he only ate pork?

(Solution)

6. We'd like to fill a 4×4 table with 0s and 2s, such that every cell has exactly one number written in it. How many different ways are there to fill it, in which in every row and column there's exactly one 0, and there's both at least one row and at least one column, where the numbers give 2022 when read from left to right, and from top to bottom?

(Solution)

7. Thor, Freya, Idun, and Loki are all taking an exam but not on the same day. All exercises in the exam are worth 1 point. Thor solved half of the exercises correctly, Freya solved exactly 40 exercises correctly and Idun solved the first quarter of the exercises. Loki takes his exam today but since he did not study for it at all, he asks for help from his three friends. Thor, Freya, and Idun tell him the solutions for all the exercises that they could solve. So, Loki knows the answer to a question in the exam if and only if at least one of his friends could solve it. How many exercises are there at most that Loki could not solve if his friends earned 130 points in their exams in total?

(Solution)

8. Ares has centaurs, humans, and pegasusses fighting in his army. Everyone has 1 head, centaurs and pegasusses have 4 legs, while humans have 2, and centaurs and humans have 2 arms, while pegasusses have 0. Upon inspecting his army Ares sees 193 heads, 666 legs, and 244 arms. How many centaurs are there in Ares' army?

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(Solution)

1.1.3 Category E

1. Zeus draws a large regular polygon on a huge piece of paper for the mortals. The polygon is too big for them to walk around it but they could measure it that its angles are 160°. How many sides does the polygon have?

(Solution)

2. Jóska, Miki, and Viktor went to the Hungry Trojan Horse Restaurant. Jóska ordered 2 plates of gyros and 3 baklavas for 3660 forints while Miki paid 5330 forints for 3 plates of gyros and 4 baklavas. How much did Viktor pay if he ordered 5 plates of gyros and 6 baklavas?

(Solution)

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(Solution)

4. Sasha, the cat starts a special diet because he became allergic to pork. He will eat three different meals from now on: chicken, beef, and fish. According to his diet, he cannot eat the same meal on two consecutive days and if he ate fish then he cannot eat it in the following two days. In how many different ways can he plan the first week of his diet if he wants to eat fish on the third day and in the two days before the diet, he only ate pork?

(Solution)

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(Solution)

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(Solution)

7. We call a positive integer *halved* if it is a double-digit number and it is divisible by the product of its two digits. What is the sum of all halved numbers?

Note: no number is divisible by 0.

(Solution)

8. In Persephone's garden a house, a bush, and a tree are located in the following way: The house is 9 meters west from the tree, while the bush is 40 meters north from the house. Persephone made a circular flowerbed centered at the bush, and its circumference is the same distance from the tree, as the tree is from the house. She would like to build a straight path starting from the house, which is tangential to the circumference of the flowerbed. She chooses one of the two possible tangent lines and builds the path there. (The two end points of the path are at the house, and on the circumference of the flowerbed.) How long will the path be?

(Solution)

9. The gate of the underworld is guarded by Hades' three-headed dog, Cerberus. Orpheus can only enter if he can put all three of Cerberus' heads to sleep at the same time, so he plays his harp until all three heads fall asleep. After he started playing, the first head immediately fell asleep, then it takes turns in sleeping for 2 minutes and then being awake for 9 minutes. The second head fell asleep after 1 minute, then it takes turns in sleeping for 2 minutes and then being awake for 4 minutes. The third head fell asleep after 3 minutes, then it takes turns in sleeping for 2 minutes and then being awake for 5 minutes. At least how many minutes does Orpheus need to play for in order to be able to enter the underworld, if it takes 10 seconds for him to walk past Cerberus?

(Solution)

1.2 Regional round

1.2.1 Category C

1. Let's call a date *fabulous*, if the sum of the digits of the month and the day equals the number of the month. Today's date (18th November, so 11.18.) is fabulous because $1+1+1+8=11$. List all the fabulous dates in a year! (There are more places on the answer sheet than fabulous dates.)

(Proposed by Kartal Nagy)

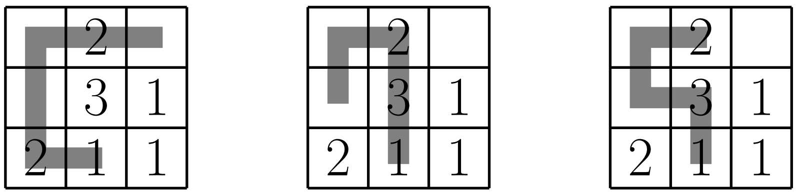
(Solution)

2. Theseus is wandering in a 4×4 square grid-shaped labyrinth in a way that he always steps on a square that he has not stepped on yet and that is adjacent with a side to the one that he is standing on at the moment. After he finished his walk, he takes the map of the labyrinth and writes on some of the squares the number of its side neighbours that he has been to during his walk. He could have been on squares that he has not written a number on as well. What could have been Theseus' route according to his notes? Draw all possibilities on the answer sheet! (There are more places on the answer sheet than correct routes.)

2			1
1		3	
	1	1	

In the figures below, you can see an example for the three possible routes in the case of a 3×3 map.

	2	
	3	1
2	1	1



(Proposed by Csenge Miklós)

(Solution)

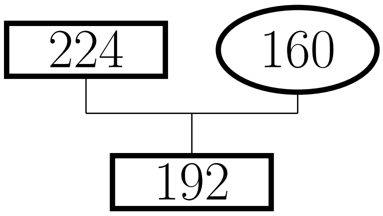
3. An ancient calculator has the following 4 buttons: $+3$, -3, $\times 3$, $\div 3$. In the beginning, the calculator displays the number 21. When we press a button, the calculator executes the selected process with the number on the display and then displays the result. Anna pushed the four buttons in such order that she got the biggest possible result starting from 21 while Béla pushed the four buttons in such order that he got the smallest possible result starting from 21. They both pushed each of the four buttons exactly once. What numbers did Anna and Béla get?
For example, if we push the buttons $+3$, -3, $\div 3$, $\times 3$ in this order, the result is $(((21 + 3) - 3) \div 3) \times 3 = 21$.

(Solution)

4. The great-great-grandparents of the young Theodorus were among the firsts who settled down in a Greek island, Samos. Some of them were giants, the others were dwarfs. Among the great-great-grandparents, the dwarfs were 128 pechys tall while the giants were 256 pechys tall. Everyone's height equals the average of their father's and mother's height. Someone is called a dwarf, if all of their ancestors were dwarfs and someone is called a giant, if all of their ancestors were giants. We have the following information about the ancestors of Theodorus:

- Theodorus has 2 parents, 4 grandparents, 8 great-grandparents and 16 great-great-grandparents.
- Theodorus is 168 pechys tall.
- Theodorus has one more maternal giant great-great-grandparents than paternal giant great-great-grandparents.
- Theodorus' father and paternal grandfather are the same height.
- One of Theodorus' grandmothers is 128 pechys tall.
- One of Theodorus' great-grandmothers is a giant.
- Among the maternal great-great-grandparents of Theodorus, there are more female giants than male ones.
- With one exception, all of Theodorus' father's male ancestors are dwarfs.

Write all of the heights of the family members in pechys in the family tree on the answer sheet!
In the family tree, the rectangles mark the men, the ovals mark the women. In the example below, there is the family tree of a family with three members where the father is 224 *pechys, the mother is* 160 *and their son is* 192 *pechys tall.*



(Proposed by Csongor Beke)

(Solution)

5. A group of students play the following game: they are counting one by one from 00 to 99 taking turns, but instead of every number they only say one of its digits. (The numbers in order are 00, 01, 02, ..., meaning that one-digit numbers are regarded as two-digit numbers with a first digit 0.) One way of starting the counting could be for example 0, 1, 2, 0, 4, 0, 6, 7, 8, 9, 1, 1, 2, 1, 1, 5, 6, 1, 8, 1, 0, 2 etc. When they reach 99, the counting restarts from 00. At some point Csongor enters the room and after listening to the counting for a while, he discovers that he is able to tell what number the counting is at. How many digits has Csongor heard at least?

(Proposed by Benedek Kovács)

(Solution)

6. Orsi fills a 5×5 table with single-digit numbers in such a way that she does not write any zeros in the squares of the leftmost column and the top row. After that, Csenge reads the digits in every row from left to right, gets 5 five-digit numbers, and calculates their sum. Similarly, Anett reads the digits in every column from top to bottom, gets 5 five-digit numbers, and calculates their sum. What can be the biggest difference between Csenge's and Anett's numbers?

(Proposed by András Imolay)

(Solution)

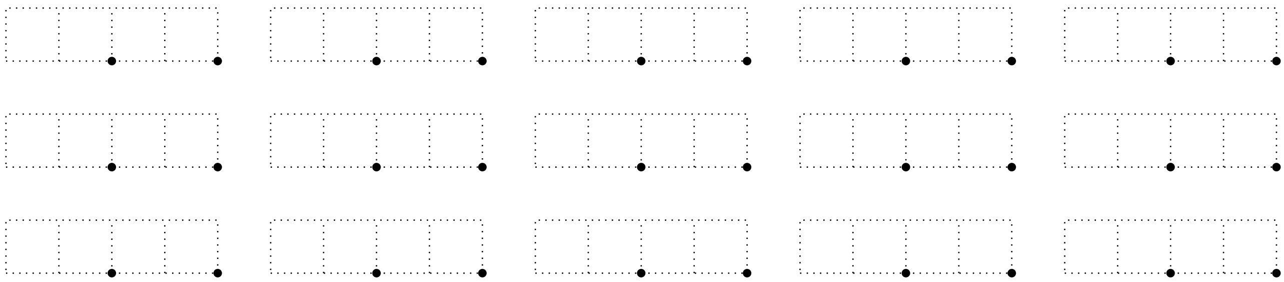
7. Let $ABCDE$ be a regular pentagon. We drew two circles around A and B with radius AB. Let F mark the intersection of the two circles that is inside the pentagon. Let G mark the intersection of lines EF and AD. What is the degree measure of angle AGE?

(Proposed by József Osztényi)

(Solution)

Answer sheet - Category C

1. Fabolous dates



2. Labyrinth

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

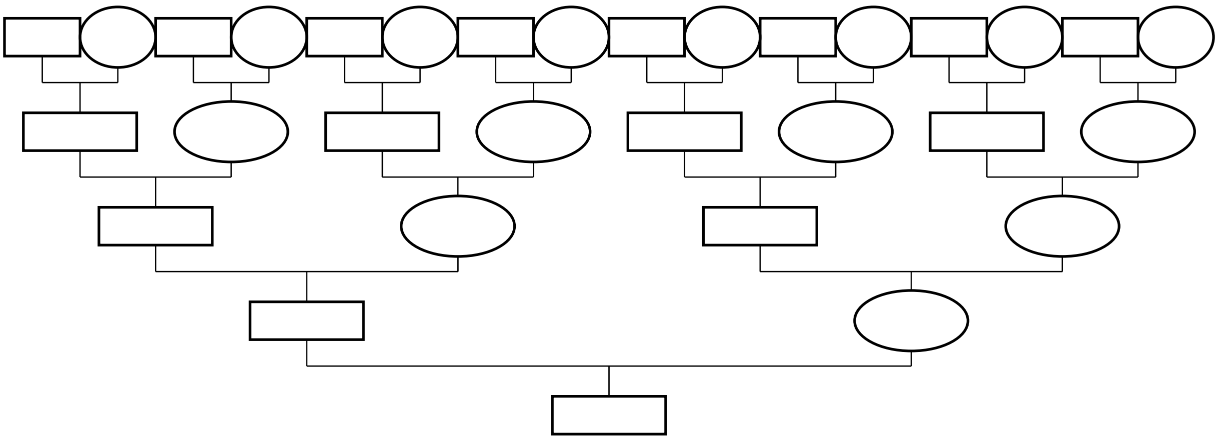
2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

3. Ancient calculator

Anna's number: .... Béla's number: ....

4. Family tree



1.2.2 Category D

1. An ancient calculator has the following 4 buttons: $+3$, -3, $\times 3$, $\div 3$. In the beginning, the calculator displays the number 21. When we press a button, the calculator executes the selected process with the number on the display and then displays the result. Anna pushed the four buttons in such order that she got the biggest possible result starting from 21 while Béla pushed the four buttons in such order that he got the smallest possible result starting from 21. They both pushed each of the four buttons exactly once. What numbers did Anna and Béla get?
For example, if we push the buttons $+3$, -3, $\div 3$, $\times 3$ in this order, the result is $(((21+3)-3)\div 3)\times 3 = 21$.

(Solution)

2. Theseus is wandering in a 4×4 square grid-shaped labyrinth in a way that he always steps on a square that he has not stepped on yet and that is adjacent with a side to the one that he is standing on at the moment. After he finished his walk, he takes the map of the labyrinth and writes on some of the squares the number of its side neighbours that he has been to during his walk. He could have been on squares that he has not written a number on as well. What could have been Theseus' route according to his notes? Draw all possibilities on the answer sheet! (There are more places on the answer sheet than correct routes.)

2			1
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In the figures below, you can see an example for the three possible routes in the case of a 3×3 map.

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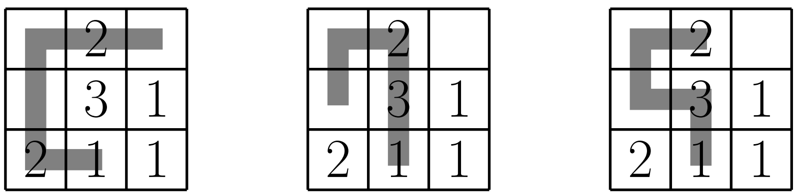


(Proposed by Csenge Miklós)

(Solution)

3. The organizers of the Dürer Competition travel to 6 different locations for the regional round of the competition. 62 teams participate in the competition and every team needs 6 sheets of paper. The organizers store paper in packages and there are 50 sheets of paper in each package. They can only deliver unopened packages to the different locations but they do not deliver more packages than needed. How many packages of paper will the organizers need at most for the regional round? Find all the possible cases for the distribution of the teams among the different locations when the most packages are needed! For each case, write the number of teams at each location in a descending sequence. (There are more spaces on the answer sheet than correct solutions.)
For example, if in a case, there are 11, 2, 25, 9, 11 *and* 4 *teams competing at the different locations, then write the series* $25, 11, 11, 9, 4, 2$ *on the answer sheet.*

(Proposed by Kristóf Zólomy)

(Solution)

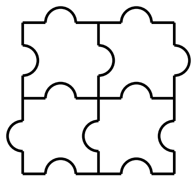
4. After Theseus' last adventure he finds himself on a 5×5 sized labyrinth's top left field. He wants to get to the bottom right field, as in every move he can only step on an adjacent field in which the number is a divisor of or a multiplication of the current number. Find all the paths where Theseus does not step twice on any of the fields. (On the answer sheet there are more spaces than possible paths.)

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

(Proposed by András Imolay)

(Solution)

5. We would like to make a table out of puzzle pieces so that no two overlap each other and there is no gap between any two pieces. How many ways can we do this, if we can only use pieces on which two adjacent sides have holes and on the other two sides there is a cam, if the table size is
a) 1×4?
b) 4×5?
All pieces are the same and only one side of each piece can look up. Two solutions are the same if they cover each other without rotating. On the edges holes or cams may remain. The only condition is that two pieces with the same edge on the common edge one should have a cam, the other should have a hole. The figure shows an example of a 2×2 sized table.



(Proposed by Eszter Szabó and Zoltán Kalocsai)

(Solution)

6. Let $ABCDE$ be a regular pentagon. We drew two circles around A and B with radius AB. Let F mark the intersection of the two circles that is inside the pentagon. Let G mark the intersection of lines EF and AD. What is the degree measure of angle AGE?

(Proposed by József Osztényi)

(Solution)

7. In a Greek village of 100 inhabitants in the beginning there lived 12 Olympians and 88 humans, they were the first generation. The Olympians are 100% gods while humans are 0% gods. In each generation they formed 50 couples and each couple had 2 children, who form the next generation (also consisting of 100 members). From the second generation onwards, every person's percentage of godness is the average of the percentages of his/her parents' percentages. (For example the children of 25% and 12.5% gods are 18.75% gods.)
a) Which is the earliest generation in which it is possible that there are equally many 100% gods as 0% gods?
b) At most how many members of the fifth generation are at least 25% gods?

(Proposed by Csongor Beke)

(Solution)

Answer sheet - Category D

1. Ancient calculator

Anna's number: .... Béla's number: ....

2. Labyrinth

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

2			1
1		3	
	1	1	

3. Competition

Number of packages needed: ....

Locations	Number of teams								
1st location									
2nd location									
3rd location									
4th location									
5th location									
6th location									

4. Divisor-multiplier path

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

1.2.3 Category E

1. A group of students play the following game: they are counting one by one from 00 to 99 taking turns, but instead of every number they only say one of its digits. (The numbers in order are 00, 01, 02, ..., meaning that one-digit numbers are regarded as two-digit numbers with a first digit 0.) One way of starting the counting could be for example 0, 1, 2, 0, 4, 0, 6, 7, 8, 9, 1, 1, 2, 1, 1, 5, 6, 1, 8, 1, 0, 2 etc. When they reach 99, the counting restarts from 00. At some point Csongor enters the room and after listening to the counting for a while, he discovers that he is able to tell what number the counting is at. How many digits has Csongor heard at least?

(Proposed by Benedek Kovács)

(Solution)

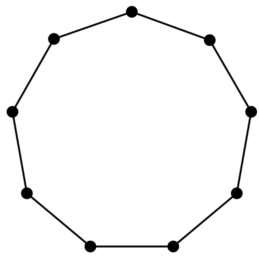
2. In a Greek village of 100 inhabitants in the beginning there lived 12 Olympians and 88 humans, they were the first generation. The Olympians are 100% gods while humans are 0% gods. In each generation they formed 50 couples and each couple had 2 children, who form the next generation (also consisting of 100 members). From the second generation onwards, every person's percentage of godness is the average of the percentages of his/her parents' percentages. (For example the children of 25% and 12.5% gods are 18.75% gods.)
a) Which is the earliest generation in which it is possible that there are equally many 100% gods as 0% gods?
b) At most how many members of the fifth generation are at least 25% gods?

(Proposed by Csongor Beke)

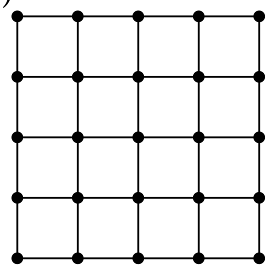
(Solution)

3. Pythagoras drew some points in the plane and connected some of these with segments. Now Tortillagoras wants to write a positive integer next to every point, such that one number divides another number if and only if these numbers are written next to points that Pythagoras has connected. Can Tortillagoras do this for the following drawings?

a)



b)



In part b), vertices in the same row or column but not adjacent are not connected.

(Proposed by András Imolay)

(Solution)

4. Let k be a circle with diameter AB and centre O. Let C be an arbitrary point on the circle different from A and B. Let D be the point for which O, B, D and C (in this order) are the four vertices of a parallelogram. Let E be the intersection of the line BD and the circle k, and let F be the orthocenter of the triangle OAC. Prove that the points O, D, E, C, F lie on a circle.

(Proposed by Anett Kocsis)

(Solution)

5. Let $n \geq 3$ be an integer. Timi thought of n different real numbers and then wrote down the numbers which she could produce as the product of two different numbers she had in mind. At most how many different positive prime numbers did she write down (depending on n)?

(Proposed by Csongor Beke)

(Solution)

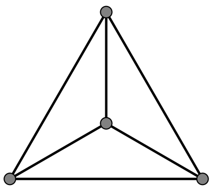
1.2.4 Category E^{+}

1. Find all positive integers n such that $\lfloor\sqrt{n}\rfloor + \left\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\right\rfloor > 2\sqrt{n}$.
If k is a real number, then $\lfloor k\rfloor$ means the floor of k, this is the greatest integer less than or equal to k.

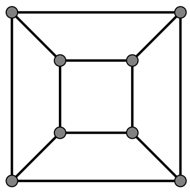
(Proposed by Erik Füredi)

(Solution)

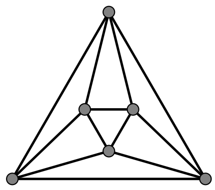
2. We say that a graph G is *divisive*, if we can write a positive integer on each of its vertices such that all the integers are distinct, and any two of these integers divide each other if and only if there is an edge running between them in G. Which Platonic solids form a divisive graph?



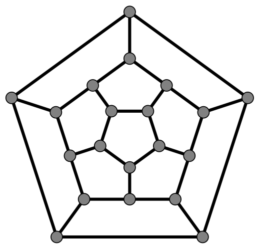
tetrahedron



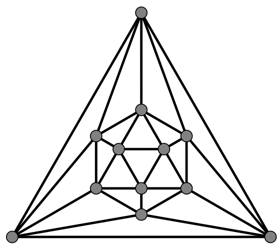
hexahedron



octahedron



dodecahedron



icosahedron

(Proposed by András Imolay)

(Solution)

3. Let $n \geq 3$ be an integer and A be a subset of the real numbers of size n. Denote by B the set of real numbers that are of the form $x \cdot y$, where $x, y \in A$ and $x \neq y$. At most how many distinct positive primes could B contain (depending on n)?

(Proposed by Csongor Beke)

(Solution)

4. We are given an angle $0° < \varphi \leq 180°$ and a circular disc. An ant begins its journey from an interior point of the disc, travelling in a straight line in a certain direction. When it reaches the edge of the disc, it does the following: it turns clockwise by the angle φ, and if its new direction does not point towards the interior of the disc, it turns by the angle φ again, and repeats this until it faces the interior. Then it continues its journey in this new direction and turns as before every time when it reaches the edge. For what values of φ is it true that for any starting point and initial direction the ant eventually returns to its starting position?

(Proposed by Benedek Váli)

(Solution)

5. Consider an acute triangle ABC. Let D, E and F be the feet of the altitudes through vertices A, B and C. Denote by A', B', C' the projections of A, B, C onto lines EF, FD, DE, respectively. Further, let H_D, H_E, H_F be the orthocenters of triangles $DB'C'$, $EC'A'$, $FA'B'$. Show that

$$H_DB^2 + H_EC^2 + H_FA^2 = H_DC^2 + H_EA^2 + H_FB^2.$$

(Proposed by Áron Bán-Szabó)

(Solution)

1.3 Final round – day 1

1.3.1 Category C

1. a) Six demigods took part in a theater festival. They played several shows every night and everyone sat in on one of the performances. n the evenings, all performances took place at the same time. Is it possible that any two of the demigods watched a show together exactly once, if every night was there an even number of performances that at least one demigod saw?
b) Is it possible with seven demigods?
c) Is it possible with eight demigods?

(Proposed by Anett Kocsis)

(Solution)

2. Kartal fills a 5×5 table with positive integers. Then Benedek writes at the end of each row and coloumn the sum of the numbers there. Finally Andris writes, how many of the numbers written by Benedek are even. What number Andris could write down?

(Proposed by András Imolay)

(Solution)

3. ABC is an isosceles triangle. The base BC is 1 cm long, and legs AB and AC are 2 cm long. Let the midpoint of AB be F, and the midpoint of AC be G. Additionally, k is a circle, that is tangent to AB and AC, and it's points of tangency are F and G accordingly. Prove, that the intersection of CF and BG falls on the circle k.

(Proposed by András Imolay)

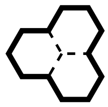
(Solution)

4. When Andris entered the room, there were the numbers 3 and 24 on the board. In one step, if there are the (not necessarily different) numbers k and n on the board already, then Andris can write the number $kn + k + n$ on the board, too. **a)** Can Andris write the number 9999999 on the board after a few moves?
b) What if he wants to get 99999999?
c) And what about 48999999?

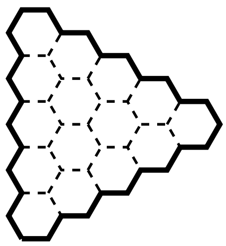
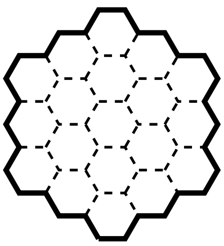
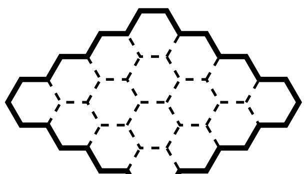
(Proposed by András Imolay)

(Solution)

5. Jóska wants to turn the shapes below entirely black. In one step, he can place the shape on the right, composed of three hexagons, anywhere and the fields under it will change colour (either from white to black or from black to white). Which shapes can Jóska turn completely black in this way?



We can rotate the shape on the right but it always needs to cover exactly three whole hexagons when Jóska puts it down.

a)  **b)**  **c)** 

(Proposed by József Osztényi)

(Solution)

6. Game: We have a 3×3 table. In each step, the player in turn writes 1, 2, or 3 in an empty cell in a way that there should be no row or column in which there are more than one of the same number. If all the 9 cells of the table are filled, the first player wins but if that becomes impossible at one point, the second player wins.
Beat the organisers in this game two times in a row! You can decide if you want to be the first or the second player.

