XVII. DÜRER COMPETITION

E CATEGORY

PROBLEMS

DATE: 24TH NOVEMBER 2023.

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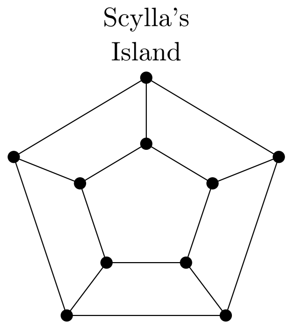
1. a) This is the map of the ten islands of the Düreric Ocean. There is a treasure hidden on one of the islands. Two islands are connected on the map if there is a direct ship connection between them. Leila, who has a friend on each of the islands, wants to find the treasure. Before she visits the archipelago, she wants to make sure she knows where to go, therefore she calls some of her friends on the phone. Any friend she calls can only tell her how many direct ship trips are needed to reach the treasure island from his/her own island. How many people does Leila need to call in order to be able to tell the location of the treasure with certainty, if first she calls Lily who lives on Scylla's Island?



b) This time Leila is visiting another archipelago made up of five islands and similarly to the previous part, one island holds a treasure. Leila managed to find out which islands are connected by direct ship connections. After a bit of thinking she discovered that she could definitely determine the location of the treasure by calling not more than two of her friends living on different islands. Based on this, what is the maximum number of direct ship connections between the islands?

The conditions are the same as in the first part: on each island, she has a friend whom she can call, and the friend will tell her how many direct ship trips are needed to reach the treasure island from his/her own island. Between two islands, there are at most one direct ship routes, and ships travel in both directions. We also know that every island is reachable from every island via ship trips.

2. Let $ABCD$ be a parallelogram, and let E be the midpoint of side CD. Denote the intersection of segments AE and BD by F. Suppose that the angle AEB is a right angle and $EB = ED$. Calculate the angle AFB.

3. There are 100 people seated around a round table: 50 knights who always tell the truth and 50 knaves who always lie. Mark enters the room, chooses someone sitting at the table, and starting from that person, moving clockwise, asks each person the question: "Among the answers given so far, was the number of 'yes' answers even?" Can the people be seated in such a way that no matter who Mark asks first, he always gets the same number of 'yes' answers?

4. Let $a_1, a_2, \dots, a_{2023}$ be real numbers such that

- $a_{2023} = a_1$,
- and for every $n \geq 3$ we have $a_n = \frac{a_{n-1}+a_{n-2}}{2} - 1$, so from the third number onwards, each number is one less than the average of the two preceding numbers.

Prove that $a_n \geq a_1$ holds for all $1 \leq n \leq 2023$.

5. A round table is surrounded by $n \geq 2$ people, each assigned one of the integers $0, 1, \dots, n-1$ such that no two people received the same number. In each round, everyone adds their number to their right neighbour's number, and their new number becomes the remainder of the sum when divided by n. We call an initial configuration of the integers *glorious* if everyone's number remains the same after some finite number of rounds, never changing again.

a) For which integers $n \geq 2$ is every initial configuration glorious?

b) For which integers $n \geq 2$ is there no glorious initial configuration at all?

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.

Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVII. Dürer Competition