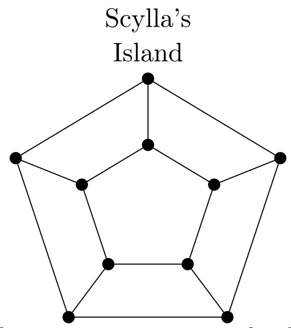
XVII. DÜRER COMPETITION

E CATEGORY

SOLUTIONS

DATE: 24TH NOVEMBER 2023.



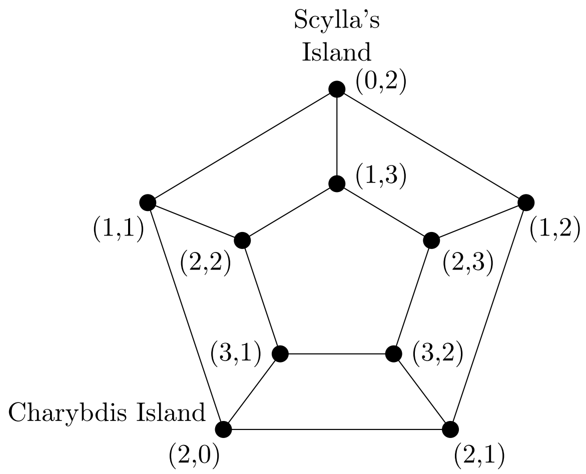
E1. **a)** This is the map of the ten islands of the Düreric Ocean. There is a treasure hidden on one of the islands. Two islands are connected on the map if there is a direct ship connection between them. Leila, who has a friend on each of the islands, wants to find the treasure. Before she visits the archipelago, she wants to make sure she knows where to go, therefore she calls some of her friends on the phone. Any friend she calls can only tell her how many direct ship trips are needed to reach the treasure island from his/her own island. How many people does Leila need to call in order to be able to tell the location of the treasure with certainty, if first she calls Lily who lives on Scylla's Island?

b) This time Leila is visiting another archipelago made up of five islands and similarly to the previous part, one island holds a treasure. Leila managed to find out which islands are connected by direct ship connections. After a bit of thinking she discovered that she could definitely determine the location of the treasure by calling not more than two of her friends living on different islands. Based on this, what is the maximum number of direct ship connections between the islands?

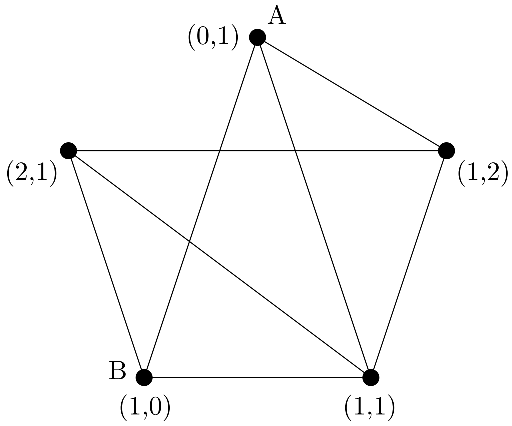
The conditions are the same as in the first part: on each island, she has a friend whom she can call, and the friend will tell her how many direct ship trips are needed to reach the treasure island from his/her own island. Between two islands, there is at most one direct ship route, and ships travel in both directions. We also know that every island is reachable from every island via ship trips.

Solution: a) It is not enough for Leila to only call Lily: if, for instance, the treasure is just one ship trip away from Scylla's Island, she would not be able to determine which neighbouring island it is on.

Let's call the bottom left island on the map Charybdis Island. We will see that it is sufficient for Leila to only call Scylla and Charybdis Islands, and based on the obtained information, she can determine the treasure island uniquely. The treasure island can be determined uniquely if and only if for any two islands, it holds that they are not the same distance away from either Scylla or Charybdis. On the map, we have written two numbers for each island, the first being the minimum number of ship trips needed from Scylla to get there, and the second being the number of trips needed from Charybdis. It can be seen that there are indeed no two islands for which we wrote the same pair of numbers. Therefore, by calling only Scylla and Charybdis Islands, Leila can indeed determine the treasure island uniquely, hence the answer to the problem is two.



b) Consider the following configuration:



If we label the top island as A and the bottom left one as B, then similarly to the previous part of the problem, by assigning pairs of coordinates based on these, each island will have different coordinates (see diagram). Thus, it is sufficient to call these two islands to determine the location of the treasure. In this setup, there are 8 ship connections, and it only remains to show that there cannot be more than this.

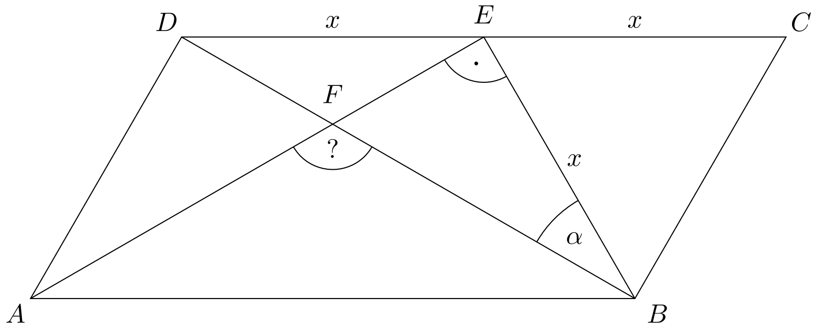
Having 10 ship connections is clearly impossible because no matter which two islands we choose to call, the other three islands are at a distance of 1 from both, hence they cannot be distinguished.