(Proposed by Kartal Nagy)

(Solution)

1.3.2 Category D

1. Give all integers greater than 1 that has exactly one less divisor than it's other (than itself) divisors have in total!
For example number 6 *with four divisors is such a number, because number* 2 *and number* 3 *have two, whilst number* 1 *has one divisor.*

(Proposed by Kristóf Zólomy)

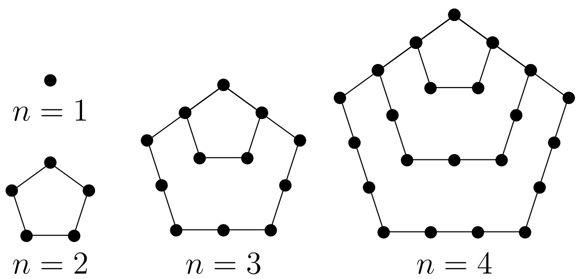
(Solution)

2. Let $ABCD$ be a isosceles trapezoid. Base AD is 11 cm long while the other three sides are each 5 cm long. We draw the line that is perpendicular to BD and contains C and the line that is perpendicular to AC and contains B. We mark the intersection of these two lines with E. What is the distance between point E and line AD?

(Proposed by Dániel Hegedűs)

(Solution)

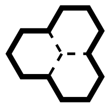
3. Prove that there are infinitely many *pentagonal numbers* that can be written up as the sum of two other (not necessarily different) pentagonal numbers.
The first pentagonal number is 1*. The nth pentagonal number (for $n \geq 2$) is the number of points on the graph we obtain from adding an 'outer pentagon' on the previous graph, with points on it such that there are n equidistant points on each outer edge, as shown on the figure. The first four pentagonal numbers are* 1*,* 5*,* 12 *and* 22*.*



(Proposed by József Osztényi)

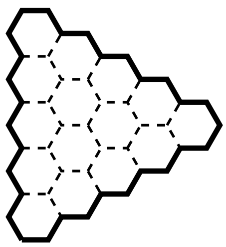
(Solution)

4. Jóska wants to turn the shapes below entirely black. In one step, he can place the shape on the right, composed of three hexagons, anywhere and the fields under it will change colour (either from white to black or from black to white). Which shapes can Jóska turn completely black in this way?

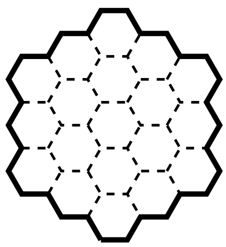


We can rotate the shape on the right but it always needs to cover exactly three whole hexagons when Jóska puts it down.

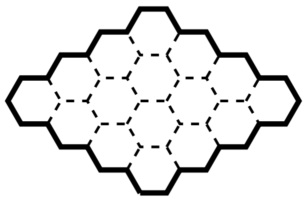
a)



b)



c)



(Proposed by József Osztényi)

(Solution)

5. Hanga, the ant stands on one of a square grid's grid points, in which the length of the sides of the unit squares are 1 cm, and looks towards one of the grid lines. She turns 15° left, then she starts walking straight in the direction she is looking at. Then, whenever she crosses a grid line (or grid point), she turns 30° left, then continues her way in that direction. What distance will she be from her starting point at the moment of the tenth 30° turn?

(Proposed by Anita Páhán)

(Solution)

6. Game: We have two piles of pucks. In each step, the player in turn adds at least one puck to one of the piles and takes away two times as many pucks from the other pile. The one who cannot make a new step, loses.

Beat the organisers in this game two times in a row! Knowing the number of pucks in each pile at the start, you can decide if you would rather be the first or the second player.

(Proposed by András Imolay)

(Solution)

1.3.3 Category E

1. ABC is an isosceles triangle. The base BC is 1 cm long, and legs AB and AC are 2 cm long. Let the midpoint of AB be F, and the midpoint of AC be G. Additionally, k is a circle, that is tangent to AB and AC, and it's points of tangency are F and G accordingly. Prove, that the intersection of CF and BG falls on the circle k.

(Proposed by András Imolay)

(Solution)

2. When Andris entered the room, there were the numbers 3 and 24 on the board. In one step, if there are the (not necessarily different) numbers k and n on the board already, then Andris can write the number $kn + k + n$ on the board, too. **a)** Can Andris write the number 9999999 on the board after a few moves?
b) What if he wants to get 99999999?
c) And what about 48999999?

(Proposed by András Imolay)

(Solution)

3. **a)** Four merchants want to travel from Athens to Rome by cart. On the same day, but different times they leave Athens and arrive on another day to Rome, but in reverse order. Every day, when the evening comes, each merchant enters the next inn on the way. When some merchants sleep in the same inn at night, then on the following day at dawn they leave in reverse order of arrival, because they can only park this way on the narrow streets next to the inns. They cannot overtake each other, their order only changes after a night spent together in the same inn. Eventually each merchant arrives in Rome while they sleep with every other merchant in the same inn exactly once. Is it possible, that the number of the inns they sleep in is even every night?
b) Is it possible if there are 8 merchants instead of 4 and every other condition is the same?

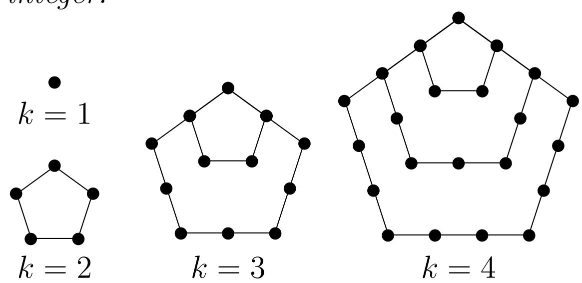
(Proposed by Anett Kocsis)

(Solution)

4. Prove that for all $n \geq 3$ there are an infinite number of n-sided polygonal numbers which are also the sum of two other (not necessarily different) n-sided polygonal numbers!

The first n-sided polygonal number is 1. *The k^{th} n-sided polygonal number for $k \geq 2$ is the number of different points in a figure that consists of all of the regular n-sided polygons which have one common vertex, are oriented in the same direction from that vertex and their sides are l cm long where $1 \leq \ell \leq k-1$ cm and l is an integer.*

In this figure, what we call points are the vertices of the polygons and the points that break up the sides of the polygons into exactly 1 cm long segments. For example, the first four pentagonal numbers are 1, 5, 12, and 22, like it is shown in the figure.



(Proposed by József Osztényi)

(Solution)

5. At the end of the first quarter of the Greece-Egypt basketball game, the score was 26-25. During the first quarter, Áron wrote down the total number of points of the Greeks after every Greek basket, while Benedek wrote down the total number of points of the Egyptians after every Egyptian basket. In the break they noticed that there is no number that both of them wrote down. In how many ways could they have written down the numbers, if there were 21 baskets, and every basket was a 2-pointer or a 3-pointer?
Two options are different if at least one of them wrote down different numbers.

(Proposed by Kartal Nagy)

(Solution)

6. Game: There are four piles of discs given, numbered from 1 to 4. Every turn the current player chooses integers m and n that satisfy $1 \le m < n \le 4$ and takes m discs from pile number n and distributes them into the piles $n-1, n-2, \ldots, n-m$ by adding one disc to every pile. The player that has no available moves loses.
Beat the organisers in this game twice in a row! Based on the number of discs in the piles you can decide if you would like to be the first or the second player.

(Proposed by Ádám Fraknói)

(Solution)

1.3.4 Category E$^+$

1. Show that for every positive real number r, the perimeter of a rectangle of size $1 \times r$ can be covered by pairwise non-intersecting circles of radius 1.
The circles can be tangent to each other.

(Proposed by Benedek Váli)

(Solution)

2. a) Find all solutions to the equation $a^2+b^2+c^2 = abc$, where a, b and c are positive primes.
b) Prove that for every positive integer N there exist integers $a, b, c \ge N$ that satisfy the equation $a^2 + b^2 + c^2 = abc$.

(Proposed by Csongor Beke)

(Solution)

3. At the end of the first quarter of the Greece-Egypt basketball game, the score was 26-25. During the first quarter, Áron wrote down the total number of points of the Greeks after every Greek basket, while Benedek wrote down the total number of points of the Egyptians after every Egyptian basket. In the break they noticed that there is no number that both of them wrote down. In how many ways could they have written down the numbers, if there were 21 baskets, and every basket was a 2-pointer or a 3-pointer?
Two options are different if at least one of them wrote down different numbers.

(Proposed by Kartal Nagy)

(Solution)

4. For an integer $n \geq 2$, the n-level pyramid consists of $1^2 + 2^2 + 3^2 + \cdots + n^2$ cubes of size 1 m $\times$ 1 m $\times$ 1 m, and each cube is made of marble or sandstone. On the kth level, the cubes are arranged in a square grid of size of $(n+1-k) \times (n+1-k)$, and the centers of these grids fall on the same vertical line for all $1 \leq k \leq n$. In addition, the cube faces are parallel, hence each cube of the pyramid is either on the ground or stands on 4 other cubes. The top cube is made of marble, and to ensure the stability of the building, it is true for every marble cube that it is either on the ground or at least 3 out of the 4 cubes on which it stands are marble. What is the least possible number of marble cubes in the pyramid?

(Proposed by Csongor Beke)

(Solution)

5. Let ABC be an acute triangle and let O be its circumcentre. Let O_A, O_B and O_C be the circumcentres of triangles BCO, CAO and ABO respectively. Prove that lines AO_A, BO_B and CO_C are concurrent.

(Proposed by Benedek Váli)

(Solution)

6. Game: There are four piles of discs given, numbered from 1 to 4. Every turn the current player chooses integers m and n that satisfy $1 \leq m < n \leq 4$ and takes m discs from pile number n and distributes them into the piles $n-1, n-2, \ldots, n-m$ by adding one disc to every pile. The player that has no available moves loses.
Beat the organisers in this game twice in a row! Based on the number of discs in the piles you can decide if you would like to be the first or the second player.

(Proposed by Ádám Fraknói)

(Solution)

1.4 Final round – day 2

1.4.1 Category C

1. Benedek and Bence ate a box of gooseberries. A quarter of the fruit was eaten by Bence, half of the fruit plus seven by Benedek. How many gooseberries were in the box, provided it was a whole number?

(Solution)

2. One year, the first of December falls on a Wednesday. Sara and Lili both have a Dürer T-shirt, and they have decided that on certain days they will go to school in it. Sára wears it every third weekday, and Lili wears it every fourth weekday. If they both wore their Dürer T-shirts on December 1st, on what day of December will they both wear these T-shirts again?

(Solution)

3. Five friends, Lili, Dalma, Eszti, Balázs and Áron are lining up in order of height. Their heights in descending order are 173 cm, 171 cm, 166 cm, 165 cm and 162 cm. Everyone sees those lower than them and their own neighbors. We know that:

- Áron has only one neighbor.
- Balázs only sees girls.
- Eszti is the tallest girl.
- Lili does not see Dalma.

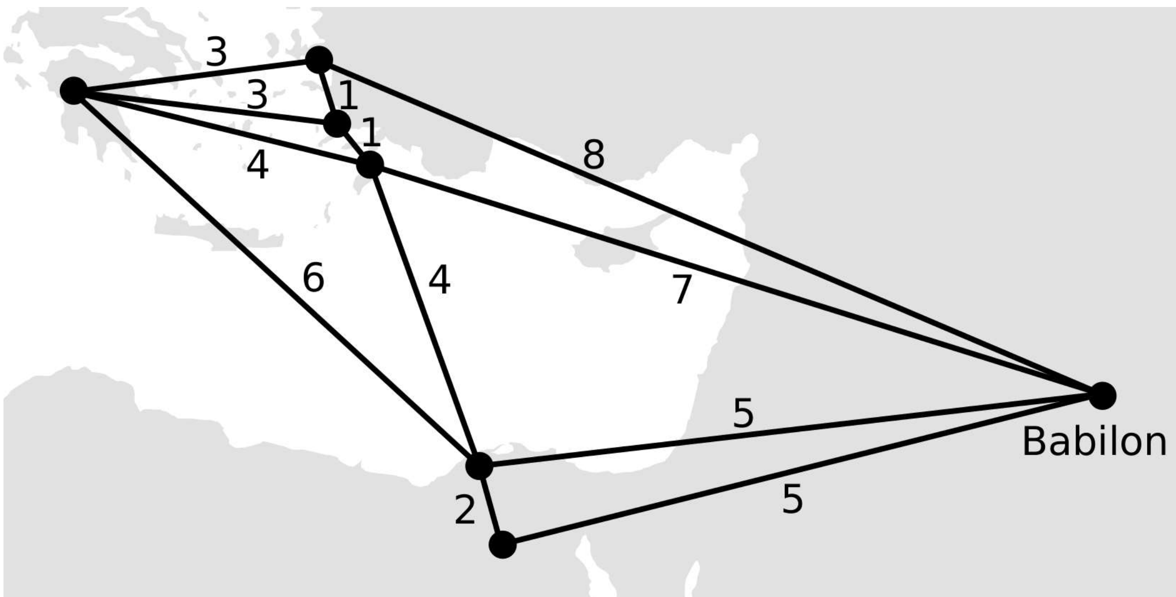
How many centimeters is the difference between the heights of Lili and Dalma?

(Solution)

4. Four Mayan Gods who don't like math are planning to tighten the use of numbers. Chaac can't stand primes, so he would ban the use of prime numbers. Itzamna doesn't like numbers that can be written as the product of two even numbers. Pawahtun would ban numbers that are divisible by three. And Yum Kaax doesn't like numbers that can be written as the sum of two different positive prime numbers. How many numbers could be used among the numbers $1, 2, ..., 20$ if all four prohibitions were to apply?

(Solution)

5. Hermes wants to visit the 7 wonders of the world by starting from Babylon and ending up there, visiting every other wonder exactly once. He can travel on the lines in the figure, each one lasting as many hours as it can be seen. In how many hours at least can you complete your journey? *The dots represent the wonders. Assume that Hermes moves on immediately after each wonder.*

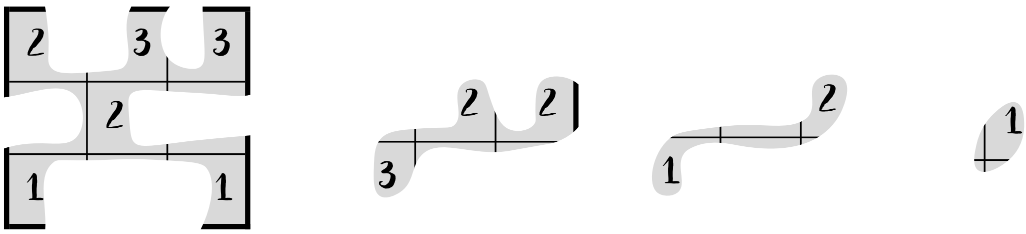


(Solution)

6. How many double-digit numbers are there which have both of their digits among their divisors?

(Solution)

7. Lilla has written the numbers 11, 12, 13, 21, 22, 23, 31, 32, 33 in a 3×3 table in a random order. She took 4 photos of it so she wouldn't forget to fill it in. Unfortunately, the grid was lost and the pictures were damaged, leaving only the details shown in the figure below. What was the sum of the three numbers in the bottom row?



(Solution)

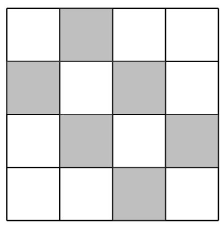
8. Timi was born in 1999. Ever since her birth how many times has it happened that you could write that day's date using only the digits 0, 1 and 2? *For example, 2022.02.21. is such a date.*

(Solution)

9. The AB side of the convex quadrilateral $ABCD$ is 18 cm long. The area of triangles ABC and ABD is 63 cm^2 and the area of triangle BCD is 42 cm^2. How long is the side CD?

(Solution)

10. Zoli wants to fill the given 4×4 table with the digits 1, 2, 3 and 4, such that in every row and column, and also in the diagonal going from the top left cell to the bottom right, each digit appears exactly once. What is the highest possible value of the sum of the digits in the six grey cells?



(Solution)

11. The diagonal of a rectangle is 14 cm long, and its area is 64 cm^2. How many centimeters is the perimeter of the rectangle?

(Solution)

12. Benedek wrote down the following numbers: 1 piece of one, 2 pieces of twos, 3 pieces of threes, ..., 50 piecies of fifties. How many even digits did Benedek write down?

(Solution)

13. The *binary sudoku* is a puzzle in which a table should be filled with digits 0 and 1 such that in each row and column, the number of 0s is equal to the number of 1s. Furthermore, there cannot exist three adjacent cells in a row or in a column such that they have the same digit written in them. Solving the given binary sudoku, what is the sum of the numbers in the two diagonals?

							1
			0			1	1
	1						
1				0			1
				0	0		
						0	
			0				
				1		0	

(Solution)

14. In his large square notebook, Goliat constructed a circle of radius 555cm that is centered at the centre of one of the squares. How many squares' interiors does the circle cross through, if the sides of the squares are 1cm long?

(Solution)

15. Andris was on his way home from school, but he left his book there. Anett wants to bring it to him. There are three different straight lines starting from the exit of the school. Anett doesn't know which way did Andris go, but she knows, that he walks in a straight line at 6 km/h velocity. At least how many minutes will it take for Anett to catch up to Andris, if she can run at 12 km/h, see 100 meters ahead, and she knows that Andris left 5 minutes ago? *The only way Anett can switch to another path is to go back to the school.*

(Solution)

16. Odysseus wants to put 21 chests in total in the 4 storages on his ship that are numbered from 1 to 4. In order for the ship to have an adequate weight distribution, to stay balanced, there has to be at least as many chests in storage 2 as in storage 1, at least as many chests in storage 3 as in storage 4 and at least one chest in each storage. In how many different ways can Odysseus pack the chests? *The chests are all exactly the same and the storages can fit any number of chests.*

(Solution)

1.4.2 Category D

1. Benedek and Bence ate a box of gooseberries. A quarter of the fruit was eaten by Bence, half of the fruit plus seven by Benedek. How many gooseberries were in the box, provided it was a whole number?

(Solution)

2. Four Mayan Gods who don't like math are planning to tighten the use of numbers. Chaac can't stand primes, so he would ban the use of prime numbers. Itzamna doesn't like numbers that can be written as the product of two even numbers. Pawahtun would ban numbers that are divisible by three. And Yum Kaax doesn't like numbers that can be written as the sum of two different positive prime numbers. How many numbers could be used among the numbers $1, 2, ..., 20$ if all four prohibitions were to apply?

(Solution)

3. Five friends, Lili, Dalma, Eszti, Balázs and Áron are lining up in order of height. Their heights in descending order are 173 cm, 171 cm, 166 cm, 165 cm and 162 cm. Everyone sees those lower than them and their own neighbors. We know that:

- Áron has only one neighbor.
- Balázs only sees girls.
- Eszti is the tallest girl.
- Lili does not see Dalma.

How many centimeters is the difference between the heights of Lili and Dalma?

(Solution)

4. Odysseus wants to pack 7 chests into 4 storages on his ship, numbered 1 to 4. For the weight distribution of the ship to be correct, storage 2 must have at least as many chests as storages 1, and storage 3 must have at least as many as storage 4. How many ways can you arrange the chest if you put at least one in each storage? *There is no distinction between the chests, and you can fit as many chests as you like in a storage.*

(Solution)

5. Which is the largest four-digit number that has all four of its digits among its divisors and its digits are all different?

(Solution)

6. The AB side of the convex quadrilateral $ABCD$ is 18 cm long. The area of triangles ABC and ABD is 63 cm^2 and the area of triangle BCD is 42 cm^2. How long is the side CD?

(Solution)

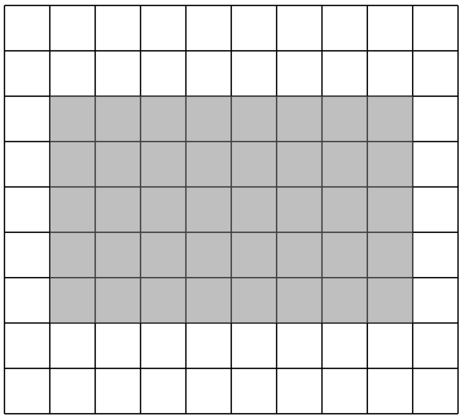
7. In Eldorado a year has 20 months, and each month has 20 days. One day Brigi asked Adél who lives in Eldorado what day her birthday is. Adél answered that she is only going to tell her the product of the month and the day in her birthday. (For example, if she was born on the 19^{th} day of the 4^{th} month, she would say $4 \cdot 19 = 76$.) From this, Brigi was able to tell Adél's birthday. Based on this information, how many days of the year can be Adél's birthday?

(Solution)

8. We are given a triangle ABC and two circles (k_1 and k_2) so the diameter of k_1 is AB and the diameter of k_2 is AC. Let the intersection of BC line segment and k_1 (that isn't B) be P, and the intersection of BC line segment and k_2 (that isn't B) be Q. We know, that $AB = 3003$ and $AC = 4004$ and $BC = 5005$. What is the distance between P and Q?

(Solution)

9. A city consists of 90 square-based houses (as shown in the figure). The grey area in the figure marks the downtown which has 40 houses while the remaining part is the suburb which has 50 houses. The city is divided into thirty coherent districts which all consist of 3 houses. Each district will have one representative who belongs to either the downtown or the suburb depending on which side has a majority in the given district. How many representatives can belong to the downtown at most? *We call a district consisting of three houses coherent if there is a house in it that shares a side with both of the other houses.*

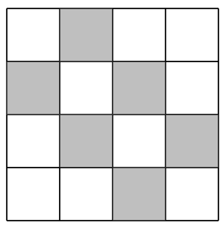


(Solution)

10. The area of a rectangle is 64 cm^2, and the radius of its circumscribed circle is 7 cm. What is the perimeter of the rectangle in centimetres?

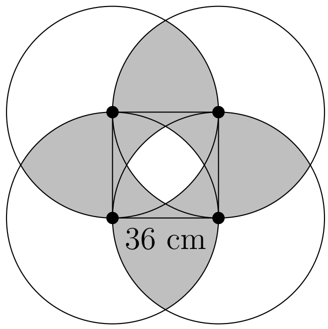
(Solution)

11. Zoli wants to fill the given 4×4 table with the digits 1, 2, 3 and 4, such that in every row and column, and also in the diagonal going from the top left cell to the bottom right, each digit appears exactly once. What is the highest possible value of the sum of the digits in the six grey cells?



(Solution)

12. Archimedes drew a square with side length 36 cm into the sand and he also drew a circle of radius 36 cm around each vertex of the square. If the total area of the grey parts is $n \cdot \pi$ cm^2, what is the value of n?
Do not disturb my circles!



(Solution)

13. The *binary sudoku* is a puzzle in which a table should be filled with digits 0 and 1 such that in each row and column, the number of 0s is equal to the number of 1s. Furthermore, there cannot exist three adjacent cells in a row or in a column such that they have the same digit written in them. Solving the given binary sudoku, what is the sum of the numbers in the two diagonals?

							1
			0			1	1
	1						
1				0			1
				0	0		
						0	
			0				
				1		0	

(Solution)

14. One day Mnemosyne decided to colour all natural numbers in increasing order. She coloured 0, 1 and 2 in brown, and her favourite number, 3, in gold. From then on, for any number whose sum of digits (in the decimal system) was a golden number less than the number itself, she coloured it gold, but coloured the rest of the numbers brown. How many four-digit numbers were coloured gold by Mnemosyne?

The set of natural numbers includes 0.

(Solution)

15. King Minos divided his rectangular island of Crete between his 3 sons as follows: he built a wall along one diagonal of the island and gave one half of the island to his eldest son. Then, in the remaining triangular area, from the right-angled vertex he built a wall perpendicular to the other wall. Of the two areas thus obtained, the larger was given to the middle son and the smaller to the youngest. Each of the three sons had the largest possible square palace built on his own land. How many times is the area of the eldest son's palace larger than the area of the youngest son's palace if the side lengths of the island are 30 m and 210 m?

(Solution)

16. Csongi bought a 12-sided convex polygon-shaped pizza. The pizza has no interior point with three or more distinct diagonals passing through it. Áron wants to cut the pizza along 3 diagonals so that exactly 6 pieces of pizza are created. In how many ways can he do this? *Two ways of slicing are different if one of them has a cut line that the other does not have.*

(Solution)

1.4.3 Category E

1. Csenge and Eszter ate a whole basket of cherries. Csenge ate a quarter of all cherries while Eszter ate four-sevenths of all cherries and forty more. How many cherries were in the basket in total?

(Solution)

2. Timi was born in 1999. Ever since her birth how many times has it happened that you could write that day's date using only the digits 0, 1 and 2? *For example, 2022.02.21. is such a date.*

(Solution)

3. Which is the largest four-digit number that has all four of its digits among its divisors and its digits are all different?

(Solution)

4. Benedek wrote down the following numbers: 1 piece of one, 2 pieces of twos, 3 pieces of threes, $\dots$, 50 piecies of fifties. How many digits did Benedek write down?

(Solution)

5. We are given a triangle ABC and two circles (k_1 and k_2) so the diameter of k_1 is AB and the diameter of k_2 is AC. Let the intersection of BC line segment and k_1 (that isn't B) be P, and the intersection of BC line segment and k_2 (that isn't B) be Q. We know, that $AB = 3003$ and $AC = 4004$ and $BC = 5005$. What is the distance between P and Q?

(Solution)

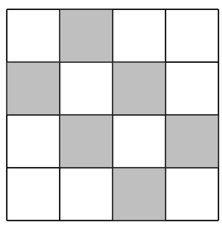
6. In Eldorado a year has 20 months, and each month has 20 days. One day Brigi asked Adél who lives in Eldorado what day her birthday is. Adél answered that she is only going to tell her the product of the month and the day in her birthday. (For example, if she was born on the 19^{th} day of the 4^{th} month, she would say $4 \cdot 19 = 76$.) From this, Brigi was able to tell Adél's birthday. Based on this information, how many days of the year can be Adél's birthday?

(Solution)

7. The area of a rectangle is 64 cm^2, and the radius of its circumscribed circle is 7 cm. What is the perimeter of the rectangle in centimetres?

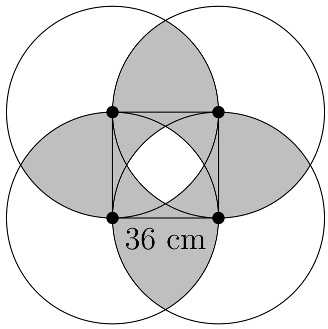
(Solution)

8. Zoli wants to fill the given 4×4 table with the digits 1, 2, 3 and 4, such that in every row and column, and also in the diagonal going from the top left cell to the bottom right, each digit appears exactly once. What is the highest possible value of the sum of the digits in the six grey cells?



(Solution)

9. Archimedes drew a square with side length 36 cm into the sand and he also drew a circle of radius 36 cm around each vertex of the square. If the total area of the grey parts is $n \cdot \pi$ cm^2, what is the value of n?
Do not disturb my circles!



(Solution)

10. One day Mnemosyne decided to colour all natural numbers in increasing order. She coloured 0, 1 and 2 in brown, and her favourite number, 3, in gold. From then on, for any number whose sum of digits (in the decimal system) was a golden number less than the number itself, she coloured it gold, but coloured the rest of the numbers brown. How many four-digit numbers were coloured gold by Mnemosyne?

The set of natural numbers includes 0.

(Solution)

11. The *binary sudoku* is a puzzle in which a table should be filled with digits 0 and 1 such that in each row and column, the number of 0s is equal to the number of 1s. Furthermore, there cannot exist three adjacent cells in a row or in a column such that they have the same digit written in them. Solving the given binary sudoku, what is the sum of the numbers in the two diagonals?

							1
			0			1	1
	1						
1				0			1
				0	0		
						0	
			0				
				1		0	

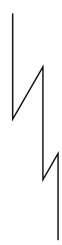
(Solution)

12. Marvin really likes pancakes, so he asked his grandma to make pancakes for him. Every time Grandma sends pancakes, she sends a package of 32. When Marvin is in the mood for pancakes, he eats half of the pancakes he has. Marvin ate 157 pancakes for lunch today. At least how many times has Grandma sent pancakes to Marvin so far? *Marvin does not necessarily eat an integer number of pancakes at once, and he is in the mood for pancakes at most once a day.*

(Solution)

13. A country has 2023 cities and there are flights between these cities. Each flight connects two cities in both directions. We know that you can get from any city to any other using these flights, and from each city there are flights to at most 4 other cities. What is the maximum possible number of cities in the country from which there is a flight to only one city?

(Solution)



14. Zeus's lightning is made of a copper rod of length 60 by bending it 4 times in alternating directions so that the angle between two adjacent parts is always 60°. What is the minimum value of the square of the distance between the two endpoints of the lightning?
All five segments of the lightning lie in the same plane.

(Solution)

15. Csongi bought a 12-sided convex polygon-shaped pizza. The pizza has no interior point with three or more distinct diagonals passing through it. Áron wants to cut the pizza along 3 diagonals so that exactly 6 pieces of pizza are created. In how many ways can he do this? *Two ways of slicing are different if one of them has a cut line that the other does not have.*

(Solution)

16. For the Dürer final results announcement, four loudspeakers are used to provide sound in the hall. However, there are only two sockets in the wall from which the power comes. To solve the problem, Ádám got two extension cords and two power strips. One plug can be plugged into an extension cord, and two plugs can be plugged into a power strip. Gábor, in his haste before the announcement of the results, quickly plugs the 8 plugs into the 8 holes. Every possible way of plugging has the same probability, and it is also possible for Gábor to plug something into itself. What is the probability that all 4 speakers will have sound at the results announcement? *For the solution, give the sum of the numerator and the denominator in the simplified form of the probability. A speaker sounds when it is plugged directly or indirectly into the wall.*

(Solution)

1.4.4 Category E$^+$

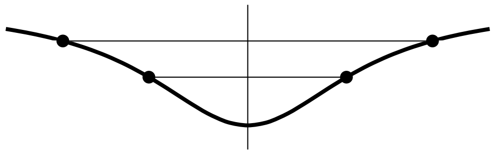
1. Nüx has three moira daughters, whose ages are three distinct prime numbers, and the sum of their squares is also a prime number. What is the age of the youngest moira?

(Solution)

2. The area of a rectangle is 64 cm^2, and the radius of its circumscribed circle is 7 cm. What is the perimeter of the rectangle in centimetres?

(Solution)

3. Hapi, the god of the annual flooding of the Nile is preparing for this year's flooding. The shape of the channel of the Nile can be described by the function $y = \frac{-1000}{x^2+100}$ where the x and y coordinates are in metres. The depth of the river is 5 metres now. Hapi plans to increase the water level by 3 metres. How many metres wide will the river be after the flooding? *The depth of the river is always measured at its deepest point.*



(Solution)

4. In Eldorado a year has 20 months, and each month has 20 days. One day Brigi asked Adél who lives in Eldorado what day her birthday is. Adél answered that she is only going to tell her the product of the month and the day in her birthday. (For example, if she was born on the 19^{th} day of the 4^{th} month, she would say $4 \cdot 19 = 76$.) From this, Brigi was able to tell Adél's birthday. Based on this information, how many days of the year can be Adél's birthday?

(Solution)

5. King Minos divided his rectangular island of Crete between his 3 sons as follows: he built a wall along one diagonal of the island and gave one half of the island to his eldest son. Then, in the remaining triangular area, from the right-angled vertex he built a wall perpendicular to the other wall. Of the two areas thus obtained, the larger was given to the middle son and the smaller to the youngest. Each of the three sons had the largest possible square palace built on his own land. How many times is the area of the eldest son's palace larger than the area of the youngest son's palace if the side lengths of the island are 30 m and 210 m?

(Solution)

6. Archimedes drew a square with side length 36 cm into the sand and he also drew a circle of radius 36 cm around each vertex of the square. If the total area of the grey parts is $n \cdot \pi$ cm^2, what is the value of n?
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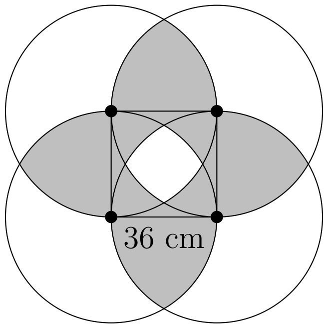


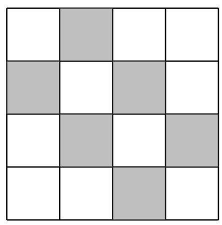
(Solution)

7. One day Mnemosyne decided to colour all natural numbers in increasing order. She coloured 0, 1 and 2 in brown, and her favourite number, 3, in gold. From then on, for any number whose sum of digits (in the decimal system) was a golden number less than the number itself, she coloured it gold, but coloured the rest of the numbers brown. How many four-digit numbers were coloured gold by Mnemosyne?

The set of natural numbers includes 0.

(Solution)

8. Zoli wants to fill the given 4×4 table with the digits 1, 2, 3 and 4, such that in every row and column, and also in the diagonal going from the top left cell to the bottom right, each digit appears exactly once. What is the highest possible value of the sum of the digits in the six grey cells?



(Solution)

9. The *binary sudoku* is a puzzle in which a table should be filled with digits 0 and 1 such that in each row and column, the number of 0s is equal to the number of 1s. Furthermore, there cannot exist three adjacent cells in a row or in a column such that they have the same digit written in them. Solving the given binary sudoku, what is the sum of the numbers in the two diagonals?

							1
			0			1	1
	1						
1				0			1
				0	0		
						0	
			0				
				1		0	

(Solution)

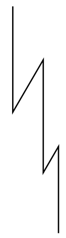
10. Marvin really likes pancakes, so he asked his grandma to make pancakes for him. Every time Grandma sends pancakes, she sends a package of 32. When Marvin is in the mood for pancakes, he eats half of the pancakes he has. Marvin ate 157 pancakes for lunch today. At least how many times has Grandma sent pancakes to Marvin so far? *Marvin does not necessarily eat an integer number of pancakes at once, and he is in the mood for pancakes at most once a day.*

(Solution)

11. A country has 2023 cities and there are flights between these cities. Each flight connects two cities in both directions. We know that you can get from any city to any other using these flights, and from each city there are flights to at most 4 other cities. What is the maximum possible number of cities in the country from which there is a flight to only one city?

(Solution)

12. Zeus's lightning is made of a copper rod of length 60 by bending it 4 times in alternating directions so that the angle between two adjacent parts is always $60°$. What is the minimum value of the square of the distance between the two endpoints of the lightning?
All five segments of the lightning lie in the same plane.



(Solution)

13. Csongi bought a 12-sided convex polygon-shaped pizza. The pizza has no interior point with three or more distinct diagonals passing through it. Áron wants to cut the pizza along 3 diagonals so that exactly 6 pieces of pizza are created. In how many ways can he do this? *Two ways of slicing are different if one of them has a cut line that the other does not have.*

(Solution)

14. For the Dürer final results announcement, four loudspeakers are used to provide sound in the hall. However, there are only two sockets in the wall from which the power comes. To solve the problem, Ádám got two extension cords and two power strips. One plug can be plugged into an extension cord, and two plugs can be plugged into a power strip. Gábor, in his haste before the announcement of the results, quickly plugs the 8 plugs into the 8 holes. Every possible way of plugging has the same probability, and it is also possible for Gábor to plug something into itself. What is the probability that all 4 speakers will have sound at the results announcement? *For the solution, give the sum of the numerator and the denominator in the simplified form of the probability. A speaker sounds when it is plugged directly or indirectly into the wall.*

(Solution)

15. What is the biggest positive integer which divides $p^4 - q^4$ for all primes p and q greater than 10?

(Solution)

16. What is the remainder of $2025\string^(2024\string^(2022\string^(2021\string^(2020\string^\ldots\string^(2\string^1)\ldots)))))$ when it is divided by 2023?

Here ^ *is the exponential symbol, for example* $2\string^(3\string^2) = 2\string^9 = 512$. *The power tower contains the integers from* 2025 *to* 1 *exactly once, except that the number* 2023 *is missing.*

(Solution)

2 Solutions

2.1 Online round

2.1.1 Tables

#	ANS	Problem	P
C-1	4	In Dionysus' garden,	3p
C-2	48	As it is commonly known,	3p
C-3	700	Morpheus, the god of dreams,	4p
C-4	18	Zeus draws a large regular polygon	4p
C-5	5	A flea is sitting on the 0	4p
C-6	16	Sasha, the cat starts a special diet	5p
C-7	24	We'd like to fill a 4×4 table	5p
C-8	69	Ares has centaurs, humans, and pegasusses	6p
C-9	44	The gate of the underworld	6p

#	ANS	Problem	P
D-1	6	In Dionysus' garden,	3p
D-2	700	Morpheus, the god of dreams,	3p
D-3	8670	Jóska, Miki, and Viktor	4p
D-4	7	A flea sits at the point 0	4p
D-5	16	Sasha, the cat starts a special diet	4p
D-6	24	We'd like to fill a 4×4 table	5p
D-7	60	Thor, Freya, Idun, and Loki	5p
D-8	69	Ares has centaurs, humans, and pegasusses	6p
D-9	44	The gate of the underworld	6p

#	ANS	Problem	P
E-1	18	Zeus draws a large regular polygon	3p
E-2	8670	Jóska, Miki, and Viktor	3p
E-3	7	A flea sits at the point 0	4p
E-4	16	Sasha, the cat starts a special diet	4p
E-5	24	We'd like to fill a 4×4 table	4p
E-6	69	Ares has centaurs, humans, and pegasusses	5p
E-7	98	We call a positive integer *halved*	5p
E-8	24	In Persephone's garden	6p
E-9	67	The gate of the underworld	6p

2.1.2 Category C

1. It takes 4 days for the entire bunch to become moldy.

(Back to problems)

2. The wooden horse has 4 legs, so the Troyans let $25 \cdot 4 = 100$ legs into their city. Of those 100 legs, $100 - 4 = 96$ did not belong to the wooden horse, so $\frac{96}{2} = 48$ Greek soldiers could get into Troy.

(Back to problems)

3. We have to subtract the areas of 4 isosceles right-angled triangles which have 10 m long legs from the area of $ABCD$. The total area of these triangles is $T_{\text{triangles}} = \dfrac{4 \cdot 10^2}{2} m^2 = 200m^2$. The area of $ABCD$ is $T_{\text{square}} = 30^2 m^2 = 900m^2$. So, the area of the octagon is $T_{\text{square}} - T_{\text{triangles}} = 900m^2 - 200m^2 = 700m^2$.

(Back to problems)

4. We know that if the polygon has n sides, then $\frac{(n-2)180^\circ}{n} = 160^\circ$ from which we can conclude that $n = 18$. So, the polygon has 18 sides.

(Back to problems)

5. The flea can reach 7 in 5 steps if it jumps left in the fourth step and jumps right in all other steps, because $1+2+3-4+5 = 7$. n cannot be smaller than 5 because, with 3 or fewer steps, the flea cannot get further than 6 units, and with 4 steps, it can be easily checked that it cannot reach precisely point 7. With 4 steps, the flea will jump $1+2+3+4 = 10$ units in total, an even number of units which means that it will arrive at an even point, no matter what direction it chose at each step, but 7 is an odd number, so it cannot arrive to 7 in 4 steps.

(Back to problems)

6. We know that Sasha eats fish on the third day, so he cannot eat that on the first, second, fourth and fifth days. So, on the first and the second days he eats chicken and beef in some order, and on the fourth and fifth days too. On the sixth and seventh days, he can eat anything that he didn't eat on the previous day. So, in total, he can plan his diet in $2 \cdot 2 \cdot 2 \cdot 2 = 16$ different ways.

(Back to problems)

7. The second row and column will have one 0, since this stands for every row or column. Then, the column of the 0 in the second row will satisfy the condition for the columns, it will read 2022. Similarly, the row of the 0 in the second column will satisfy the other condition for the rows. As such, every possible way to fill the table will satisfy these conditions.

We have 24 such ways to fill the table. The 0 in the upper row can be in 4 different cells, in the second row we can only choose from the remaining 3 ones, and so on and so forth. These are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possibilities in total.

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8. Let the number of humans be H, the number of pegasusses P and the number of centaurs C. Then we can translate the problem to the following equations:

$$193 = H + P + C$$
$$666 = 2H + 4P + 4C$$
$$244 = 2H + 2C$$

Taking the first equation four times, and subtracting the second from it, we get that there are 53 humans. Substituting this to the third equation we get that there are 69 centaurs. We can check that according to the first equation there are 71 pegasusses and these three numbers satisfy all conditions.

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9. If we express it with remainders, the task is to find the smallest positive integer which has 0 or 1 as a remainder if we divide it by 11, 2 or 3 if we divide it by 6 and 4 or 5 if we divide it by 8.

We can easily see that 44 fulfills these conditions, 44 minutes after Orpheus starts playing, all three heads will be sleeping for a minute. The first head only slept four times before that, so we only need to check those four time periods to see if 44 really is the smallest possible solution.

In the first 2 minutes, the third head is not sleeping, after 11, 12, 22, and 23 minutes, the second head is awake and after 34 and 35 minutes, the third head is not sleeping. So Orpheus needs at least 44 minutes to be able to enter the underworld.

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2.1.3 Category D

1. It takes 6 days for the entire bunch to become moldy.

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2. For the solution, see Category C Problem 3.

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3. If we triple Miklós' order and subtract Jóska's order from it two times, we get Viktor's order. So, Viktor paid $3 \cdot 5330\text{Ft} - 2 \cdot 3660\text{Ft} = 8670\text{Ft}$.

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4. It can get there in seven steps: for example by jumping to the left in the third and fifth steps, and to the right in all the others: $1+2-3+4-5+6+7=12$. It's not difficult to check that a smaller number of jumps won't suffice. There's a quicker way to argue this: If we can get to 12 in k steps, then the sum $1+2+\ldots+k$ needs to be even. This is true because if the sum of the steps to the left is l, then $12=1+2+\ldots+k-2l$, since instead of l steps to the right we need to count l steps to the left. This means that 5 or 6 steps won't suffice, since for them the sums are 15 or 21. Additionally, 4 or fewer steps are not enough, since even if we jumped to the right every time, we wouldn't get to point 12. As such, we proved that we need at least 7 steps to reach it, and that's sufficient as well.

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5. For the solution, see Category C Problem 6.

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6. For the solution, see Category C Problem 7.

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7. If f marks the number of exercises in the exam, then $\frac{f}{2}+\frac{f}{4}+40=130$, so $f=120$. If his friends solved the first 60, first 40 and first 30 exercises, then Loki will only be able to solve 60 exercises. But since he would be able to solve the 60 exercises that Thor solved in any case, then Loki will solve at least 60. Since there are 120 exercises in the exam, there will be 60 exercises at most that Loki cannot solve.

Tudjuk, hogyha a feladatok számát f-nek nevezzük, akkor $\frac{f}{2}+\frac{f}{4}+40=130$, amiből $f=120$. Ekkor viszont lehet, hogy Loki társai az első 60, az első 40 és az első 30 feladatot csinálták meg, ami azt jelenti, hogy Loki szintén csak az első 60 feladatot tudja megoldani. Ennyit pedig tényleg meg is tud oldani, hiszen Thor 60 feladatát mindenképp meg tudja oldani. Mivel a feladatok száma 120, így legfeljebb 60 feladatot nem tud megoldani Loki.

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8. For the solution, see Category C Problem 8.

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9. For the solution, see Category C Problem 9.

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2.1.4 Category E

1. For the solution, see Category C Problem 4.

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2. For the solution, see Category D Problem 3.

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3. For the solution, see Category D Problem 4.

(Back to problems)

4. For the solution, see Category C Problem 6.

(Back to problems)

5. For the solution, see Category C Problem 7.

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6. For the solution, see Category C Problem 8.

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7. Mark double-digit numbers as $\overline{ab} = 10 \cdot a + b$ where a and b are digits and a is not 0. Then, $\overline{ab} = 10 \cdot a + b = x \cdot a \cdot b$ is true for all halved numbers where x is a positive integer. We can convert this equation into $10 \cdot a = (xa - 1)b$. a and $(xa - 1)$ have to be coprime, so $(xa - 1) \mid 10$. So, the value of $(xa - 1)$ is 10, 5, 2 or 1. If $(xa - 1) = 10$, then based on the equation, $a = b$ and since $xa = 11$, $a \mid 11$, so $a = b = 1$, which means that the only solution in this case is 11. If $(xa - 1) = 5$, then based on the equation, $b = 2a$ and since $xa = 6$, $a \mid 6$, which means that 12, 24 and 36 are the solutions in this case. If $(xa - 1) = 2$, then based on the equation, $b = 5a$ and since $xa = 3$, $a \mid 3$, so the only solution in this case is 15. If $(xa - 1) = 1$, then based on the equation, $b = 10a$, which means that there is no solution in this case. The found numbers are all indeed halved, their sum is $11 + 12 + 24 + 36 + 15 = 98$.

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8. Let A mark the spot of the tree, B that of the bush, and C that of the house. We know ABC has a right angle, so according to the Pythagorean theorem $AB = 41$m. So if we call the intersection of AB and the flowerbed D, then $BD = BA - CA = 41\text{m} - 9\text{m} = 32$m, the radius of the flowerbed. If E is the point of tangency in question, then $BEC\sphericalangle$ is a right angle, so once again using the Pythagorean theorem we get that $BE = 24$m.

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9. If we express it with remainders, the task is to find the smallest positive integer which has 0 or 1 as a remainder if we divide it by 11, 1 or 2 if we divide it by 6 and 3 or 4 if we divide it by 7.

We can easily see it that 67 fulfills these conditions, 67 minutes after Orpheus starts playing, all three heads will be sleeping for a minute. The first head only slept six times before that, so we only need to check those six time periods to see if 67 really is the smallest possible solution.

In the first 2 minutes, the third head is not sleeping, after 11, 12, 22, 23, 33, 34, 45, and 66 minutes, the second head is awake and after 44, 55, and 56 minutes, the third head is not sleeping. So, Orpheus needs at least 67 minutes to be able to enter the underworld.

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2.2 Regional round

2.2.1 Category C

1. A date cannot be fabulous if the month's number is a single-digit number because by adding the digits of the day to the number of the month, the sum will have to be larger than the number of the month. So, fabulous dates can only occur in October, November, or December. Since $10 - (1 + 0) = 9$, $11 - (1 + 1) = 9$, and $12 - (1 + 2) = 9$, a date can only be fabulous if the sum of the digits of the day is 9. So, the day can be the 9., the 18. or the 27. So, we have 9 solutions in total, 10.09., 10.18., 10.27., 11.09., 11.18, 11.27, 12.09., 12.18. and 12.27.

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2. Let's find the squares on the map which Theseus surely visited and which he surely didn't!

He visited both neighbours of the 2 in the top left corner so we found the only neighbour of the 1 in the left column which was visited by Theseus so he definitely did not visit its two other neighbours. So, now we know which neighbour of the 3 was not visited by Theseus which means that he must have visited all of its other neighbours. Now we know which neighbour of the 1 on the left in the row at the bottom has been visited by Theseus.

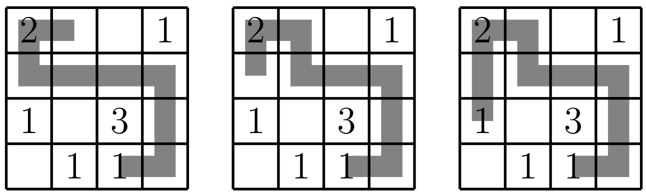
If we mark on the map the places which were definitely visited by Theseus with an O and mark the places which were definitely not with an X, then this is how the map looks currently:

2	O		1
O		O	
1	X	3	O
X	1	1	

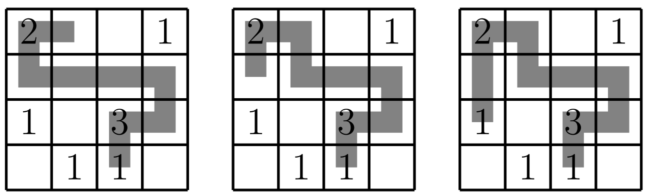
Then, we separate three cases based on which neighbour of the 1 on the right in the row at the bottom has been visited by Theseus.

1. If he visited the square containing the 1 on the left, he would not have been able to move to other squares from these two, which is not possible, so there are no solutions in this case.

2. If he visited the bottom right corner, then the 1 in question is one of the endpoints of his journey. Since he didn't visit the square containing the 3, he must have continued his journey upwards, so the neighbour of the top right corner on the left has not been visited by Theseus. Then, he must have turned left and stayed in this direction for the next two steps. After that, he could either take a third step in the same direction or turn upwards. If he continued in the same direction, then after that step, he must have turned upwards, then right after one step and finish his journey after that step, otherwise he could not have reached the square on the right of the 2 in the top left corner and he cannot go further from there. If he turned upwards, then after that step, he must have turned left and after that step, he must have turned down. He either took one or two steps in that direction and then he must have finished his journey. So, these are the three possible routes he could take in this case:



3. If he visited the square containing the 3, then one of the endpoints of his journey was the square containing the 1 in question in this case as well. After stepping on the square containing 3, he must have turned right, since he has to visit that square as well but he could not have returned to it if he went upwards first. After that step, he must have turned upwards since he had no alternative. After that step, he is in the same position as he was in the second case, so he must have finished his journey in one of the three possible ways that we already examined previously:



So, there are 6 possible routes that Theseus could have taken.

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3. In total, there are 24 different ways in which the four buttons can be pushed, since $4 \cdot 3 \cdot 2 \cdot 1 = 24$. So, we calculate the results for all these 24 possibilities. (We do not indicate the parentheses.)

If the first operation is multiplication:

$$\begin{aligned}
21 \times 3 \div 3 + 3 - 3 \quad &\Rightarrow \quad 21,\\
21 \times 3 \div 3 - 3 + 3 \quad &\Rightarrow \quad 21,\\
21 \times 3 + 3 - 3 \div 3 \quad &\Rightarrow \quad 21,\\
21 \times 3 + 3 \div 3 - 3 \quad &\Rightarrow \quad 19,\\
21 \times 3 - 3 + 3 \div 3 \quad &\Rightarrow \quad 21 \text{ and}\\
21 \times 3 - 3 \div 3 + 3 \quad &\Rightarrow \quad 23.
\end{aligned}$$

If the first operation is division:

$$\begin{aligned}
21 \div 3 \times 3 + 3 - 3 \quad &\Rightarrow \quad 21,\\
21 \div 3 \times 3 - 3 + 3 \quad &\Rightarrow \quad 21,\\
21 \div 3 + 3 \times 3 - 3 \quad &\Rightarrow \quad 27,\\
21 \div 3 + 3 - 3 \times 3 \quad &\Rightarrow \quad 21,\\
21 \div 3 - 3 + 3 \times 3 \quad &\Rightarrow \quad 21 \text{ and}\\
21 \div 3 - 3 \times 3 + 3 \quad &\Rightarrow \quad 15.
\end{aligned}$$

If the first operation is addition:

$$\begin{aligned}
21 + 3 \times 3 \div 3 - 3 \quad &\Rightarrow \quad 21,\\
21 + 3 \times 3 - 3 \div 3 \quad &\Rightarrow \quad 23,\\
21 + 3 \div 3 \times 3 - 3 \quad &\Rightarrow \quad 21,\\
21 + 3 \div 3 - 3 \times 3 \quad &\Rightarrow \quad 15,\\
21 + 3 - 3 \div 3 \times 3 \quad &\Rightarrow \quad 21 \text{ and}\\
21 + 3 - 3 \times 3 \div 3 \quad &\Rightarrow \quad 21.
\end{aligned}$$

If the first operation is subtraction:

$$\begin{aligned}
21 - 3 \times 3 + 3 \div 3 \quad &\Rightarrow \quad 19,\\
21 - 3 \times 3 \div 3 + 3 \quad &\Rightarrow \quad 21,\\
21 - 3 + 3 \times 3 \div 3 \quad &\Rightarrow \quad 21,\\
21 - 3 + 3 \div 3 \times 3 \quad &\Rightarrow \quad 21,\\
21 - 3 \div 3 + 3 \times 3 \quad &\Rightarrow \quad 27 \text{ and}\\
21 - 3 \div 3 \times 3 + 3 \quad &\Rightarrow \quad 21.
\end{aligned}$$

So, Anna got 27 and Béla got 15.

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4. Since everyone's height is the average of their parents' heights, Theodorus' height is the average of all of his great-great-grandparents. So, based on his height, we can calculate how many of his great-great-grandparents are dwarfs and how many are giants. If k indicates the number of giants, then $\dfrac{k \cdot 256 + (16-k) \cdot 128}{16} = 168$, so $2k + 16 - k = 21$ which means that $k = 5$.

Since Theodorus has one more maternal giant great-great-grandparents than paternal giant great-great-grandparents, he has 3 maternal giant great-great-grandparents and 2 paternal giant great-great-grandparents.

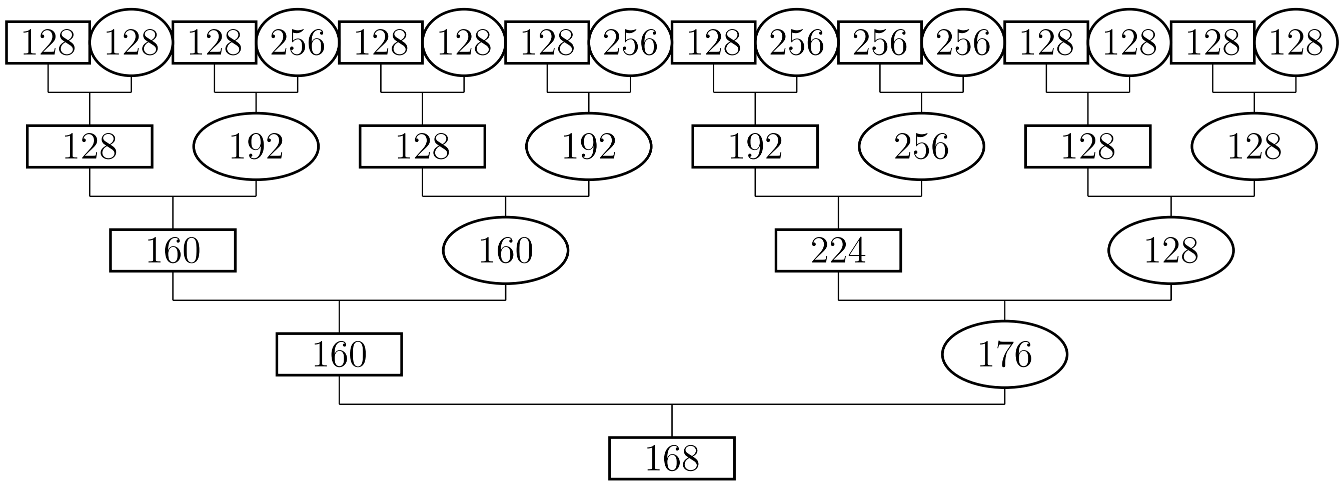
From this, we can calculate the height of Theodorus' parents' heights since their heights equal the average of their great-grandparents', Theodorus' great-great-grandparents'. Theodorus' father's height is the average of Theodorus' paternal great-great-grandparents. We know that 2 of them are giants and 6 of them are dwarfs, so Theodorus' father is $\dfrac{2 \cdot 256 + 6 \cdot 128}{8} = 160$ pechys tall. Similarly, Theodorus' mother is $\dfrac{3 \cdot 256 + 5 \cdot 128}{8} = 176$ pechys tall.

Theodorus' father and paternal grandfather are the same height, so his paternal grandmother is as well. So, all three of them are 160 pechys tall. This means that Theodorus' paternal grandfather is the one male ancestor of Theodorus' father who is not a dwarf. So, all other male ancestors of Theodorus' father are 128 pechys tall. Also, Theodorus' paternal grandfather and grandmother both have one giant ancestor among Theodorus' great-great-grandparents. Since the ancestors of dwarfs are all dwarfs, the two paternal great-great-grandparents who are giants are the mothers of the paternal great-grandmothers.

Theodorus' paternal grandmother is not 128 pechys tall, so his maternal grandmother has to be a dwarf. This means that all of her ancestors are dwarfs as well, they are all 128 pechys tall.

Theodorus has no giant paternal great-grandmothers, so his maternal grandfather's mother is a giant and so are her ancestors.

Since among the maternal great-great-grandparents of Theodorus, there are more female giants than male ones, Theodorus' maternal grandfathers' paternal grandmother is a giant.



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5. Csongor had to hear at least three digits to figure out where the counting is at. First we show that two digits are not enough. Let the two digits he heard be a and b. If $a = b$, they could be anywhere between $a+1$ and $a+9$. If $b = a+1$ (including $9+1=0$), then $a, a+1$ and $10+a, 10+(a+1)$ could be a pair of numbers. If $a \neq b$ and $a \neq b-1$, then a could be the digit in the tens place of the first number and b in the ones place of the second number, or vice versa (a is in the ones place of the first, b is in the tens place of the second). These pairs of numbers exist, we only need to show they are different. This is true, because in the first pair of numbers the first number has a in the tens place, but in the second pair of numbers the first number can only have $b-1$ or b in the tens place, because the next number has b in the tens place.

If he heard three digits, it could be enough, for example, in the case of 1, 5, 3, he knows from the first two digits that just the number pairs 14, 15 or 51, 52 are possible and knowing the third digit, he will know that the counting is on 51, 52, 53.

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6. Let's see which digits in the table have higher or lower positions in Anett's numbers than in Csenge's numbers!

b	a	a	a	a
c	b	a	a	a
c	c	b	a	a
c	c	c	b	a
c	c	c	c	b

The digits that are indicated by b, have the same positions in both girls' numbers, the digits that are indicated by a, have higher positions in Anett's numbers, and the digits that are indicated by c, have higher positions in Csenge's numbers. Then, the difference will be its largest possible value if the digits that are indicated by c, are as high as they can be and the digits that are indicated by a, are as low as they can be. If we increase a digit that is indicated by c, then Csenge's numbers will increase more than Anett's numbers, since those digits are at higher positions in Csenge's numbers than in Anett's numbers. The digits that are indicated by a behave inversely to this while the digits that are indicated by b do not have an effect on the difference between the two sums. So, the difference in question can be the biggest if we write 9 on all places that are indicated by c and 0 on all places that are indicated by a, except for the first row, because there, the lowest possible digit is 1. We can write any digits in the places that are indicated by b, for example, 5.

5	1	1	1	1
9	5	0	0	0
9	9	5	0	0
9	9	9	5	0
9	9	9	9	5

With this construction, we can reach the highest possible difference, $51111+95000+99500+99950+99995-59999-15999-10599-10059-10005 = 338895$.

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7. Since we drew circles around A and B with radius AB, segments AB, AF and BF are the same length, so triangle ABF is regular. This means that $\angle BAF = 60°$.

The internal angle of a regular pentagon is $\frac{3 \cdot 180°}{5} = 108°$, which means that $\angle EAF = \angle EAB - \angle BAF = 48°$.

Points E and F are both on the circle that is drawn around A, so triangle EAF is isosceles and from that we can calculate that $\angle FEA = \frac{180° - \angle EAF}{2} = 66°$.

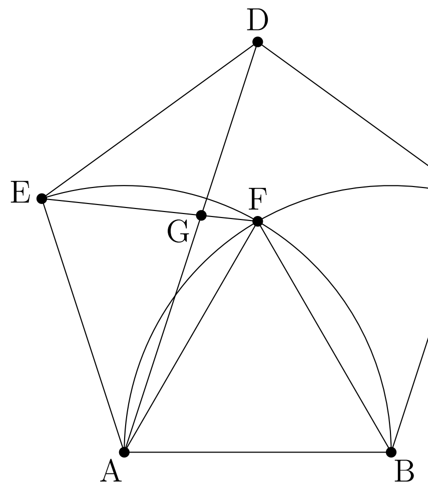
There is another isosceles triangle in the pentagon, triangle DEA, from which we can calculate the measure of $\angle EAD$ since we already know that $\angle DEA = 108°$. From that, $\angle EAD = \frac{180° - \angle DEA}{2} = \frac{180° - 108°}{2} = 36°$

Since the the sum of the internal angles of a triangle is always 180°,

$$\angle AGE = 180° - \angle FEA - \angle EAD = 180° - 66° - 36° = 78°$$

.

So, angle AGE measures 78°.



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2.2.2 Category D

1. For the solution, see Category C Problem 3.

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2. For the solution, see Category C Problem 2.

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3. If the number of teams at a given location is between 1 and 8, then they will need 1 package of paper since they will use at least $1 \cdot 6 = 6$ and at most $8 \cdot 6 = 48$ sheets of paper. Similarly, if the number of teams at a given location is between 9 and 16, they will need 2 packages, if it is between 17 and 25, they will need 3 packages and if there are at least 26 teams at a given location, they will need at least 4 packages.

If there are 17 teams at one location and 9 teams at each of the other 5 locations, then there are indeed $17 + 5 \cdot 9 = 62$ participating teams. At the first location, they will need 3 packages while they will need 2 packages at each of the other 5 locations which is $3 + 5 \cdot 2 = 13$ packages in total.

If they need k packages at a location, then it means that they need at least $(k - 1) \cdot 50 + 1$ sheets of paper. So, if there needs to be $a_1, \dots a_6$ packages at the six locations, respectively, that means that the teams need at least $(a_1 - 1 + a_2 - 1 + \dots a_6 - 1) \cdot 50 + 6 = ((a_1 + \dots a_6) - 6) \cdot 50 + 6$ sheets of paper. We know, that the 62 teams need $62 \cdot 6 = 372$ sheets of paper in total. So, $((a_1 + \dots a_6) - 6) \cdot 50 + 6 \leq 372$ which means that $((a_1 + \dots a_6) - 6) \cdot 50 \leq 366$ from which $(a_1 + \dots a_6) - 6 \leq 7$. So, $a_1 + \dots a_6 \leq 13$. Since $a_1 + \dots a_6$ is exactly the number of packages that are needed, we proved that the organizers will not need more than 13 packages in any case.

We already have a solution, in which the organizers need 13 packages, 17, 9, 9, 9, 9, 9. If we put 8 teams from a location with 9 teams over to a location with 9 teams as well, we will still need 13 packages. So, 17, 17, 9, 9, 9, 1 and 17, 17, 17, 9, 1, 1 are also correct solutions. Now, we will prove that there are no other solutions. If we do not take more than 3 packages to any location, then there are no other possible solutions. If we have to deliver k packages to a location where k is 1, 2 or 3, then there has to be at least $(k - 1) \cdot 8 + 1$ teams there. But if $k \geq 4$, there has to be at least $(k - 1) \cdot 8 + 2$ teams at a location. So, if $13 = a_1 + \dots a_6$, then at least $(a_1 - 1) \cdot 8 + 1 + \dots (a_6 - 1) \cdot 8 + 1 = (a_1 + \dots a_6) \cdot 8 - 6 \cdot 8 + 6 = 13 \cdot 8 - 6 \cdot 8 + 6 = 62$ teams are needed. But if there needs to be at least 4 packages delivered to somewhere, then our previous estimation needs to be increased by 1, so we would need 63 teams for 13 packages, but we only have 62. So, our only 3 solutions are 17, 9, 9, 9, 9, 9; 17, 17, 9, 9, 9, 1 and 17, 17, 17, 9, 1, 1.

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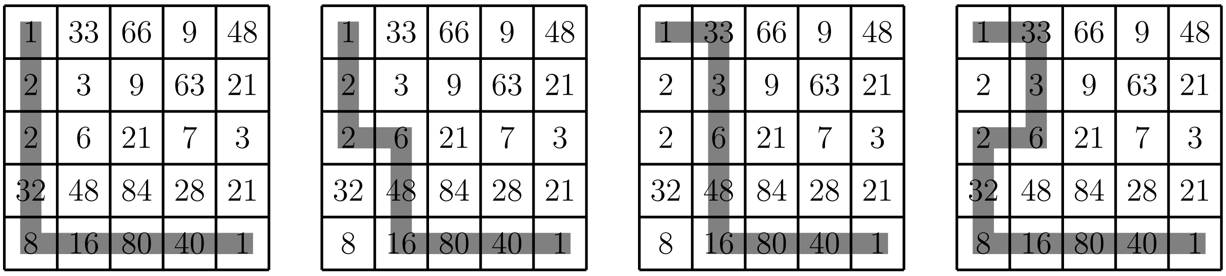
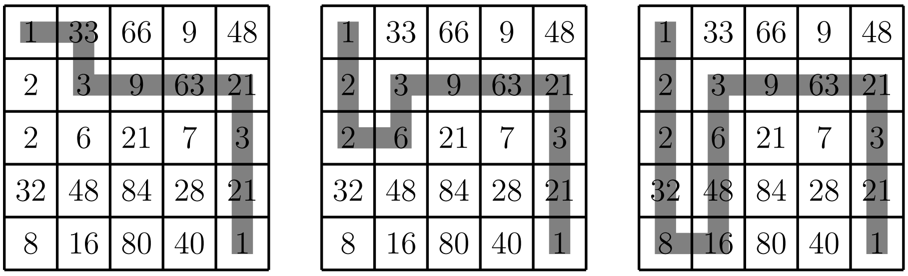
4. Thicken the borders between those fields which two don't have divisor-multiplication relation.

1	33	66	9	48
2	3	9	63	21
2	6	21	7	3
32	48	84	28	21
8	16	80	40	1

Next we have to determine, how many paths lead from the top left corner to the bottom right, which doesn't cross any of the thickened lines. The possible paths can be split up as we got there through number 40 or 21. Through number 40 there are 4 paths:



If we want to go through number 21, then 3 paths can be found:



In total, 7 paths are correct according to the conditions.

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5. If we tell that there is a hole or a cam on the left edge of one of the rows of the table, then in the given row the pieces are in a hole-cam or cam-hole position all to the end. Just like this, it is enough to tell what is at the bottom of the columns, it defines the entire column. So if we tell for each row if there is a hole or a cam at the left edge and for the bottoms of each column, then the position of each piece is determined. This will be an appropriate table, because on each piece there will be an opposite cam-hole pair, so on two adjacent sides there will be holes, on the other two cams, so there is only one way to rotate the specific piece to the correct position. So there are as many tables as many ways in which we can decide at the left edge of the rows and at the bottom of the columns how the pieces will be there.

a) Here we have 1 row and 4 columns, so we have 5 decisions with 2 possibilities for each, therefore the number of possibilities is $2^5 = 32$.

b) Here we have 4 rows and 5 columns, so we have 9 decisions with 2 possibilities for each, therefore the number of possibilities is $2^9 = 512$.

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6. For the solution, see Category C Problem 7.

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7. I'll call 100% gods gods, 0% gods humans, everyone else partly god.

a) As you go from one generation to the next, the difference between the number of humans and the number of gods always decreases by at most the number of partly gods. If two parents are the same (human, partly god, god), their children will be the same, it won't change the difference of humans and gods in the next generation. If a god and a human are the two parents, then the difference will also not change, as they will have two partly god children. If a partly god and a human or a partly god and a god are the parents, they will have two partly god children, in which case the difference between the number of humans and gods will decrease, or increase, by 1. Therefore, for the whole generation, the difference between the number of gods and humans can be reduced by at most the number of partly gods.

In the first generation there are no partly gods, in the second generation the at most 24 children of the 12 gods can be partly gods, in the third generation the at most 48 grandchildren of the god 12 can be partly gods. Originally the difference between the number of gods and humans is $88 - 12 = 76$, this can be reduced in the next three generations by at most the number of partly gods, in total $0 + 24 + 48 = 72$, so in the fourth generation the difference between the number of gods and humans is at least $76 - 72 = 4$. That means that the earliest generation where the number of gods and humans can be the same is the fifth. This is possible, for example, if in the first generation there are 12 god - human pairs and 38 human - human pairs, in the second generation there are 24 demigod - human pairs and 26 human - human pairs, in the third generation there are 48 quarter-god - human pairs and 2 human - human pairs, in the fifth generation there are 4 eighth-god - human pairs and 46 eighth-god - eighth-god pairs. Thus, in the fifth generation there will be 92 eighth-gods and 8 sixteenth-gods, there will be the same number (0) of humans and gods in this generation.

b) The sum of the parents' percentages of godness is equal to the sum of the percentages of godness of their children, since a couple has two children and the percentages of both are the average of the parents' percentages. In all 50 families this is true, so the children's generation always has the same god percentage as their parents'. So in every generation this sum is the same. In the first generation this sum is $12 \cdot 100 + 88 \cdot 0 =$1200, that is the sum in all generations. So in the fifth generation there can't be 49 or more at least 25% gods, because the sum would be $49 \cdot 25 = 1225 > 1200$, which is not possible. Thus, at most 48 members of the fifth generation can be at least 25% gods, and this is possible: for example, in the first two generations there are 6 god - god pairs and 44 human - human pairs, in the third there are 12 god - human pairs and 38 human - human pairs, in the fourth there are 24 demigod - human pairs and 26 human - human pairs, then in the 5th generation there will be 48 quarter-gods and 52 humans.

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2.2.3 Category E

1. For the solution, see Category C Problem 5.

(Back to problems)

2. For the solution, see Category D Problem 7.

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3. Let $k(x)$ be the number on the point x.

a) Answer: No.

Suppose indirectly that we can write numbers on the points so that for any two numbers on adjecent points, one divides the other. We can orient the segments so that if $k(a)$ divides $k(b)$, then the segment between them is oriented towards b.

We have a polygon with 9 sides. But in this case we must have two adjecent segments, which are oriented in the same way. This means, we have a, b, c numbers, such that $k(a) \mid k(b)$ and $k(b) \mid k(c)$. This also means, that $k(a) \mid k(c)$. So we got that any segments between these three points must be part of the drawing, but there is no triangle on the picture.

b) Answer: Yes.

Let us denote points as coordinates, such that if a point has coordinates (x, y), it means that the point lies in the xth coloumn from the left, and the yth row from the bottom.

First we write different primes on the points, let $p(x)$ be the prime on the point x. Then we take a point with even coordinate sum, and multiple this number with the numbers on the adjecent points, and write this number on the point instead. We do that for every point with even coordinate sum. This is a good constrution, so we need to prove if $k(a) \mid k(b)$ then a and b are connected. Firstly note that if two points are connected, their coordinate sums have different parities.

If the coordinate sum for both a and b is odd, they cannot be connected and $k(a)$ and $k(b)$ are different primes, so none of them divides the other.

If the coordinate sum for both a and b is even, they are again not connected, then lets look at the primes we first written to the poins, lets sign them $p(a)$ and $p(b)$. These points are not adjecent, because of the coordinates, so $p(a) \nmid k(b)$ and $p(b) \nmid k(a)$, which means $k(a) \nmid k(b)$ and vice versa.

The last case is when the coordinate sum of a is even, and the coordinate sum of b is odd. This means, that $k(b)$ is a prime, but $k(a)$ has more than one prime divisor, so $k(a) \nmid k(b)$. But it can happen that $k(b) \mid k(a)$, but this means that $p(b) \mid k(a)$, which is happens exactly when a and b are adjecent. And this is exactly what we wanted.

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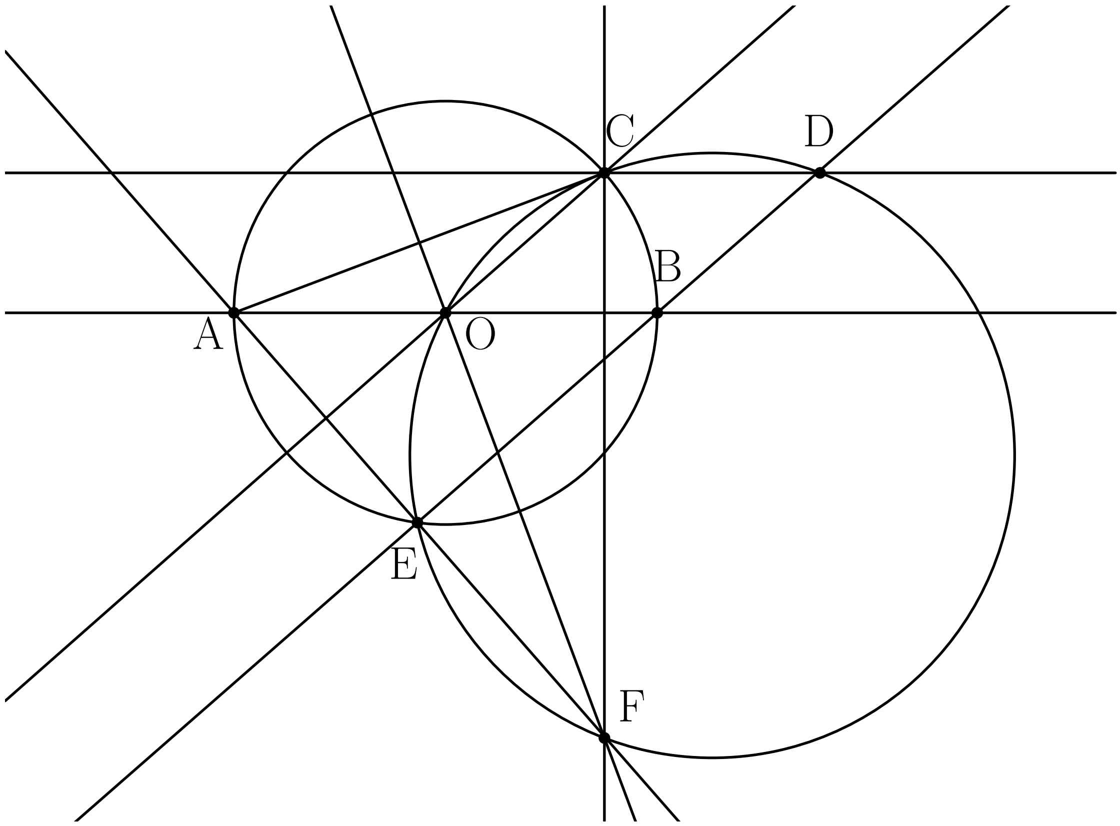
4. From the reverse of Thales's theorem, it is enough to prove that $FOD\angle = FED\angle = FCD\angle = 90°$, as all the five points would be on the circle with diameter FD.

By definition AF is an altitude in triangle AOC, so AF is perpendicular to CO. Also, as $OCDB$ is a parallelolgram, we know that CO and DB are parallel, so DB is perpendicular to AF. But as DBE is collinear, from the reverse of Thales's theorem, we know that AEF is also collinear. This implies that $DEF\angle = 90°$.

By definition CF is an altitude in triangle AOC, so CF is perpendicular to AO. Also, as $OCDB$ is a parallelogram, we know that AO and CD are parallel, so CD is perpendicular to CF. This implies that $DCF\angle = 90°$.

As OB and OC have the length of the radius of k, they have equal length. This implies that $OCDB$ is a parallelogram, with adjacent sides with equal lengths. This implies that $OCDB$ is a rhombus: its diagonals are perpendicular to each other, so BC is perpendicular to OD. As AB is a diameter in k, we know that BC is perpendicular to AC, so AC and OD are parallel. Also FO is an altitude in triangle AOC, so FO is perpendicular to AC. Then as AC and OD are paralalel, we know that FO is also perpendicular to OD, so $DOF\angle = 90°$. So we proved all the three desired right angles, which completes the proof.



(Back to problems)

5. Let the set of Timi's numbers be A and the numbers she had written down be B. The following solution uses graph theoretic notation but the only property of graphs it uses is that if a graph on m vertices has at least m edges, then it must contain a cycle (which you can prove for yourselves as an exercise). So don't get intimidated by the notation, just follow each of the logical steps carefully.

Let us regard the prime elements of set B, for all $p \in B$ pick a pair (x, y) from set A for which $xy = p$. Then we create the following graph G: the vertices are the elements of set A, and two vertices are connected by an edge if and only if (x, y) is a pair that we picked earlier. It is clear that the number of edges in G is exactly the number of prime elements in B.

Now we claim that in G there is no closed walk of even length with an edge that has been only visited once. For the contrary suppose that such a walk exists and let its vertices be $c_0, c_1, c_2 \dots c_{2k}$, where $c_0 = c_{2k}$. Let also $x_0, x_1, x_2 \dots x_{2k}$ be the numbers from A corresponding to vertices $c_0, c_1, c_2 \dots c_{2k}$ (meaning that $x_0 = x_{2k}$). Since $c_i c_{i+1}$ is an edge in G, $x_i \cdot x_{i+1}$ is prime for all $0 \le i < 2k$. Now let us consider the product $\prod_{i=1}^{2k} x_i$, we will write it two different ways:

$$\prod_{i=1}^{2k} x_i = \prod_{i=1}^{k} x_{2i-1} \cdot x_{2i} = \prod_{i=0}^{k-1} x_{2i} \cdot x_{2i+1}.$$

Since we know that the product of x_i and x_{i+1} is always prime, thus both $\prod_{i=1}^{k} x_{2i-1} \cdot x_{2i}$ and $\prod_{i=0}^{n-1} x_{2i} \cdot x_{2i+1}$ are the products of k (not necessarily different) primes. But since we assumed that there exists an edge of the walk that we only passed once, this prime would divide one of the products but not the other, which is a contradiction by the fundamental theorem of number theory.

Now we will show that G has at most n edges. For this it is enough to show that otherwise it contains a walk of even length with an edge that is visited only once. Suppose indirectly that G has at least $n+1$ edges. Then pick a component of G that has more edges than vertices. Since we can always find such a component by the pigeonhole principle, we can assume that G is connected.

It is clear that G cannot contain a cycle of even length, since then this would form an undersired walk. But since G has more than $n-1$ edges, it must contain a cycle. Let the vertices of this cycle be $c_0, c_1, c_2 \dots c_l$ where $c_0 = c_l$. Now erase the edge c_0c_1 from the graph. Since the graph will still contain at least $n-1$ edges, it will stay connected and we can find another cycle. Let the vertices of this cycle be $b_0, b_1 \dots b_k$ where $b_0 = b_k$. Since G remained connected after ereasing the edge c_0c_1, can find a path between c_0 and b_0, let its vertices be $c_0 = a_0, a_1, \dots a_m = b_0$. Since we know that G does not contain a cycle of even length, both n and k must be odd. Now consider the following closed path:

$$c_0, c_1 \dots c_l = c_0 = a_0, a_1 \dots a_m = b_0, b_1 \dots b_k = b_0 = a_m, a_{m-1}, \dots a_0 = c_0.$$

This is clearly a closed path and it has exactly $l+m+k+m$ vertices, which is even. We also know that the path passes through c_0c_1 only once since the second cycle was constructed without this edge in the graph. Therefore we have obtained a path of even length with an edge that has only been visited once, which is a contradiction.

Thus we have shown that G can have at most n edges, therefore B cannot have more than n prime elements. Now we just need to construct a set A for which B has exactly n prime elements.

Let $p_1, p_2 \dots p_n$ the first n primes and let the elements of A be as follows: $x_1 = \frac{\sqrt{p_3p_2}}{\sqrt{p_1}}, x_2 = \frac{\sqrt{p_1p_3}}{\sqrt{p_2}}$ and $x_3 = \frac{\sqrt{p_1p_2}}{\sqrt{p_3}}$. From this on define the elements of A recursively: let for every $n-1 \geq k \geq 3$ be $x_{k+1} = \frac{p_{k+1}}{x_k}$. (Since none of the x_k are 0, this recursive formula is valid). Then the products $x_2 \cdot x_3, x_3 \cdot x_1, x_1 \cdot x_2$ and $x_3 \cdot x_4, x_4 \cdot x_5 \dots x_{n-1} \cdot x_n$ equal to primes $p_1, p_2 \dots p_n$, therefore we have shown that there could be at most n distinct positive primes in B.

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2.2.4 Category E$^+$

1. Answer: Numbers of the form $x(x+1)$ and $x(x+2)$, where x is an arbitrary positive integer.

Solution: Let $\lfloor\sqrt{n}\rfloor = s$, where s is a positive integer. Then $s \leq \sqrt{n} < s+1$, so $s^2 \leq n < (s+1) = s^2+2s+1$. Since n and s are integers, $n \leq s^2+2s = s(s+2)$, so $s^2 \leq n \leq s(s+2)$, this means $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{n}{s}\rfloor$. We look for n positive integers for which $s + \lfloor\frac{n}{s}\rfloor > 2\sqrt{n}$. There are two cases depending on whether or not s divides n.

If s divides n, $s^2 \leq n \leq s(s+2)$ so $n = s^2$, $n = s(s+1)$ or $n = s(s+2)$. For the first of these $\sqrt{n} = s = \lfloor\sqrt{n}\rfloor = \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor$, so that $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = 2\sqrt{n}$, there is equality. If $n = s(s+1)$, then $\lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = \lfloor\frac{s(s+1)}{s}\rfloor = s+1$. Thus $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor = s+(s+1) = 2s+1 = \sqrt{4s^2+4s+1}$ and $2\sqrt{n} = \sqrt{4s^2+4s}$, so $\lfloor\sqrt{n}\rfloor + \lfloor\frac{n}{\lfloor\sqrt{n}\rfloor}\rfloor > 2\sqrt{n}$ for the numbers of the form $s(s+1)$, where

s is a positive integer. If $n = s(s+2)$, then $\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = \lfloor \frac{s(s+2)}{s} \rfloor = s+2$. Thus $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = s + (s+2) = 2s+2 = \sqrt{4s^2+8s+4}$ and $2\sqrt{n} = \sqrt{4s^2+8s}$, so $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor > 2\sqrt{n}$ for the numbers $s(s+2)$, where s is a positive integer. (In all cases we use that out of two positive real number, the larger has larger square.)

If s does not divide n, from $s^2 \le n \le s(s+2)$ we get that $s^2 < n < s(s+1)$ or $s(s+1) < n < s(s+2)$. In the first case $\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = \lfloor \frac{n}{s} \rfloor = s$, so $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = s+s = 2s$, while $s^2 < n$. Then $s < \sqrt{n}$, so $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = 2s < 2\sqrt{n}$. In the second case $\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = \lfloor \frac{n}{s} \rfloor = s+1$, so $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = s + (s+1) = 2s+1$, while from $s^2+s < n$, $(s+0,5)^2 = s^2+s+0,25 < n$ because s^2+s and n are integers. Thus $s+0,5 < \sqrt{n}$, therefore $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor = 2s+1 < 2\sqrt{n}$. So if s does not divide n, the right-hand side, $2\sqrt{n}$ is the greater.

In summary for a positive integer n, $\lfloor\sqrt{n}\rfloor + \lfloor \frac{n}{\lfloor\sqrt{n}\rfloor} \rfloor > 2\sqrt{n}$ is satisfied exactly if $n = x(x+1)$ for some positive integer x or $n = y(y+2)$ for some positive integer y.

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2. Direct the edges of our graph in a way such that each edge points towards the number which is divisible by the number on the start of the edge. A finite graph is divisive if and only if the edges can be directed in a way that the directed graph exactly forms the relationships of a partially ordered set. Our first observation is that a divisive graph can not contain an odd cycle of length at least 5 as an induced subgraph. This is because two adjacent edges would be directed in the same direction hence creating a new edge (a diagonal of the cycle) and so the cycle wouldn't be an induced subgraph.

Now we will answer the cases one by one:

- **Tetrahedron:** We can get a construction by writing different powers of 2 on all nodes.
- **Hexahedron:** This is a bipartite graph, so let us form two classes of nodes such that there is no edge running inside any of the classes. We write different prime numbers on the nodes of the first class, and we write the product of the three adjacent primes to the nodes of the second class.
- **Octahedron:** If the octahedron is put on one of its vertices, there is a single top and bottom vertex and there is a square in the middle layer. The four nodes of the square shall be filled with numbers $2, 12, 3, 18$ respectively. The bottom vertex shall be given number $36 \cdot 5 = 180$ and we assign $36 \cdot 7 = 252$ to the top vertex. This way, all nodes in the middle layer will divide the top and bottom vertices, but these two won't divide each other.
- **Dodecahedron:** The graph of the dodecahedron is not divise, beacause it contains a pentagon as an induced subgraph (any 5 points lying on the same face of the hexagon form a pentagon).
- **Icosahedron:** The graph of the icosahedron is not divise, beacause it contains a pentagon as an induced subgraph (the neighbours of any vertex form a pentagon in this case).

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3. Let us regard the prime elements of set B, for all $p \in B$ pick a pair (x, y) from set A for which $xy = p$. Then we create the following graph G: the vertices are the elements of set A, and two vertices are connected by an edge if and only if (x, y) is a pair that we picked earlier. It is clear that the number of edges in G is exactly the number of prime elements in B.

Now we claim that in G there is no closed walk of even length with an edge that has been only visited once. For the contrary suppose that such a walk exists and let its vertices be $c_0, c_1, c_2 \dots c_{2k}$, where $c_0 = c_{2k}$. Let also $x_0, x_1, x_2 \dots x_{2k}$ be the numbers from A corresponding to vertices $c_0, c_1, c_2 \dots c_{2k}$ (meaning that $x_0 = x_{2k}$). Since $c_i c_{i+1}$ is an edge in G, $x_i \cdot x_{i+1}$ is prime for all $0 \leq i < 2k$. Now let us consider the product $\prod_{i=1}^{2k} x_i$, we will write it two different ways:

$$\prod_{i=1}^{2k} x_i = \prod_{i=1}^{k} x_{2i-1} \cdot x_{2i} = \prod_{i=0}^{k-1} x_{2i} \cdot x_{2i+1}.$$

Since we know that the product of x_i and x_{i+1} is always prime, thus both $\prod_{i=1}^{k} x_{2i-1} \cdot x_{2i}$ and $\prod_{i=0}^{n-1} x_{2i} \cdot x_{2i+1}$ are the products of k (not necessarily different) primes. But since we assumed that there exists an edge of the walk that we only passed once, this prime would divide one of the products but not the other, which is a contradiction by the fundamental theorem of number theory.

Now we will show that G has at most n edges. For this it is enough to show that otherwise it contains a walk of even length with an edge that is visited only once. Suppose indirectly that G has at least $n + 1$ edges. Then pick a component of G that has more edges than vertices. Since we can always find such a component by the pigeonhole principle, we can assume that G is connected.

It is clear that G cannot contain a cycle of even length, since then this would form an undersired walk. But since G has more than $n - 1$ edges, it must contain a cycle. Let the vertices of this cycle be $c_0, c_1, c_2 \dots c_l$ where $c_0 = c_l$. Now erase the edge $c_0 c_1$ from the graph. Since the graph will still contain at least $n - 1$ edges, it will stay connected and we can find another cycle. Let the vertices of this cycle be $b_0, b_1 \dots b_k$ where $b_0 = b_k$. Since G remained connected after ereasing the edge $c_0 c_1$, can find a path between c_0 and b_0, let its vertices be $c_0 = a_0, a_1, \dots a_m = b_0$. Since we know that G does not contain a cycle of even length, both n and k must be odd. Now consider the following closed path:

$$c_0, c_1 \dots c_l = c_0 = a_0, a_1 \dots a_m = b_0, b_1 \dots b_k = b_0 = a_m, a_{m-1}, \dots a_0 = c_0.$$

This is clearly a closed path and it has exactly $l + m + k + m$ vertices, which is even. We also know that the path passes through $c_0 c_1$ only once since the second cycle was constructed without this edge in the graph. Therefore we have obtained a path of even length with an edge that has only been visited once, which is a contradiction.

Thus we have shown that G can have at most n edges, therefore B cannot have more than n prime elements. Now we just need to construct a set A for which B has exactly n prime elements.

Let $p_1, p_2 \dots p_n$ the first n primes and let the elements of A be as follows: $x_1 = \frac{\sqrt{p_3 p_2}}{\sqrt{p_1}}, x_2 = \frac{\sqrt{p_1 p_3}}{\sqrt{p_2}}$ and $x_3 = \frac{\sqrt{p_1 p_2}}{\sqrt{p_3}}$. From this on define the elements of A recursively: let for every $n - 1 \geq k \geq 3$ be $x_{k+1} = \frac{p_{k+1}}{x_k}$. (Since none of the x_k are 0, this recursive formula is valid). Then the products $x_2 \cdot x_3, x_3 \cdot x_1, x_1 \cdot x_2$ and $x_3 \cdot x_4, x_4 \cdot x_5 \dots x_{n-1} \cdot x_n$ equal to primes $p_1, p_2 \dots p_n$, therefore we have shown that there could be at most n distinct positive primes in B.

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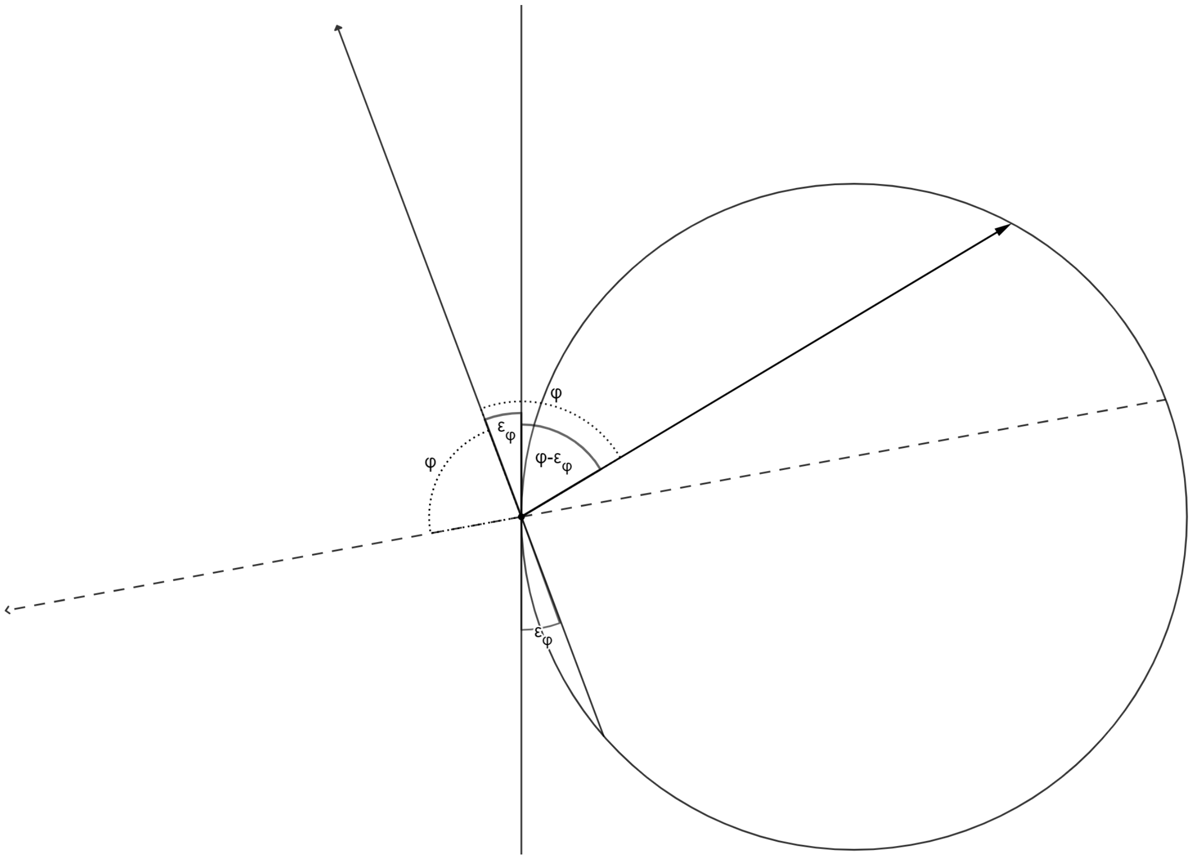
4. The answer is the ant only returns if $\varphi = 90°$ or $\varphi = 180°$. In the second case, after it reaches the edge of the disc, it goes back on the same chord, and will return to its initial position, regardless of its starting position and initial direction. If $\varphi = 90°$, we have two cases depending on whether it turns once or twice the first time it reaches the edge. It cannot turn more, as after two turns, the direction is pointing back to just where it came from. Clearly, if it turns twice, it is the same case as when $\varphi = 180°$.

In case it turns only once, its path will be the inscribed rectangle which has initial chord (on which the ant starts moving) as a side. This is because initially it is true, and whenever it reaches the edge, it can only continue its journey on the same line as it came from or on the other side of the rectangle at that point. After reaching the edge the first time it will travel in the negative direction on the perimeter of the rectangle.

Now we prove that if φ is not $90°$ or $180°$, then there is a starting point and direction so that the ant never returns there. For a contradiction assume that for a certain such φ the ant returns to its starting point regardless of the starting point and direction.

Observe that two non-identical chords intersect at most at one point, that there are uncountably many points on a chord, and that the ant's journey consists of countably many chords. If we could choose a starting chord and direction along it so that the ant would never go on that chord, then we could choose a starting point on that chord that the other chords in the journey don't contain, contradicting our assumption. Therefore for this φ, for any starting point and initial direction, the ant has to return to the chord during its journey.

Since it must return an infinite numer of times because of this, there will be two instances when it goes in the same direction through the chord. Without loss of generality, we may assume that it is the n^{th} chord when it first travels through the initial chord again in the initial direction. It is easy to see that the $(n+k)^{\text{th}}$ chord will be the same as the $(1+k)^{\text{th}}$ chord, since the chord and direction determine the next chord and direction. Because of this, whenever the ant returns to the first chord in the initial direction it must come from the same chord and direction. So if we can show a second chord, which would be the second chord for two different initial chords, we get a contradiction, as only one of those two chords can be in the cycle we get when we start the ant from the second chord. Starting from the other initial chord, the ant gets into the cycle which doesn't include that initial chord. Contradiction.



Draw the tangent to the disc from an arbitrary edge point and measure the angles as in the above figure, where ε_φ is small enough. The figure will be similar to the above (except for that the dashed and the vector chords may coincide or can switch order) when $\varphi - \varepsilon_\varphi$, ε_φ and φ are positive and $\varepsilon_\varphi + \varphi < 180°$. For example $\varepsilon_\varphi = \min\left(\frac{\varphi}{2}, \frac{180°-\varphi}{2}\right)$ works. In this case, we can see that using the dashed or solid chord as the initial chord and directing the initial direction towards the point results in the vector chord as the second chord. Contradiction.

Note: The reason why the above proof doesn't work for 90° is that that is the case when the dashed and the vector lines coincide, or equivalently when the initial chord is also the second chord. In this case, this is the chord from which it never returns *directly* to the second chord, but it agrees with it. This is only true when $2\varphi = 180°$, so no other cases are affected.

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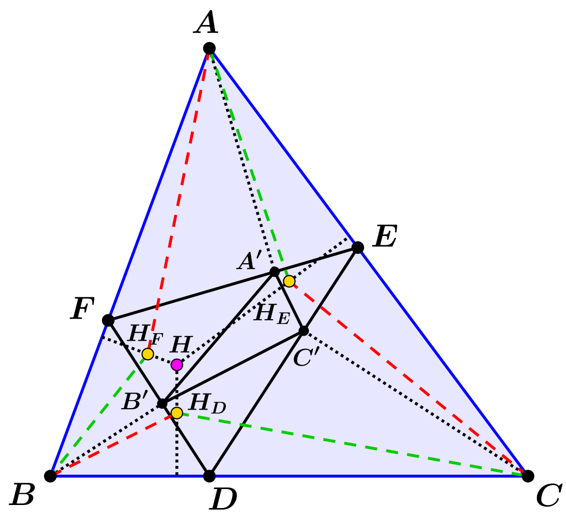
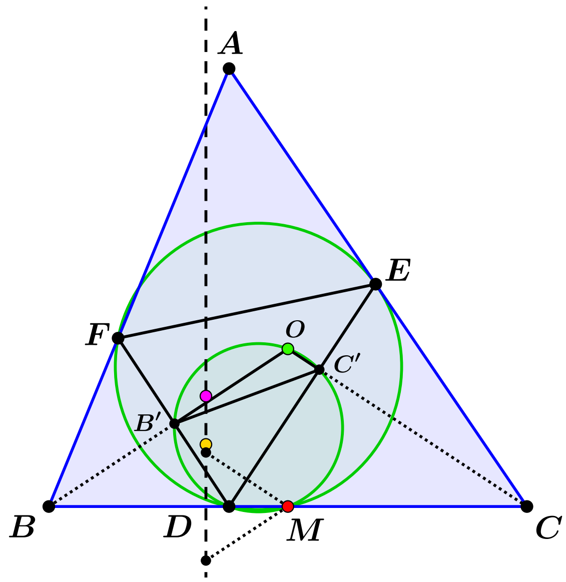
5. First of all, let's create a diagram. Denote by H the orthocenter of triangle DEF. We are going to prove that $HH_D \perp BC$, $HH_E \perp CA$ and $HH_F \perp AB$.





By symmetry, it is enough to prove that $HH_D \perp BC$. Let M be the midpoint of side BC. We will show that the Miquel point of quadrilateral $B'C'EF$ (that is the second intersection of circles $(DB'C'), (DEF)$) is in fact, point M. It is well-known that the nine-point circle of a triangle passes through the feet of the altitudes and the midpoints of the sides. Thus, circle (DEF) goes through M. Denote by O the circumcenter of triangle ABC. We will first show that lines BB', CC' intersect at O. Again by symmetry, it suffices to show that B, B', O are collinear or its equivalent form: $\angle DBB' = \angle CBO$. Since quadrilateral $ACDF$ is cyclic, applying the inscribed angles theorem we get $\angle DBB' = 90° - \angle B'DB = 90° - \angle FDB = 90° - \angle BAC = \dfrac{180° - 2\angle BAC}{2} = \dfrac{180° - \angle BOC}{2} = \angle CBO$ because $BO = CO$. (One can show the previous collinearity by noticing that in the cyclic quadrilateral $ACDF$ lines AC, DF are antiparallel and lines BO, BH' are isogonal - where H' denotes the orthocenter of triangle ABC - so $BH' \perp AC \implies BO \perp DF$). Therefore, both B' and C' lies on the circle with diameter DO, which passes through M as O lies on the bisector of segment BC.

It is well-known that in a triangle with orthocenter K, the Simson-line of a point P on the circumcircle halves segment PK. Since M lies on circles $(DB'C'), (DEF)$, the reflections of M onto lines DE, DF lie on HH_D. Also, it is known that in triangle DEF the inner angle bisectors are precisely the altitudes, while the external angle bisectors are precisely the sidelines of triangle ABC. Thus, M lies on the external angle bisector of $\angle EDF$, implying that the line connecting the reflections of M onto lines DE, DF is perpendicular to the bisector (since if we reflect any of the reflections on the bisector, we must get the other reflection). Indeed, $HH_D \perp BC$.

To finish the proof, we will use a well-know lemma stating that $PQ \perp AB \iff PA^2 - PB^2 = QA^2 - QB^2$ (the proof of which is an easy application of the Pythagorean theorem). This means that

$$\begin{array}{rcll}
HH_D \perp BC & \implies & HB^2 - HC^2 = H_DB^2 - H_DB^2 & \\
HH_E \perp CA & \implies & HC^2 - HA^2 = H_EC^2 - H_EA^2 & \\
HH_F \perp AB & \implies & HA^2 - HB^2 = H_FA^2 - H_FB^2 & + \\
\hline
 & & 0 = H_DB^2 - H_DC^2 + H_EC^2 - H_EA^2 + H_FA^2 - H_FB^2. &
\end{array}$$

Therefore, $H_DB^2 + H_EC^2 + H_FA^2 = H_DC^2 + H_EA^2 + H_FB^2$.

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2.3 Final round – day 1

2.3.1 Category C

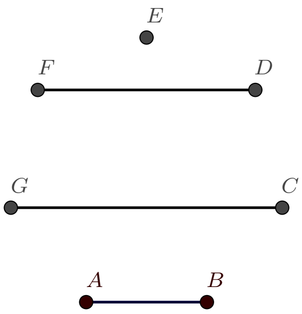
1. a) Denote the six demigods A, B, C, D, E and F. The first night there were two shows: A, B, C and D were on one, and E and F were on the other. After that, there were four performances every night: two of A, B, C and D were always alone, and two were paired with E and F. For example like this:

2. $\{A\}$, $\{B\}$, $\{C, E\}$, $\{D, F\}$
3. $\{A\}$, $\{B\}$, $\{C, F\}$, $\{D, E\}$
4. $\{C\}$, $\{D\}$, $\{A, E\}$, $\{B, F\}$
5. $\{C\}$, $\{D\}$, $\{A, F\}$, $\{B, E\}$

b) Denote the seven demigods A, B, C, D, E, F and G. Every night they go to the performances in pairs, except for one person who is alone. For example, the following schedule is good:

1. $\{A, B\}$, $\{C, G\}$, $\{D, F\}$, $\{E\}$
2. $\{B, C\}$, $\{A, D\}$, $\{G, E\}$, $\{F\}$
3. $\{C, D\}$, $\{B, E\}$, $\{A, F\}$, $\{G\}$
4. $\{D, E\}$, $\{C, F\}$, $\{B, G\}$, $\{A\}$
5. $\{E, F\}$, $\{D, G\}$, $\{C, A\}$, $\{B\}$
6. $\{F, G\}$, $\{E, A\}$, $\{D, B\}$, $\{C\}$
7. $\{G, A\}$, $\{F, B\}$, $\{E, C\}$, $\{D\}$

The figure below shows what the logic is in this construction. We marked with a line which couples went to a performance on the first night. If we start rotating these three lines to the left at the same time, we get the schedule of the second night, and so on. It can be verified that any pair of demigods will pair exactly once during the rotation.



c) Let the eight demigods be A, B, C, D, E, F, G and H. The allocation should be the same as in task **b)** , with the change that H is always associated with the lone demigod.

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2. If we add up all the numbers at the end of the rows and columns, we get twice of the sum of the numbers in the table, because each number belongs to a column and a row. For this reason, we get an even number, which means that Benedict must have written down an even number. This means that 0, 2, 4, 6, 8 or 10 is the number written by Andris. These six cases are actually possible. For example, if we consider the six tables where all the numbers in the table except one diagonal are even and there are 5, 4, 3, 2, 1 or 0 odd numbers in this diagonal, then Andris sees 0, 2, 4, 6, 8 or 10 even numbers in this order. The number Andris describes could therefore be 0, 2, 4, 6, 8 or 10.

Examples of the possible cases:

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Andris száma 0

2	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Andris száma 2

2	1	1	1	1
1	2	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Andris száma 4

2	1	1	1	1
1	2	1	1	1
1	1	2	1	1
1	1	1	1	1
1	1	1	1	1

Andris száma 6

2	1	1	1	1
1	2	1	1	1
1	1	2	1	1
1	1	1	2	1
1	1	1	1	1

Andris száma 8

2	1	1	1	1
1	2	1	1	1
1	1	2	1	1
1	1	1	2	1
1	1	1	1	2

Andris száma 10

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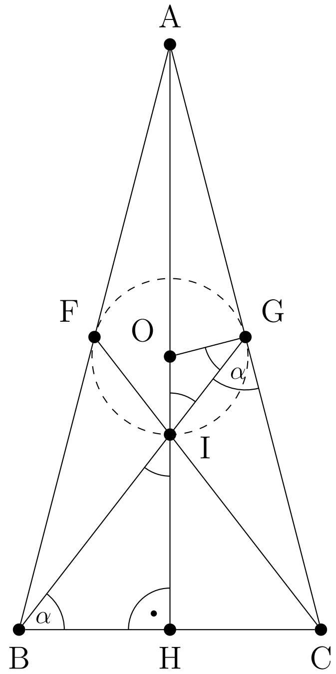
3. Let the midpoint of BC be H, and the intersection of BG and CF be I. Mark O as the center of the FGI triangle's circumscribed circle. Additionally, $\alpha = CBG\sphericalangle$. It is sufficient to prove, that $OGC\sphericalangle = 90°$, because the FGI circle will be tangent to the ABC triangle's legs, thus I falls on the circle k.
Since F and G are midpoints, $BF = CG = BC = 1$ cm. Thus the BCG tirangle is isosceles, from which we can conclude, that $IGC\sphericalangle = \alpha$.
The figure is symmetrical to line BC, so lines IH and BC are perpendicular. From these, $BIH\sphericalangle = 90° - \alpha$, thus $OIG\sphericalangle = 90° - \alpha$. Also because O is the center of the FGI triangle's circumscribed circle, $OGI\sphericalangle = OIG\sphericalangle = 90° - \alpha$. Hence, $OGC\sphericalangle = OGI\sphericalangle + IGC\sphericalangle = 90° - \alpha + \alpha = 90°$. Thus we finished our proof.



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4. Let's notice, that all the numbers that can be on the board are 1 less than a square number. 3 and 24 are such numbers, so we have to see, that if before a move, all the numbers on the board are 1 less than a square number, then in that move the number written by Andris will be like this as well. If k and n are both 1 less than a square number and they are on the board, Andris will write $kn + k + n = (k + 1)(n + 1) - 1$ on the board, then it will also be 1 less than a square number, because $k + 1$ and $n + 1$ are square numbers, so their multiplication will be too.

a) According to our proof above he cannot reach it 9999999, because $9999999+1 = 10000000$ is not a square number.

b) He can reach it through the following steps: out of 3 and 24 he gets $3 \cdot 24 + 3 + 24 = 99$, then from 99 he gets $99{\cdot}99+99+99 = 9999$, finally out of it $9999{\cdot}9999+9999+9999 = 99999999$.

c) Lets's notice, that the only prime divisors of the numbers that are exactly 1 bigger than the numbers on the board, are 2 and 5. This can be proved similarly as the fact that these numbers are also square numbers. The two starting numbers (3 and 24) fulfill the statement, and if before a move, the condition holds true for all the numbers on the board, then the number that Andris writes down in this move will be $kn + k + n = (k + 1)(n + 1) - 1$, where k and n were already on the board, and all of its prime divisors will be 2 and 5, because the prime divisors of $k + 1$ and $n + 1$ are only 2 and 5, so their multiplication's prime divisors will also only be 2 and 5. According to this, Andris cannot get $48999999 = 49000000 - 1$, because 49000000 is divisible by 7 too.

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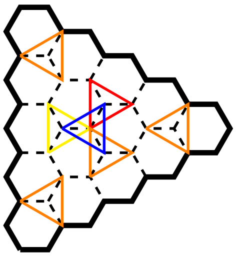
5. Jóska can turn shape **a)** completely black but he cannot turn shapes **b)** and **c)** entirely black.

a) We will indicate every step with a small triangle. Since every hexagon changed colour 1 or 3 times, all of them will be black at the end of these steps. The order of the steps does not matter, the triangles are coloured differently only to make them more transparent.

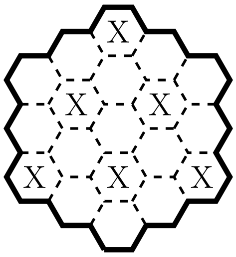
b) and **c)** We put X's in both shapes in a way that if we place our small shape of three hexagons on **b)** or **c)** in any way, it will cover exactly one field with X and two without X. At first, the shapes are empty. Whenever we take a step, we put a matchstick on every field that changed colour and does not have an X on it. So, in every step the number of matchsticks on the shape increases by two, so the total number of matchsticks on the shape will always be even.

If we could turn the entire shape black, there would be an odd number of matchsticks at every field, since we add a matchstick to a field without X if and only if the field changes colour. Since there are an odd number of fields without X in both cases, we had to put down an odd number of matchsticks in total. However, we also know, that the number of matchsticks on the shapes is always even in total. We reached a contradiction, so these shapes cannot be turned entirely black.

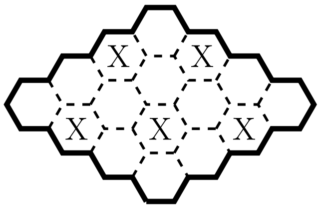
a)



b)



c)



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6. We call three cells a *bastion* if they are arranged in a way that in each row and in each column there is exactly one cell from the bastion. If during the game, there is a bastion in which the three cells contain the three different numbers, the first player wins, because we can easily see that this means that only one number would fit in each remaining cell, so the table will be completely filled. We call this type of bastion *mixed*.

1	3	2
2	1	**3**
3	**2**	1

The first player has a winning strategy since they will always be able to make a mixed bastion. For example, we assume that the first player writes a 1 in the cell in the middle in their first step. If the second player writes a 2 or a 3 in a cell in a corner, in their second step, the first player can already create a mixed bastion and wins a few steps later.

If the second player writes a 1 in a cell in a corner, then the first player has to write a 1 in the opposite cell in the corner to create a diagonal with only 1's. After that, wherever the second player writes a number, we can see, that the first player will be able to complete a mixed bastion, and a few steps later, they win.

	1	
3		1
1		**2**

If the second player writes a number in a cell that is not in the corner, then the first player should complete the row or column which already has two filled cells. Then, wherever the second player writes a number, the first player can complete a mixed bastion in their next step. So, the first player can complete a mixed bastion in all cases.

1		**3**
2		
3	**1**	

So, the first player can always win.

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2.3.2 Category D

1. Let the number we are examining be a and denote the number of its divisors by $d(a)$. Let us examine the divisors of a. Let's put them in 3 groups:

1. group: 1
2. group: a
3. group: all other divisors of the number a

The number in the first group has 1 divisor. The numbers in the last group must have two divisors, because there are themselves and the number 1.

From this we get that the sum of those divisors of the number a that are less than a is at least $1 + 2 \cdot (d(a) - 2) = 2 \cdot d(a) - 3$, that is, if $d(a) > 4$, then equality with $d(a) + 1$ can never hold. So it is enough to examine the cases $d(a) \leq 4$.

If $d(a) = 4$, then a can have two types of prime factorizations, $a = p \cdot q$ or $a = p^3$, where p and q are different prime numbers. Here, the first case works, because the number of divisors of the number smaller than itself is 5, while the second case is incorrect, because there this number is 6.

If $d(a) = 3$, then a can only have one primefactorizatiom, which is $a = p^2$, which is not a solution because it has a divisor of 3 and smaller divisors

If $d(a) = 2$, then it can have only one prime factorisation, $a = p$, for which the problem condition is not fulfilled.

If $d(a) = 1$, then a can only be 1 for which the problem condition is again not satisfied.

So we got that the numbers with a prime factorization of $p \cdot q$ are the right ones.

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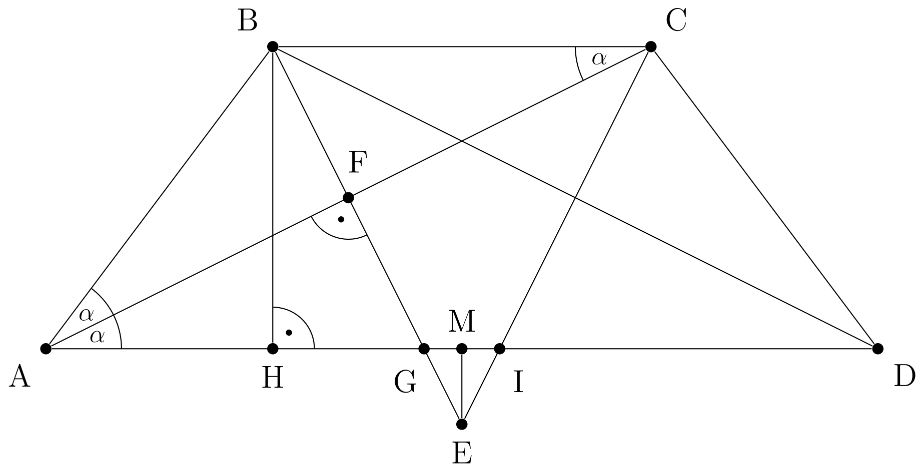
2. We mark line BE's intersection with line AC and line AD with F and G, respectively. We draw the line that is perpendicular to AD and contains B and we mark the point where it intersects AD with H. We mark the midpoint of AD with M.

Since $ABCD$ is an isosceles trapezoid, the perpendicular bisector of segment AD, the line that contains M and is perpendicular to AD, contains E. We also know that triangle ABC is isosceles, so F is the midpoint of segment AC. Because of that, triangles AFG and CFB are congruent, since one pair of their sides (AF and CF) are equal and their angles are also equal because angles FAG and FCB are alternate angles as well as angles AGF and CBF and angles AFG and BFC are vertically opposite angles. So, $AG = BC = 5cm$. Since $ABCD$ is an isosceles trapezoid, $AH = \frac{11-5}{2}cm = 3cm$. Because $AM = \frac{AD}{2} = \frac{11}{2}cm$, $MG = MA - AG = \frac{11}{2} - 5 = \frac{1}{2}cm$ and $GH = GA - AH = 5 - 3 = 2cm$.

Based on the Pythagorean theorem, in triangle ABH, $HB = 4cm$. Because of the right angles and the vertically opposite angles, triangles HBG and EMG are similar to each other. The ratio of similarity is $\frac{HG}{GM} = \frac{2}{0.5} = 4$. So, $\frac{BH}{EM} = 4$ as well which means that since $BH = 4cm$, $EM = 1cm$. Since EM is perpendicular to AD, the length of EM is the distance of point E from line AD, so the answer is 1 cm.



Alternative conclusion: we know that the length of segment GI is $11 - 2 \cdot 5 = 1cm$. BC and GI are parallel to each other, so triangles EGI and EBC are similar to each other. Their ratio of similarity is $\frac{BC}{GI} = 5$, so the ratio of their altitudes is also 5. We also know that the difference between their altitudes is the height of the trapezoid, which is 4 cm, so the bigger altitude is 5 cm and the smaller one is 1 cm.

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3. Let p_n denote the nth pentagonal number. Then we have $p_{n+1} - p_n = 3n + 1$, as the new points are the 4 new vertices of the pentagon and the $3 \cdot (n-1)$ points on the interior of the the 3 new edges, and $4 + 3 \cdot (n-1) = 3n + 1$.

This means all natural numbers that are greater than 1 and give 1 modulo 3 can be written as the difference of two neighbouring pentagonal numbers.

On the other hand, the mod 3 residue of pentagonal numbers always increase by 1 with n. The mod 3 residue of the first pentagonal number is 1, so for each $3m + 1$ for some m, p_{3m+1}

will give 1 modulo 3. As all numbers that give 1 modulo 3 can be written as a difference of two neighbouring pentagonal numbers, so can be all p_{3m+1}. So we can find a pentagonal number of any magnitude (and of form p_{3m+1}) that can be written up as a difference of two pentagonal numbers, hence there are infinitely many pentagonal numbers that can be written up as the sum of two pentagonal numbers.

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4. For the solution, see Category C Problem 5.

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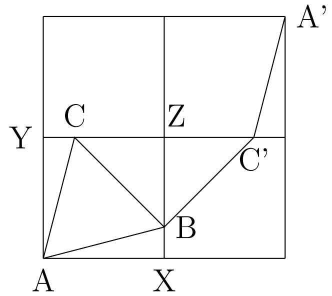
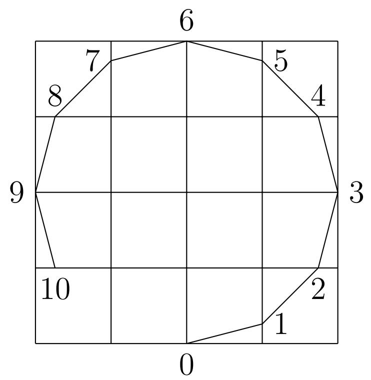
5. Let's start from an ABC triangle in a unit square, where $XAB\sphericalangle = CAY\sphericalangle = 15°$, see figure. Then ACY and ABX triangles are congruent, so $AC = AB$ and because $CAB\sphericalangle = 90° - 15° - 15° = 60°$, then ABC is a regular triangle. Furthermore because of symmetry $CBZ\sphericalangle = 45°$.

Hanga starts at the AB segment. If we reflect C on line XZ, we get C'. $ABC'\sphericalangle = 60° + 45° + 45° = 150°$, so Hanga will exactly move on segment BC'.

If we reflect segment AC on Z, the reflection of C will be C', and the reflection of A will be A'. $A'C'B\sphericalangle = A'C'Z\sphericalangle + ZC'B\sphericalangle = ACZ\sphericalangle + ZC'B\sphericalangle = 75° + 45° = 150°$, so Hanga will continue her walk on segment $C'A'$. After the next 30° turn she will be turned by 90 degrees compared to the starting position, and because she is on a grid point, we simply need to turn the table by 90 degrees to follow her path.



As we go on, the length of every segment will remain the same (because they are reflected sides of a regular triangle), and the internal angle is always 150° so all the points are on a circle. The radius of this circle is 2 cm so the distance of points 0 and 10 is also 2 cm, because we can divide the circle into 6 regular triangles, where one of the vertices of the triangles is the center of the circle, and the other vertices are adjacent even numbers.



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6. We mark with a and b the number of pucks in the two piles, and we mark this position with (a, b). We call a position winning position if by stepping on it, a player can certainly win in some steps.

Claim: Position (a, b) is a winning position if and only if $|a - b| \leq 1$.

Proof of this claim:

We have to prove that if a position is a winning position according to our claim, then the next player can only step on a position that is not a winning position according to our claim and that if a position is not a winning position, then the next player can always step on a winning position. This would result in a complete strategy for the game. If the starting position is not a winning position, then we should be the first player and only step on the winning positions and if the starting position is a winning position, then we should let the organizers go first.

1. case: $|a - b| \leq 1$.

In a step, we put m pucks into one pile and take $2m$ pucks from the other pile for some positive integer m. If we take pucks from the first pile, then from position (a, b) we get to position $(a - 2m, b + m)$, and if we take pucks from the second pile, then from position (a, b) we get to position $(a + m, b - 2m)$. In the first case, the value of $a - b$ decreases by $3m$ and in the second case, it increases by $3m$. So, in any step, it changes with at least 3 since m is a positive integer. So, if the value of $a - b$ is -1, 0 or 1 before a step, then it cannot be -1, 0 or 1 after the step.

2. case: $|a - b| \geq 2$.

Since the positions of a and b are symmetrical, we can assume that $a > b$. Let $a - b = k$ where we know that $k \geq 2$. Let $k = 3m + r$ where m is an integer and r is -1, 0 or 1. Then, $m > 0$ and if we take away $2m$ from the pile a with and add m pucks to the pile with b pucks, the difference between the two piles will be $|r|$ which is not more than 1.

With that, we proved our statement and provided a complete strategy for winning the game.

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2.3.3 Category E

1. For the solution, see Category C Problem 3.

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2. For the solution, see Category C Problem 4.

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3. **a)** Let the four merchants be A, B, C, D and suppose, that they left in alphabetical order.

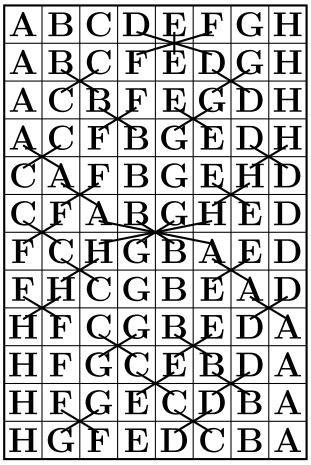
Because each night the number of the inns they sleep in is even, every night they sleep in either 2 or 4 inns. If they sleep in 4, then everyone is sleeping seperately. In this case neither of who they slept with at the same place, nor the order of the merchants change, so we can assume, that it never happens. Therefore each night they sleep in $2 + 2$ or $3 + 1$ distribution.

First assume, that in the first night they sleep in $3+1$ distribution. Because of symmetry it can be assumed, that from options $ABC+D$ and $A+BCD$ the first is correct, so their order on the next day changes to C, B, A, D. The next overtake has to be between D and A. But from A, B and C neither of them can sleep at the same place, so the next night their distribution will be $C+B+AD$, hence they are in 3 inns, which is incorrect according to the conditions.

Assume now, that in the first night they sleep in distribution $2+2$. So they have to sleep in distribution $AB+CD$, hence their order on the next day is B, A, D, C. But A and B, respectively C and D can sleep no more together, hence on the next night only $B+AD+C$ can be the distribution. But here they sleep in 3 different inns again. This way we can see, that they cannot get to Rome with these conditions.

b) Use the table below to mark the movements of the merchants. Each row shows the movements of that day. Those merchants, who changed position, are connected with a line to their position of the next day. If some of theese lines cross each other, than those merchants the night between the two days spend together in the same inn.

See the construction in the figure. It is easy to check that it is appropriate. We write a way of thinking, which can ease finding another similar construction, and on the other hand also works for even numbered merchants.



The solution's structure is the following: first we make a construction for 6 merchants; for example we get a solution, if from the contruction above we leave merchants A and H. We can easily check, that it still fulfills the problem's condition, as every night the number of the inns is even. Out of 8 (and similarly to any $2n$ merchants) we get a solution, if we give 1-1 merchants to the left and right sides of the 6 merchants (generally $(n-3)$ to each side), in this case A and H. Then until that day, when C and F, respectively B and G sleep in the same inn, we don't change the order of the merchants. Before CF and BG changes are done, we put merchants A and H by changes in the middle (generally all the added ones), then instead of the change of BG we do an order change on $ABGH$ (generally all the ones who we put in the middle, we turn them over).

Then merchants A and H (and all the added merchants) will be taken by changes to the sides, than the changes of the 6 middle merchants will be ended as we can see in the figure, at this point the positions of the merchants will no longer be modified.

This way the conditions are fulfilled, we managed to turn over the order of the merchants, if originally the number of the merchants is even and at least 6.

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4. We mark for a given n the k. n-sided polygonal number with a_k. Then, we know that $a_{k+1} = a_k + k \cdot (n-2) + 1$ since we get a_{k+1} from a_k by adding the new points from the regular n-sided polygon which has k long sides, the vertices and the points that break up the sides into k exactly 1 long segments that were not present in the figure before. We can calculate the number of these new points by adding up all of the points of this new polygon, which is $n \cdot k$, and subtracting the number of already existing points on the figure, which is $k + (k-1)$. From this, we can conclude that n-sided polygonal numbers increase strictly, so for a given n, all of the n-sided polygonal numbers are different. We can also observe that the remainder from the division $a_{k+1} : (n-2)$ is greater than the remainder from the division $a_k : (n-2)$ by 1. So, for all n and N, exists a k_1 for which the remainder of the division $a_{k_1} : (n-2)$ is 1 and $k_1 \geq N$. Let $k_2 = \frac{a_{k_1}-1}{n-2}$. Then k_2 is an integer. So, $a_{k_2+1} = a_{k_2} + a_{k_1}$, since for all k, $a_{k+1} = a_k + k \cdot (n-2) + 1$ which means that for k_2, $a_{k_2+1} = a_{k_2} + k_2 \cdot (n-2) + 1$ and the equation has to be true because of the definition of k_2. We know that there are an infinite number of k_1's which fulfill these conditions, and to different k_1's, different k_2's belong, so there are indeed an infinite number of n-sided polygonal numbers that fulfill the conditions.

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5. Write down the numbers from 1 to 26 and colour red the numbers, that Áron has written down and colour blue the numbers, that Benedek has written down. We know that there is no number, that was written down by both Áron and Benedek, so every number was coloured at most once. Then erase the numbers that haven't been coloured. We know that there were 21 buckets, so we have 5 erased numbers.

It is easy to see that the remaining numbers are coloured red and blue by alternating pattern. Let's take a look at the erased numbers. We know that 1 is erased, therefore 2 and 3 must be coloured, and also the numbers 25 and 26 are coloured. Two erased numbers must have a difference of at least 3, because otherwise there would be two adjacent numbers with the same colour with a difference of at least 4.

If we know the erased numbers we can reconstruct what numbers they have written down. We must colour the numbers alternating in such way, that the last number, 26 is red. If the difference of any two erased numbers is at least 3, then this will be possible.

Therefore, we need to select 4 numbers from 4 to 24, such that every two numbers has a difference at least 3. This is equivalent to the problem that we must select 4 numbers from 4 to 18, because if we selected the numbers $a < b < c < d$ for the second problem, then the $a, b+2, c+4, d+6$ selection will be good for the first problem, and vice versa. Hence the answer is $\binom{15}{4}$.

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6. The second player has a winning strategy if and only if the number of discs in pile number 2 is even and in pile number 4 is congruent to 0 or 2 modulo 5 or if there are odd in pile number 2, and congruent to 1 modulo 5 in pile number 4. We only need to see that from a winning state we cannot step into a winning one but can always from a losing one.

If we are not in a winning state, then we can always get to a winning state as if the number of discs in pile 4 is congruent to 3 or 4 modulo 5, then by taking away 2 or 3 discs from this pile then the parity of the number of discs will change in pile 2, and we can choose which of $5k$ and $5k+1$ or $5k+1$ and $5k+2$ fits our strategy. If in pile 4 the number of discs is congruent to 0 or 2, then there is an odd number in pile 2, therefore by taking away one from there, we get to a winning state. And lastly if there is 1 modulo 5 in pile 4, then by taking away one disc from pile 4 we again get to a winning state as there has to be an even number of discs in pile 2.

From a winning state we cannot get to a winning state as if they take away from piles 2 or 3 then pile 4 does not change but the parity changes in pile 2. If they take away from pile 4 not resulting in 3 or 4 modulo 5, then if there was 0 modulo 5, then 3 has to be taken away, and that changes the parity of pile 2, and if there was 1 or 2 modulo 5, then by taking away only one disc, the parity of pile 2 does not change, and by taking away 2 the parity of pile 2 changes, so it is easy to see that we always get to a losing state.

The game is clearly finite, so we are done.

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2.3.4 Category E$^+$

1. Consider the rectangle on a grid with vertices at points $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, r)$, $(\frac{1}{2}, r)$.

Let the centres of the unit circles be the points of the form $(-\frac{1}{2}, (2k-\frac{1}{2})\sqrt{3})$, $(\frac{1}{2}, (2k-\frac{1}{2})\sqrt{3})$ and $(0, (2k+\frac{1}{2})\sqrt{3})$ where k is an integer.

These circles cover the side of length r since they are tangent to each other at points of the form $(-\frac{1}{2}, k\sqrt{3})$, $(\frac{1}{2}, k\sqrt{3})$, $(2k\sqrt{3}, 0)$ where (k is an integer). This means that the circles cover the lines $y = -\frac{1}{2}$ and $y = \frac{1}{2}$ and the longer sides of the rectangle are on this line.

Now we only have to cover the sides of length 1. The $(-\frac{1}{2}, 0)(\frac{1}{2}, 0)$ side can be covered by the circle with centre $(0, \frac{1}{2}\sqrt{3})$. Let us call a positive number z good if there exists an integer k such that $|z - (2k + \frac{1}{2})\sqrt{3}| \leq \frac{\sqrt{3}}{2}$. This is equivalent to the circles covering the segment $(-\frac{1}{2}, z)(\frac{1}{2}, z)$. The good numbers are a union of closed intervals of length $\sqrt{3}$ and the non good numbers are a union of open intervals of length $\sqrt{3}$.

If r is good, then we are done, but if r is not good, then there exists a real number $0 < l < \sqrt{3}$ for which $|(r+l) - (2k+\frac{1}{2})\sqrt{3}| \leq \frac{\sqrt{3}}{2}$ because of the lengths of the intervals. If now we translate all of the cirlces by l parallel to the y axis in the positive direction, then we obtain a desired covering of the perimeter of the rectangle.

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2. **a)** Consider the equation modulo 3. If $p \neq 3$ prime, then $p^2 \equiv 1 \pmod 3$. Therefore if none of the primes are 3, the left hand side is divisible by 3, it is a contradiction. So at least one of the primes is equal to 3, and the right hand side is divisible by 3. By the above, the left hand side can only be divisible by 3 if $a = b = c = 3$. These numbers satisfy the equation.

b) Suppose that (a, b, c) is a solution, with $a \leq b \leq c$. Then

$$f(x) = x^2 - bcx + b^2 + c^2$$

is a monic quadratic polynomial in $\mathbb{Z}[x]$ with an integer root. Then the other root a' is integer as well as the sum of the two roots is bc by the Viéte formulas. Hence (a', b, c) is a solution to the original equation as well. We also have $a' = \frac{b^2+c^2}{a} > a$, again by Viéte formulas, so we found another solution where we managed to increase the minimal value of (a, b, c) and leave the other two values the same. Starting from the solution $(3, 3, 3)$, and repeating the above $3N - 9$ times we get a solution with $a, b, c \geq N$.

The method used to solve part b) is called Viéte jumping.

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3. For the solution, see Category E Problem 5.

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4. Answer: There are at least n^2 marble cubes.

Construction for n^2 marble cubes: On all levels put marble cubes on the diagonal starting from the southwest corner and going to the northeast corner, also put marbles on the diagonal on the east side of this diagonal. Then we have n^2 marble cubes in total and all of the properties are satisfied.

Proof that we can't have less than n^2 marble cubes: We will prove that on the level with $k \times k$ cubes, we have at least $2k-1$ marble stones. We prove 2 propositions first:

Proposition 1: In an optimal construction, on each level, the set of marble cubes forms a connected set, where we consider 2 cubes adjacent if they share a side. We prove this by induction going from top to bottom. It is clear that the top level is connected. Using the fact that each marble stone has at least 3 marble cubes below it, we clearly get the induction step. Note that if we had a marble cube with no marble cube on top, we could replace it with sandstone and get another allowed construction with less marble.

Proposition 2: For the set of $k \times k$ cubes, there is at least 1 marble stone on each side. We prove this by induction again. For the top layer, this is clear. If we have a cube on the side from the $k-1$-th layer, then there are 2 cubes below it in the next level, being on the same side. At least one is marble, so we are done by induction.

Combining the two propositions, we know that on the level of $k \times k$ cubes, there is a path from south to north and a path from east to west using only marble stones. These both have at least $k-1$ south-north and $k-1$ east-west steps. We count the number of sides of the marble stones on this level that is not horizontal. From these steps, we get $2 \cdot 2(k-1)$ of them. Looking at the level from the four directions, we see k sides from each direction, each of them is disjoint from each other. Hence there are at least $4k + 2 \cdot 2(k-1)$ not horizontal sides on this level, corresponding to $2k-1$ marble cubes.

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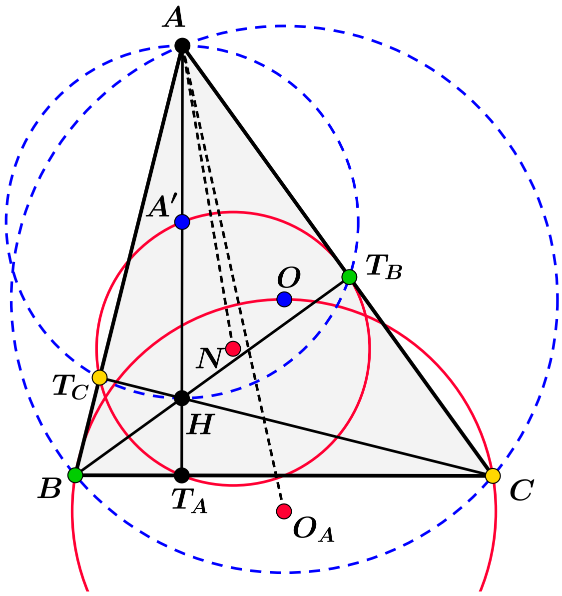
5. Denote the feet of altitudes of triangle ABC by T_A, T_B and T_C, respectively. Let H be the orthocentre, A' be the midpoint of AH and N be the centre of the nine-point circle of $\triangle ABC$. Define f as the composition of the homothety with centre A and ratio $\frac{AT_B}{AB}$ and the reflection over the angle bisector of $\angle BAC$.

It is easy to see that f takes B to T_B. Since $\triangle ABC \sim \triangle AT_CT_B$ (as they have equal angles) and since f is a similarity, f takes C to T_C. Observe that T_C and T_B lie on the circle with diameter AH, thus the circumcentre of AT_CT_B triangle is A'. However, f is a similarity, so it takes the circumcentre of triangle ABC, O, to the circumcentre of triangle AT_CT_B, which is A'. So f takes O to A'.

By the above, f takes triangle BCO and its circumcircle to triangle T_BT_CA' and its circumcircle. Knowing that A' is on the nine-point circle of triangle ABC, circle BCO is sent to the nine-point circle of ABC by f. So f takes O_A to N.

As f takes A to itself, the image of line AO_A is AN. Since a homothety with center A maps AO_A to itself, we have that lines AO_A, AN are symmetric w.r.t. the angle bisector of $\angle BAC$.

Thus, line AO_A passes through the isogonal conjugate of N. The argument above holds for the other two lines, BO_B and CO_C, hence they all pass through the isogonal conjugate of the nine-point circle.



Note: The point of concurrency has a name too, Kosnita point. (In the Encyclopedia of Triangle Centers it is denoted by X_{54}.) As we have seen from the proof above, it is the isogonal conjugate of the nine-point centre. The statement of the problem (also known as Kosnita's theorem) is due to the Romanian mathematician, Cezar Coşniţă. **Second solution:**

First of all, note that the segment bisector of AO passes through points O_B, O_C as they are circumcentres of two circles having segment AO as a side. Thus, $AO \perp O_BO_C$. Similarly, $BO \perp O_CO_A$ and $CO \perp O_AO_B$. This means that $\angle O_CO_AB = 90° - \angle O_ABO$. However, OBO_AC is a kite (because $OB = OC$ and $O_AB = O_AC$ as they are radii), so $90° - \angle O_ABO = 90° - \angle O_ACO = \angle O_BO_AC$. Similarly, $\angle O_AO_BC = \angle O_CO_BA$ and $\angle O_BO_CA = \angle O_AO_CB$

Let $A' = AO_A \cap O_BO_C$, $B' = BO_B \cap O_CO_A$ and $C' = CO_C \cap O_AO_B$. Further, denote by α, β, γ the equal angles proven previously around points O_A, O_B, O_C, respectively (in the figure they are shown by different colours).

Using the sine rule first on triangles AO_BA', AO_CA', then on triangles AO_AO_B, AO_AO_C, we get

$$\frac{O_BA'}{O_CA'} = \frac{\frac{\sin\angle A'AO_B}{\sin\beta}\cdot AA'}{\frac{\sin\angle A'AO_C}{\sin\gamma}\cdot AA'} = \frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin\angle A'AO_B}{\sin\angle A'AO_C} =$$

$$= \frac{\sin\gamma}{\sin\beta}\cdot\frac{\frac{O_AO_B}{AO_A}\cdot\sin\angle AO_BO_A}{\frac{O_AO_C}{AO_A}\cdot\sin\angle AO_CO_A} = \frac{\sin\gamma}{\sin\beta}\cdot\frac{O_AO_B\cdot\sin\angle AO_BO_A}{O_AO_C\cdot\sin\angle AO_CO_A}.$$

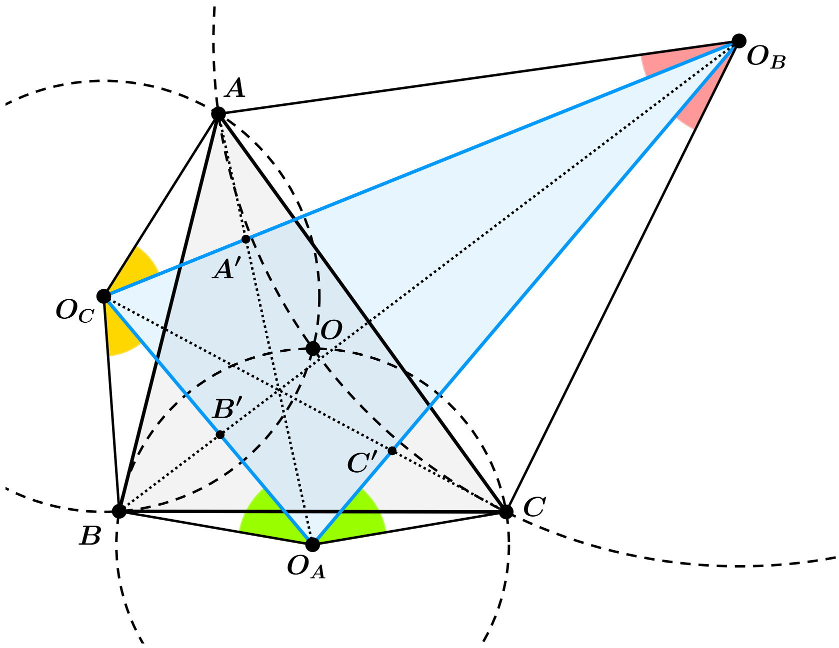
Similarly, we have

$$\frac{O_CB'}{O_AB'} = \frac{\sin\alpha}{\sin\gamma}\cdot\frac{O_BO_C\cdot\sin\angle BO_CO_B}{O_BO_A\cdot\sin\angle BO_AO_B} \quad\text{and}\quad \frac{O_AC'}{O_BC'} = \frac{\sin\beta}{\sin\alpha}\cdot\frac{O_CO_A\cdot\sin\angle CO_AO_C}{O_CO_B\cdot\sin\angle CO_BO_C}.$$

However, due to the equal angles we get $\angle AO_BO_A = \angle CO_BO_C$, $\angle BO_CO_B = \angle AO_CO_A$ and $\angle CO_AO_C = \angle BO_AO_B$, so multiplying the three equations from before yields

$$\frac{O_BA'}{O_CA'} \cdot \frac{O_CB'}{O_AB'} \cdot \frac{O_AC'}{O_BC'} = 1.$$

Therefore, due to the converse of Ceva's theorem we have proved the desired concurrency.



Note: The proof above basically shows Jakobi's theorem.

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6. For the solution, see Category E Problem 6.

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2.4 Final round – day 2

2.4.1 Tables

#	ANS	Problem	P
C-1	28	Benedek and Bence ate a box of gooseberries.	3p
C-2	17	One year, the first of December falls on a Wednesday.	3p
C-3	4	Five friends, Lili, Dalma, Eszti, Balázs and Áron	3p
C-4	1	Four Mayan Gods	3p
C-5	25	Hermes wants to visit the 7 wonders of the world	4p
C-6	14	How many double-digit numbers	4p
C-7	76	Lilla has written the numbers	4p
C-8	360	Timi was born in 1999.	4p
C-9	12	The AB side of the convex quadrilateral $ABCD$	5p
C-10	17	Zoli wants to fill the given 4×4 table	5p
C-11	36	The diagonal of a rectangle is 14 cm long,	5p
C-12	1340	Benedek wrote down the following numbers:	5p
C-13	5	The *binary sudoku* is a puzzle	6p
C-14	4440	In his large square notebook,	6p
C-15	69	Andris was on his way home from school,	6p
C-16	330	Odysseus wants to put 21 chests	6p

#	ANS	Problem	P
D-1	28	Benedek and Bence ate a box of gooseberries.	3p
D-2	1	Four Mayan Gods	3p
D-3	4	Five friends, Lili, Dalma, Eszti, Balázs and Áron	3p
D-4	8	Odysseus wants to pack 7 chests	3p
D-5	9864	Which is the largest four-digit number	4p
D-6	12	The AB side of the convex quadrilateral $ABCD$	4p
D-7	13	In Eldorado a year has 20 months,	4p
D-8	0	We are given a triangle ABC	4p
D-9	19	A city consists of 90 square-based houses	5p
D-10	36	The area of a rectangle is 64 cm^2,	5p
D-11	17	Zoli wants to fill the given 4×4 table	5p
D-12	1296	Archimedes drew a square	5p
D-13	5	The *binary sudoku* is a puzzle	6p
D-14	1000	One day Mnemosyne decided to colour all natural numbers	6p
D-15	50	King Minos divided his rectangular island of Crete	6p
D-16	6732	Csongi bought a 12-sided convex polygon-shaped pizza.	6p

#	ANS	Problem	P
E-1	224	Csenge and Eszter ate a whole basket of cherries.	3p
E-2	360	Timi was born in 1999.	3p
E-3	9864	Which is the largest four-digit number	3p
E-4	2505	Benedek wrote down the following numbers:	3p
E-5	0	We are given a triangle ABC	4p
E-6	13	In Eldorado a year has 20 months,	4p
E-7	36	The area of a rectangle is 64 cm^2,	4p
E-8	17	Zoli wants to fill the given 4×4 table	4p
E-9	1296	Archimedes drew a square	5p
E-10	1000	One day Mnemosyne decided to colour all natural numbers	5p
E-11	5	The *binary sudoku* is a puzzle	5p
E-12	12	Marvin really likes pancakes,	5p
E-13	1349	A country has 2023 cities	6p
E-14	900	Zeus's lightning	6p
E-15	6732	Csongi bought a 12-sided convex polygon-shaped pizza.	6p
E-16	4	For the Dürer final results announcement,	6p

#	ANS	Problem	P
E$^+$-1	3	Nüx has three moira daughters,	3p
E$^+$-2	36	The area of a rectangle is 64 cm^2,	3p
E$^+$-3	40	Hapi, the god of the annual flooding of the Nile	3p
E$^+$-4	13	In Eldorado a year has 20 months,	3p
E$^+$-5	50	King Minos divided his rectangular island of Crete	4p
E$^+$-6	1296	Archimedes drew a square	4p
E$^+$-7	1000	One day Mnemosyne decided to colour all natural numbers	4p
E$^+$-8	17	Zoli wants to fill the given 4×4 table	4p
E$^+$-9	5	The *binary sudoku* is a puzzle	5p
E$^+$-10	12	Marvin really likes pancakes,	5p
E$^+$-11	1349	A country has 2023 cities	5p
E$^+$-12	900	Zeus's lightning	5p
E$^+$-13	6732	Csongi bought a 12-sided convex polygon-shaped pizza.	6p
E$^+$-14	4	For the Dürer final results announcement,	6p
E$^+$-15	240	What is the biggest positive integer	6p
E$^+$-16	1514	What is the remainder of	6p

2.4.2 Category C

1. Let the number of gooseberries be x. According to the text, Bence ate $\frac{x}{4}$, Benedek ate $\frac{x}{2}+7$ gooseberries. So $\frac{x}{4}+\frac{x}{2}+7=x$. Rearranging, $7=x-\frac{x}{4}-\frac{x}{2}=\frac{x}{4}$. Therefore $x=28$, i.e. 28 gooseberries were in the box.

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2. Denote in a calendar with **bold** the weekdays in December on which Sára went to school in the Durer T-shirt, and denote with *italic* the respective days for Lili. The appropriate weekdays for both are marked in below, and we can see that the first day which is both marked bold and italic is the 17th, providing the answer.

Monday	Tuesday	Wednesday	Thursday	Friday
		1	2	3
6	*7*	8	**9**	10
13	**14**	15	16	***17***
20	21	**22**	*23*	24
27	28	*29*	**30**	31

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3. Lili does not see Dalma, so Lili is shorter and someone has to be between them. This person cannot be Eszti, as she is taller than them, and also cannot be Áron, as he has only one neighbour - so only Balázs can be between Lili and Dalma. As Áron has only one neighbor and Balázs only sees girls, it follows that Áron is the tallest. Eszti is the tallest girl, Lili and Dalma are shorter than her, but also Balázs, as he is between them. This means that the only possible order from shortest to tallest is: Lili, Balázs, Dalma, Eszti, Áron, and this ordering satisfies all the given conditions. Hence the height difference between Lili and Dalma is 4cm.

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4. If Chaac prohibition were to apply, then 2, 3, 5, 7, 11, 13, 17 and 19 will be banned. If Pawahtun's laws are also enacted 6, 9, 12, 15 and 18 will no longer be available. Because of Itzamna, $4(= 2\cdot2)$, $8(= 2\cdot4)$, $16(= 2\cdot8)$ and $20(= 2\cdot10)$ would also be banned. Since $7+3 = 10$ and $11+3 = 14$, Yum Kaax would no longer allow the use of these numbers either. But neither of them would ban the use of 1. Thus, of the numbers 1, 2, ..., 20, only 1 could be used after the prohibitions take effect.

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5. There are only two roads to the southernmost (lower) point, so Hermes must use these two roads. Hence, at the easternmost point, exactly one of roads 7 and 8 is used. If he takes route 7, the only way he can enter the vertices is to go north twice on edge 1, then south on edge 3 and 6. This route takes 25 hours. If you take Route 8, you can get to the second southernmost point in three ways, by $3 + 3 + 1 + 4$, $1 + 3 + 4 + 4$ or $1 + 1 + 4 + 6$ hour trips, but in these cases the trips are 26, 27 or 27 hours, so the shortest trip is 25 hours.

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6. We know that the second digit of such a number cannot be 0 since no number is divisible by 0. It is worth searching for the appropriate numbers by sorting them based on their first digits. The correct numbers are 11, 12, 15, 22, 24, 33, 36, 44, 48, 55, 66, 77, 88 and 99. So, there are 14 numbers that fulfill the condition.

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7. Since the first picture contains the whole 3×3 table, let's complete it. The rows are numbered from bottom to top, the columns from left to right. The second picture shows two rows and three columns, where the 2nd digit in column 3 is 2. Since there is another row below 2, it can be either in row 3 or in row 2. It cannot be in row 3 because the 2nd digit in column 3 is 3. We get the following table:

2_	_3	_3
___	22	_2
13	___	_1

We can see that numbers 13 and 22 are complete. We have 3 numbers ending in 3 and the table shows all three three-endings. So in the top row, 2 and 3 will come before the two 3's in some order. In picture 4, the 1 can be put in four places, in the last two columns of the top row and the last two columns of the middle row, for first digit. It cannot go to the top, nor to the 2nd column of the middle row, so it can only go to column 3. This gives us the 12. The items of picture 3 can now be clearly placed, because 2 cannot come in either row 1 or row 2. From these we get the following figure:

2_	33	23
1_	22	12
13	___	_1

This shows that the numbers starting with 2 are all included in the table (by exclusion, 21 is in the top-left corner). The 3 numbers starting with 1 are also included, with 11 in the middle row in column 1. The numbers left are placed in row 1 is 32 and 31 (in that order). So the sum of the numbers in the bottom row is $13 + 32 + 31 = 76$. The final figure:

21	33	23
11	22	12
13	32	31

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8. The possible years are 2000, 2001, 2002, 2010, 2011, 2012, 2020, 2021, 2022 (9 instances)

The possible months are 01, 02, 10, 11, 12 (5 instances)

The possible days are 01, 02, 10, 11, 12, 20, 21, 22 (8 instances)

Any combination of these years, months and days will yield a valid date. Hence in total, there were $9 \cdot 5 \cdot 8 = 360$ such dates which could be written down only using the digits 0, 1 and 2.

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9. Triangles ABC and ABD have a common base, side AB, and their area is the same. Since the area of a triangle equals half of the product of a side of the triangle and its corresponding altitude, the altitudes corresponding to vertices C and D are both 7 cm in triangles ABC and ABD. So, $ABCD$ is a trapezoid, since sides AB and CD are parallel to each other, and its height is 7 cm. The area of triangle DCB is 42 cm^2. By applying the same formula for this area, we get that $CD = 12$ cm.

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10. The answer is 17.

4	3	1	2
2	1	3	4
3	4	2	1
1	2	4	3

No number can appear more than two times in the grey cells because if there would be such number, then we would not be able to write that number in any of the cells in the main diagonal. So, the only way the sum in question can be bigger than 17, is if there are two 2's, two 3's and two 4's in the grey cells. But then, the numbers that are in the grey cells that are on the main diagonal and we can easily see that this would mean that we would not be able to write these numbers in four cells in total, so this case is not possible.

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11. Let one side of the rectangle be a cm and the other b cm. Then the area of the rectangle is $T = ab$, its perimeter is $K = 2(a+b)$, and the square of the length of its diagonal is $14^2 = a^2+b^2$ based on the Pythagorean theorem. Then $(a+b)^2 = a^2+b^2+2ab = 14^2+2\cdot T = 196+2\cdot 64 = 324$. So the square of half the circumference measured in centimeters is 324. Since the circumference of a shape can only be positive, half of the circumference is therefore 18 cm, so the perimeter of the rectangle is 36 cm.

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12. Lets look at the digits according to their position! On the local value of the tens he wrote two down $20+21+22+\cdots+29 = 245$ times, and he wrote four down $40+41+\cdots+49 = 445$ times.

In each local value, he wrote down 0 during the tens 10 times, during the twenties 20 times, and so on, finally during the fifties 50 times.

Thus, he wrote down zero $10 + 20 + 30 + 40 + 50 = 150$ times in each local value. We can count this similarly for each even digits as well:

- he wrote two $2 + 12 + 22 + 32 + 42 = 110$ times,

- he wrote four $4+14+24+34+44=120$ times,
- he wrote six $6+16+26+36+46=130$ times,
- he wrote eight $8+18+28+38+48=140$ times.

In total Benedek wrote down $245+445+150+110+120+130+140=1340$ even digits.

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13. If there are two adjacent cells in the table with the same digits in them, then their adjacent cells in the same row or column on the sides of those two have to have the other digit in them, otherwise, there would be three of the same digit in three adjacent cells in a row or a column. Similarly, if there is the same digit in two cells, that are in the same row or column and that have only one other cell between them, then in the cell between them, there has to be the other digit (as shown in the figure, the digits in the grey cells were already known while the digits in the white cells were just written down).

1	0	0	1

1
0
1

Using these two types of steps and paying attention to the condition that there has to be 4 0's and 4 1's in each row and column, the table can be filled to this degree:

				0			1
			0	1	0	1	1
	1			1			0
1				0			1
			1	0	0	1	
				1		0	
		1	0	0	1	1	0
				1		0	x

In the cell marked with x, we can only write 0, otherwise, it would contain the 4. 1 in that column, so we would have to write only 0's in the remaining empty cells in that column, but then there would be 3 0's in adjacent cells, which is not possible.

After writing 0 in the bottom right cell, we can fill more cells:

			1	0	1	0	1
y			0	1	0	1	1
0	1	0	1	1	0	1	0
1	0	1	0	0	1	0	1
			1	0	0	1	0
		0	1	1	0	0	1
		1	0	0	1	1	0
			0	1	1	0	0

If we would write 1 in the cell marked with y, then that would be the 4. in its row and the remaining cells would be filled with 0's that would result in 3 0's in adjacent cells in that row, so we can only write 0 in the cell in question.

After writing 0 in that cell, we can fill more cells:

1	0	0	1	0	1	0	1
0	0	1	0	1	0	1	1
0	1	0	1	1	0	1	0
1	0	1	0	0	1	0	1
	z		1	0	0	1	0
		0	1	1	0	0	1
		1	0	0	1	1	0
			0	1	1	0	0

If we would write 0 in the cell marked with z, then that would be the 4. 0 in that column, so there would be 3 1's in that column in adjacent cells. So, we can only write 1 in that cell.

After writing 1 in that cell, we can fill the entire table:

1	0	0	1	0	1	0	1
0	0	1	0	1	0	1	1
0	1	0	1	1	0	1	0
1	0	1	0	0	1	0	1
1	1	0	1	0	0	1	0
0	1	0	1	1	0	0	1
1	0	1	0	0	1	1	0
0	1	1	0	1	1	0	0

So, the sum of the numbers in the two main diagonals is 5.

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14. First we show that there are no grid points on the circle. Suppose for contradiction that there are. So there exist x and y integers for which $(x+\frac{1}{2})^2+(y+\frac{1}{2})^2=555^2$ according to the Pythagorean theorem. Multiplying by 4, we get $4x^2+4x+4y^2+4y+2=555\cdot 4$ - here all terms except 2 are divisible by 4 leading to a contradiction. Hence there are no grid points on the circle.

If we traverse from the top of the circle to the bottom on either side we cross 1110 horizontal lines, similarly 1110 vertical lines if we traverse from the rightmost to the leftmost point on either side. As the circle does not cross any grid points, each time we cross a (vertical or horizontal) line we enter a new square that has not been crossed before. While traversing through the whole circle, we are crossing a of 2220 horizontal and 2220 vertical lines, so we cross a total of 4440 circles.

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15. Andris travels $\frac{5}{60}\cdot 6=\frac{1}{2}$ kilometres in 5 minutes, so Anett can't see which way he went. Since she can only guess, her first two guesses might be wrong and she might find him only on her third try. Thus, first she has to make sure he is not on the first two roads and then catch up to him on the third.

On the first route, she has to run until she's 100 metres from Andris' supposed position. Andris's speed is $6\cdot\frac{1000}{60}=100$ metres/minute, so after t minutes he will be $100\cdot t$ metres from the school. Anett, on the other hand, is travelling at 200 metres/minute, but starts 5 minutes later, so after t minutes on her first try she travels $(t-5)\cdot 200$ metres. Anett can make sure that Andris is not there when there is only 100 metres between these two numbers:

$$100t=(t-5)\cdot 200+100,$$

so after $t=9$ minutes.

Then, she must turn back to the school with the same speed. At this point, Andris has been walking for 13 minutes, so after t time, Andris would be $100t$ metres away on the second route, while Anett would be $(t-13)\cdot 200$ away. Again, she needs to be within 100 meters of Andris' hypothetical position:

$$100t=(t-13)\cdot 200+100,$$

thus, after $t=25$ minutes, she can turn back for the second time. She will be running on this route for $25-13=12$ minutes, so she will return to the school after $25+12=37$ minutes.

After that, she's definitely on the right track, so now she just needs to catch up. Andris is still $100t$ away after t time and Anett is $(t-37)\cdot 200$ meters away. Now she has to catch up completely, so the time that passes till they meet will be the solution of the equation

$$100t=(t-37)\cdot 200,$$

which is $t=74$ minutes. However, t is the number of minutes that passed since Andris has left, so to get the number of minutes that the run took Anett, we need to subtract 5 from t, so the solution is $74-5=69$ minutes.

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16. At first, we will calculate the number of possible ways in which we can put k chests in the first two storages. This is $\left\lfloor \frac{k}{2} \right\rfloor$ because we can put at least 1 and at most $\left\lfloor \frac{k}{2} \right\rfloor$ chests in storage 1. This holds true for the last two storages as well. Then, we can separate cases based on how many chests are stored in the first two and how many in the last two storages. If 2 chests are stored in the first two storages and 19 in the last two, there are $1 \cdot 9$ possibilities, if 3 are in the first two and 18 in the last two, there are $1 \cdot 9$ possibilities and so on. These are $1{\cdot}9+1{\cdot}9+2{\cdot}8+2{\cdot}8+3{\cdot}7+3{\cdot}7+4{\cdot}6+4{\cdot}6+5{\cdot}5+5{\cdot}5+6{\cdot}4+6{\cdot}4+7{\cdot}3+7{\cdot}3+8{\cdot}2+8{\cdot}2+9{\cdot}1+9{\cdot}1 = 330$ possibilities in total.

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2.4.3 Category D

1. For the solution, see Category C Problem 1.

(Back to problems)

2. For the solution, see Category C Problem 4.

(Back to problems)

3. For the solution, see Category C Problem 3.

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4. Let's call the 2nd and 3rd storages the middle storages, and the 1st and 4th storages the outer storages. Since there is at least one chest in each storage, it's enough to see how many ways we can place the remaining 3 chests according to the rules.

The possible cases:

- All three chests go into one storage. Then this must be the middle storage. $\implies$ 2 possibilities
- Two chests go into the same storage, the third one into another: the two chests must go into one of the middle storages. The last chest thus must go into either the other middle storage or the nearest outer storage. $\implies$ $2 \cdot 2 = 4$ cases
- Every chest goes into different storages. Then only one outer storage can remain empty $\implies$ 2 cases

This gives Odysseus a total of 8 different ways to place the chests.

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5. Since the digits of the number in question are all different, it cannot be bigger than 9876. If the solution is more than 8999, then it has to be divisible by 9, so we will look at the numbers between 9000 and 9876 that are divisible by 9. 9873 is not divisible by 8 but 9864 is divisible by all of its digits, so that is the biggest number that fulfills the conditions.

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6. For the solution, see Category C Problem 9.

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7. Notice if Adél had been born in the ath month on the bth day, she would have said the same number to Brigi as if she had been born in the bth month on the ath day. So given that Brigi could infer the exact day from the number she heard, the numbers of the month and the day in Adél's birthday are the same.

- Adél could have been born in the 1st month on the 1st day as the number 1 cannot be written up as a product of two other natural numbers.
- She could not have been born in the 2nd month on the 2nd day as $2 \cdot 2 = 4 \cdot 1$.
- She could not have been born in the 3rd month on the 3rd day as $3 \cdot 3 = 9 \cdot 1$.
- She could not have been born in the 2nd month on the 2nd day as $4 \cdot 4 = 16 \cdot 1$.
- Adél could have been born in the pth month on the pth day (where p is a prime bigger than 3, i.e. 5, 7, 11, 13, 17, or 19), as then she said p^2 to Brigi, which can only be written up as $p \cdot p, 1 \cdot p^2$ és $p^2 \cdot 1$ but $p^2 > 20$.
- She could not have been born in the 6th month on the 6th day as $6 \cdot 6 = 3 \cdot 12$.
- She could not have been born in the 8th month on the 8th day as $8 \cdot 8 = 4 \cdot 16$.
- She could have been born in the 9th month on the 9th day as $9 \cdot 9$ doesnt have a divisor that is larger than 9 but smaller than 20.
- She could not have been born in the 10th month on the 10dth day as $10 \cdot 10 = 5 \cdot 20$.
- She could not have been born in the 12th month on the 12th day as $12 \cdot 12 = 9 \cdot 16$.
- For $n > 12$, Adél could have been born in month n and on day n, as for non of these ns will $n \cdot n$ have a divisor greater than n but smaller than 20 20.

So counting up the cases above (1, 5, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20) alltogether there are 13 possibilities for Adéls birthday.

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8. The distance between P and Q is 0, because the altitude drawn from A intersects BC in T, which will be on k_1 and k_2 because of the Thales-theorem. Also T will not coincide with B or C. Thus T must be P and Q, hence the distance is 0.

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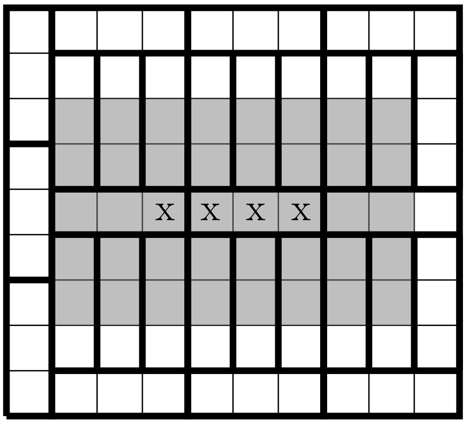
9. There could be 19 representatives from downtown, for example in this arrangement:



However, there cannot be 20 or more representatives from downtown. If that would be the case, then there must be at least two houses from downtown in at least twenty districts which would mean that every house in the downtown should be part of a district that has two houses from the downtown and one house from the suburb. However, there are 4 houses in the downtown (marked by $\times$ in the figure) that can never be in such a district, so this is not possible.

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10. The diagonal of a rectangle is also the diameter of its circumscribed circle. Since the length of the diameter of a circle is twice the length of its radius, the diagonal of the rectangle is $7 \cdot 2 = 14$ units. Let one side of the rectangle be a cm and the other b cm. Then the area of the rectangle is $T = ab$, its perimeter is $K = 2(a + b)$, and the square of the length of its diagonal is $14^2 = a^2 + b^2$ based on the Pythagorean theorem. Then $(a + b)^2 = a^2 + b^2 + 2ab = 14^2 + 2 \cdot T = 196 + 2 \cdot 64 = 324$. So the square of half the circumference measured in centimeters is 324. Since the circumference of a shape can only be positive, half of the circumference is therefore 18 cm, so the perimeter of the rectangle is 36 cm.

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11. For the solution, see Category C Problem 10.

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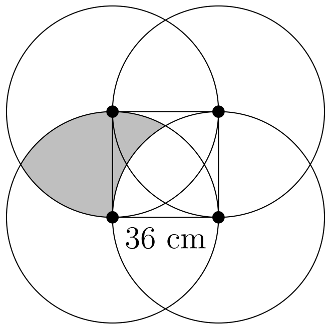
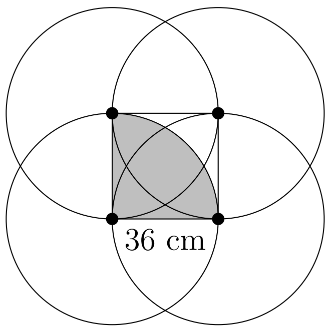
12. Notice that the grey part can be produced as four congruent images of the colored part below, therefore its area is four times the area of this:



Move the fish scale on the left is to an area that is congruent with it, so we get a shape with the same area:

helyezzük át egy vele egybevágó területre, így egy azonos területű alakzatot kapunk:



This determines a quarter circle with radius 36 cm, so the area is $\frac{1}{4} \cdot 36 \cdot 36 \cdot \pi$ cm$^2 = 324\pi$ cm^2. Therefore the area of the original grey part is $4 \cdot 324 = 1296\pi$ cm^2, i.e. $n = 1296$.

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13. For the solution, see Category C Problem 13.

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14. If we divide a number by 9, we get the same remainder that we get by dividing the sum of the digits of the number by 9. For all integers, that are more than 9, the sum of their digits will be less than the original number.

We assume that there is a number that has the form $9k + 3$ and is not coloured gold for some natural number k. We mark the smallest number that has these properties with m. m cannot be a single-digit number, since 3 is coloured gold and there are no other single-digit numbers that have the form $9k+3$. However, m cannot have more than one digit as well since then the sum of its digits has the form $9k + 3$ as well, but it must be coloured gold because m is the smallest such number which is not coloured gold. But that means that m is also coloured gold which leads to a contradiction. So, all numbers that have the form $9k + 3$ are coloured gold.

But if a number does not have the form $9k + 3$, it cannot be coloured gold. To prove this, we assume that $n = 9k + l$, where $0 \leq l < 9$, $n \neq 0$ és k is a natural number, is the smallest number that does not have the form $9k+3$ but is coloured gold. n cannot be a single-digit number but if it has multiple digits, then the sum of its digits also has the form $9k+l$ but it is smaller than n and it also has to be coloured gold. This leads to a contradiction, so there is no such number n because it cannot be the smallest.

So, we only need to count how many four-digit numbers have the form $9k+3$. The smallest k for which $9k+3$ is a four-digit number is 111, and the biggest is 1110. Since there are 1000 natural numbers between 111 and 1110, there are 1000 four-digit numbers that were coloured gold by Mnemosyne.

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15. Let the rectangle be $ABCD$ and the diagonal with the first wall AC, and let the foot point of the second wall on the AC diagonal be T. Then triangles ABC and CAD are congruent and triangles BCT, ABT and ABC are similar. Therefore the areas obtaioned are similar, so the ratio of the palaces equals the squares of the similarity ratios. The hypotenuse of the smallest area is the smaller side of the rectangle, the hypotenuse of the largest area on the other hand is the diagonal of the rectangle. The latter equals $\sqrt{210^2+30^2}$ cm according to the Pythagorean theorem, hence the ratio in question is $\left(\frac{\sqrt{210^2+30^2}}{30}\right)^2 = 50$.

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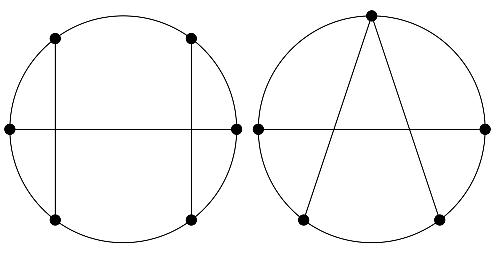
16. Three lines that intersect each other at three different points divide a plane into seven parts. In order to cut the pizza into 6 parts by cutting through 3 diagonals, the chosen diagonals must have exactly two intersections in the pizza. If they would not have any intersections, they would cut the pizza into 4 parts, if they would have only one intersection, they would cut it into 5 parts.

There are two different cases:

If the three diagonals have six different endpoints in total, so the endpoints form a hexagon, then there will only be two intersections if one of the diagonals connects two opposite vertices of the hexagon and the other two connect vertices that have only one other vertex between them (see figure 1). For that, we need to select 6 vertices out of 12 and then choose one from the three pairs of opposite vertices which will be connected by a diagonal. That is $3 * \binom{12}{6} = 2772$ possibilities in total.

If two of the chosen diagonals have a common endpoint (see figure 2), then we have to choose 5 from 12 vertices and 1 from those 5 vertices where two diagonals have a common endpoint and that defines the position of the diagonals since this vertex needs to be connected by diagonals with the vertices that are not its neighbours and the remaining two vertices need to be connected by the third diagonal in order to produce two intersections in the pizza. This is $3 * \binom{12}{6} = 2772$ possibilities in total.

So, the pizza can be cut into 6 pieces in $2772 + 3960 = 6732$ different ways.



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2.4.4 Category E

1. We mark the number of all the cherries with x. We know, that Csenge ate $\frac{x}{4}$ cherries while Eszter ate $\frac{4}{7}x + 40$ cherries. We know that $\frac{x}{4} + \frac{4}{7}x + 40 = x$, so $40 = x - \frac{x}{4} - \frac{4}{7}x = \frac{5}{28}x$ which means that $x = 224$. So, there were 224 cherries in the basket.

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2. For the solution, see Category C Problem 8.

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3. For the solution, see Category D Problem 5.

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4. The sum of the numbers from 1 to is $\frac{n(n+1)}{2}$, so the number of single-digit numbers is: $\frac{9 \cdot 10}{2} = 45$. The number of two-digit numbers is: $\frac{50 \cdot 51}{2} - \frac{9 \cdot 10}{2} = 1230$. The sum of the numbers from 10 to 50 can be calculated as summing up the numbers from 1 to 50, than subtracting the sum of the numbers from 1 to 9. Since Benedek didn't write down a more than two digit number, in total $1 \cdot 45 + 2 \cdot 1230 = 2505$ digit was written down.

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5. For the solution, see Category D Problem 7.

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6. For the solution, see Category D Problem 6.

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7. For the solution, see Category D Problem 9.

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8. For the solution, see Category C Problem 10.

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9. For the solution, see Category D Problem 12.

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10. For the solution, see Category D Problem 14.

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11. For the solution, see Category C Problem 13.

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12. Since Marvin ate 157 pancakes the last time, he had 314 pancakes before that. So, our goal is to find a path from 0 to 314 by only using the following two steps: A *grandma step* adds 2^5 to the current number and a *Marvin step* halves the current number.

One *grandma step* and n *Marvin steps*, not necessarily directly after that, add 2^{5-n} to the result. n marks the number of *grandma steps* after which there are n *Marvin steps* in total. Then, $314 = a_0 \cdot 2^5 + a_1 \cdot 2^4 + \ldots + a_5 \cdot 2^0 + \ldots$ and our goal is to find the smallest possible value of $a_0 + a_1 + a_2 + \ldots + a_5 + \ldots$.

We assume that in one of the constructions where the sum has its lowest value, there is an $i \geq 1$ that $a_i \geq 2$. Then, if we increase a_{i-1} by 1 and decrease a_i by 2, then the first equation is still true but the sum in question decreased by 1, so this construction is actually not a good solution. So, for all $i \geq 1$, a_i is either 0 or 1 in the constructions that we are looking for. Since the sum is an integer and for all $i \geq 6$ $a_i \leq 1$, $a_i = 0$ for all $i \geq 6$. We convert the equation into the binary numeral system. 314 in the binary numeral system is 100111010 while the other side of the equation almost represents a formula in the binary numeral system, but there are no positions higher than 32, they are substituted by increasing the value of a_0 adequately. So, the construction $a_0 = 9$, $a_1 = 1$, $a_2 = 1$, $a_3 = 0$, $a_4 = 1$ and $a_5 = 0$ gives the smallest possible value of the sum, 12. This is a possible arrangement, the $gMMgMnMgggggggggM$ sequence fulfills it where g marks a *grandma step* and M marks a *Marvin step*. So, Grandma sent pancakes to Marvin at least 12 times.

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13. Consider the graph representation of the problem. It is required to be connected, hence it contains a spanning tree, therefore it has at least 2022 edges, and so its degree sum is at least 4044. Let the value in question, i.e. the number of vertices with degree 1 be a. The degree sum is maximum $a + 4(2023 - a)$, as the maximum degree for any vertex is 4. So from the above we get:

$$4044 \leq a + 4(2023 - a) = 8092 - 3a.$$

Reordering, $a \leq \frac{4048}{3}$ and as a is a whole number, $a \leq 1349$. An example where $a = 1349$ can be constructed as follows: take a path of length 674 and 'hang' two leaves from every vertex on it, and in addition one leaf from one of the endpoints. Hence the solution is 1349.

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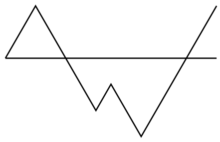
14. Set the x-axis to be the line in the plane of the lightning that passes through one endpoint of the lightning and forms an angle of 60° with all lines given by the segments of the lightning. Then the x-coordinate of the other endpoint is $60 \cdot \frac{1}{2} = 30$ or -30, we may suppose it's 30, since we'll have semi-equilateral triangles under each segment of the lightning. On the line $x = 30$, the closest point from the point $(0, 0)$ is $(30, 0)$, then the square of the distance is 900. To have this, you just need to make the y coordinate of the other end point 0, and this can be achieved, for example, if the sections are 10, 20, 10, 10 and 10 units long in that order. So the solution is 900.



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15. For the solution, see Category D Problem 16.

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16. If there would not be extension cords, the answer would be the same, because for any construction we can take the one where we omit the extension cords and we connect the plugs that were originally connected to the extension cords to the sockets in which the extension cords were plugged into. We can easily see that this does not change the probabilities. For example, if we plug the extension cords randomly and then the remaining 6 plugs and 6 sockets randomly as well, we get the same results if we would have left out the extension cords. So, we will solve the problem without the extension cords. The only way in which all four loudspeakers will work is if both power strips are plugged into the wall. The only way in which this does not happen is if a power strip is plugged into itself or if the power strips are plugged into each other. The probability of the former case is

$$\frac{1}{3} + \frac{1}{3} - \frac{1}{3} \cdot \frac{2}{5}$$

according to the sieve theory, since the possibility of plugging a power strip into itself is $\frac{1}{3}$ and we have two power strips, so we have to subtract the possibility of plugging both in itself from $2 \cdot \frac{1}{3}$, which is $\frac{1}{3} \cdot \frac{2}{5}$, because the possibility of plugging the first in itself is $\frac{1}{3}$ and after that, the possibility of plugging the second in itself is $\frac{2}{5}$. The other case is when we plug the power strips into each other. The possibility of that is $\frac{1}{3} \cdot \frac{2}{5}$, which has a similar explanation as the previous case. There is a $\frac{1}{3}$ possibility that the first power strip will be plugged into the other power strip and after that, there is a $\frac{2}{5}$ possibility that the other power strip will be plugged into the the first one. In total, there is a $\frac{2}{3}$ possibility that one of the loudspeakers will not work. So, there is a $\frac{1}{3}$ possibility that all loudspeakers will work, so the answer is $1 + 3 = 4$.

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2.4.5 Category E$^+$

1. Let the three ages be p, q and r, for which we know that $p^2+q^2+r^2$ is also a prime number. We know that each term in this sum is at least 4, that is, since it is prime, it cannot be divided by 3 or 2. But then if none of p, q, r were divisible by three, then their squares would all have a remainder of 1 divided by 3, which is a contradiction, since then the sum would be divisible by 3. So we can assume that $r = 3$. Then, in order for the sum of squares to be odd, either p and q are even, or both are odd. In the former case, however, two siblings would be 2 years old, which is not allowed. Therefore there is no 2-year-old sibling, but there is 3-year-old, so the youngest moira is 3-year-old. (The conditions of the task can be fulfilled, for example, in the case of $p = 3$, $q = 5$, $r = 7$, because $p^2 + q^2 + r^2 = 83$ is also prime.)

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2. For the solution, see Category D Problem 9.

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3. The depth of the river will be $5 + 3 = 8$ meters after the flood. At point $x = 0$, the Nile bed is at $y = \frac{-1000}{x^2+100} = \frac{-1000}{100} = -10$, so the water level after the flood is $aty = -10 + 8 = -2$. To determine the width of the river, we want to find which 2 points of the riverbed are at this height, i.e. we want to solve the equation $-2 = y = \frac{-1000}{x^2+100}$. Rearranged $2x^2 + 200 = 1000$, so $x^2 = 400$, so $x = \pm 20$, so the width of the river is $20 - (-20) = 40$ meters.

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4. For the solution, see Category D Problem 6.

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5. For the solution, see Category D Problem 15.

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6. For the solution, see Category D Problem 12.

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7. For the solution, see Category D Problem 14.

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8. For the solution, see Category C Problem 10.

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9. For the solution, see Category C Problem 13.

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10. For the solution, see Category E Problem 12.

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11. For the solution, see Category E Problem 13.

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12. For the solution, see Category E Problem 14.

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13. For the solution, see Category D Problem 16.

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14. For the solution, see Category E Problem 16.

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15. We mark the number in question with N. Let prime r be a divisor of N. If $r > 10$ and we choose p and q in a way that $p > r$ and $q = r$, then $N \mid p^4 - q^4$ but $r \nmid p^4 - q^4$ which is a contradiction. So, the possible prime divisors of N are 2, 3, 5, and 7. Note that $p^4 - q^4 = (p^2 + q^2)(p + q)(p - q)$. If $p \equiv 1 \mod 7$ and $q \equiv 2 \mod 7$, for example if $p = 29$ and $q = 23$ then because of the previous equation, $7 \nmid p^4 - q^4$. So, $7 \nmid N$.

Since p and q are prime numbers that are greater than 10, they are not divisible by 2, 3, or 5. So, the remainders of the divisions $p : 32$ and $q : 32$ can only be 1, 3, 5, 7, 9, 11, 13, 15, $-15, \ldots, -5$, -3 or -1 so the remainders of the divisions $p^2 : 32$ and $q^2 : 32$ can only be 1, 9, 25 or 17 which means that the remainders of the divisions $p^4 : 32$ and $q^4 : 32$ can only be 1 or 17. So, for all p and q, $16 \mid p^4 - q^4$ but for example, in the case of $p = 97$, $q = 67$, $32 \nmid p^4 - q^4$.

For any s square number, the remainder of the division $s : 3$ is either 0 or 1 and p and q are not divisible by 3, so $p^4 \equiv q^4 \equiv 1 \mod 3$. However, $19^4 - 11^4 \equiv 1 - 2^4 \not\equiv 0 \mod 9$, so $p^4 - q^4$ is not always divisible by 9.

For any s square number, the remainder of the division $s : 5$ is 0, 1 or 4, and p and q are not divisible by 5, so $p^4 \equiv q^4 \equiv 1 \mod 5$. However, $29^4 - 23^4 \equiv 4^4 - (-2)^4 \not\equiv 0 \mod 25$.

So, $N = 16 \cdot 3 \cdot 5 = 240$.

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16. Since $2023 = 7 \cdot 17^2$,we can get the remainder of the power tower when divided by 2023, if we calculate its reaminder when divided by 7 and 17^2.

We want to use the Euler-Fermat Theorem to decrease the exponent of the power tower so that we can calculate the above mentioned remainders. So we want to calculate the remainder of $2024^{2022^{2021^{\cdot^{\cdot^{\cdot^{1}}}}}}$ when divided by $\varphi(7) = 6$ and $\varphi(17^2) = 16 \cdot 17$. It's clear that the remainder of the smallest tower when divided by 16 is 0 and since $17 \mid 2023$ its remainder when divided by 17 is 1. Since $3 \nmid 2024$ and it's raised to the even power, so its remainder when divided by 3 is 1.

Based on these, the remainder of the smaller power tower when divivded by 6 is 4 and its remainder when divided by $16 \cdot 17$ is -16, i.e. 256. So the original power tower's remainder when divided by 7 is $2^4 = 16 \equiv 2 \mod 7$. And its remainder when divide by 17^2 is

$$2^{256} = 256^{32} \equiv 33^{32} \equiv 1089^{16} \equiv (-67)^{16} \equiv 4489^8 \equiv 154^8 \equiv 23716^4 \equiv 18^4 \equiv 35^2 \equiv 1225 \equiv 69 \mod 289.$$

Since $289 \equiv 2 \mod 7$ and the original power tower can be written in the form of $289 \cdot m + 69$ for an integer m, in order for the power tower to have a remainder of 2 when divided by 7 we must have $m \equiv 5 \mod 7$. Therefore $m = 7k + 5$ for an integer k, so

$$2025^{2024^{2022^{\cdot^{\cdot^{\cdot^{1}}}}}} = 2023k + 5 \cdot 289 + 69 \equiv 1514 \mod 2023.$$

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