Similarly, 9 ship connections cannot work because calling any two islands would result in another pair of islands, each at a distance of 1 from both of the called islands. (There is one pair of islands not connected, say A and B, and one has to check three cases depending on whether we call both of A and B, or just one of them and one other island, or two other islands.) Therefore, the location of the treasure cannot be determined uniquely in this case either.

Therefore the answer to the problem is 8.

E2. Let $ABCD$ be a parallelogram, and let E be the midpoint of side CD. Denote the intersection of segments AE and BD by F. Suppose that the angle AEB is a right angle and $EB = ED$. Calculate the angle AFB.

Solution:



Let $\angle DBE = \alpha$. As $EB = ED = x$, triangle DEB is isosceles, therefore $\angle BDE = \angle DBE = \alpha$. As $\angle BDE$ and $\angle ABD$ are alternate angles, we have $\angle ABD = \alpha$.

Since E is the midpoint of side CD, we have $ED = EC = x$, from which $CD = 2x$. As $ABCD$ is a parallelogram, we have $AB = CD = 2x$, so in the right-angled triangle AEB, the leg EB is half as long as the hypotenuse AB, therefore AEB is half of a regular triangle. Because of this, $ABE\sphericalangle = 2\alpha = 60°$, so $\alpha = 30°$, meaning that $EAB\sphericalangle = 30°$.

Now considering the angles of triangle AFB and rearranging the equality we get that

$$AFB\sphericalangle = 180° - 2 \cdot 30° = 120°.$$

E3. There are 100 people seated around a round table: 50 knights who always tell the truth and 50 knaves who always lie. Mark enters the room, chooses someone sitting at the table, and starting from that person, moving clockwise, asks each person the question: "Among the answers given so far, was the number of 'yes' answers even?" Can the people be seated in such a way that no matter who Mark asks first, he always gets the same number of 'yes' answers?

Solution: Let us examine how the parity of the total number of 'yes' answers depends the last person.

If the last person to answer is a knight and the number of 'yes' answers before his response is even, then he answers 'yes'. This results in an odd number of 'yes' answers overall. On the other hand, if the number of 'yes' answers before the last knight's response is odd, then the answer is 'no'. This also leads to an odd number of 'yes' answers overall.

However, if the last person to answer is a knave, and the number of 'yes' answers before his response is even, then he answers 'no'. This results in an even number of 'yes' answers overall. Conversely, if the number of 'yes' answers before the last knave's response is odd, then the last answer is 'yes'. This also leads to an even number of 'yes' answers overall.

Since there are both knights and knaves among the people, there will be arrangements where the last responder is a knight and others where it is a knave. However, in one case, the number of 'yes' answers is odd, while in the other case, it is even. Therefore, there is no arrangement where, regardless of the starting person, Mark receives the same number of 'yes' answers.

In conclusion, there is no seating for which Mark gets the same number of 'yes' answers regardless of whom he asks first.

E4. Let $a_1, a_2, \ldots, a_{2023}$ be real numbers such that

- $a_{2023} = a_1$,
- and for every $n \geq 3$ we have $a_n = \frac{a_{n-1}+a_{n-2}}{2} - 1$, so from the third number onwards, each number is one less than the average of the two preceding numbers.

Prove that $a_n \geq a_1$ holds for all $1 \leq n \leq 2023$.

Solution: We will prove the claim by contradiction: let us suppose there exists an a_n such that $a_n < a_1$, and we will show that it leads to a contradiction.

First we show by induction that if there is a value k for which $a_k < a_1$ and $a_{k+1} < a_1$ both hold, then $a_{2023} < a_1$, which is a contradiction. After this we will show that if there is an a_n such that $a_n < a_1$, then $a_{n+1} < a_1$ holds as well, which is sufficient to complete the proof.

From the first assumption let us show that for all $l > k+1$, $a_l < a_1$ holds. The assumption was $a_k < a_1$ and $a_{k+1} < a_1$, now we get that $a_{k+2} = \frac{a_k+a_{k+1}}{2} - 1 < \frac{2a_1}{2} - 1 < a_1$. So we have that if two consecutive elements are smaller then a_1, then all later elements are smaller than a_1. So this would mean $a_{2023} < a_1$, which is a contradiction.

Now we only have to show that if n is the smallest for which $a_n < a_1$, then $a_{n+1} < a_1$ holds as well. If $n = 2$, then

$$a_3 = \frac{a_1 + a_2}{2} - 1 < \frac{2a_1}{2} - 1 < a_1.$$

If $n > 2$, then we know that $a_n < a_1 \leq a_{n-2}$. And $a_{n+1} = \frac{a_{n-1}+a_n}{2} - 1 < \frac{a_{n-1}+a_{n-2}}{2} - 1 = a_n < a_1$.

So we showed the following, which is sufficient to prove that $a_n \geq a_1$ for all $1 \leq n \leq 2023$. If there is an n for which $a_n < a_1$, then there is also an n for which $a_n < a_1$ and $a_{n+1} < a_1$. And if there is such an n, then for all $k > n$ we have $a_k < a_1$, namely $a_{2023} < a_1$, which is a contradiction.

E5. A round table is surrounded by $n \geq 2$ people, each assigned one of the integers $0, 1, \ldots, n-1$ such that no two people received the same number. In each round, everyone adds their number to their right neighbour's number, and their new number becomes the remainder of the sum when divided by n. We call an initial configuration of the integers *glorious* if everyone's number remains the same after some finite number of rounds, never changing again.
a) For which integers $n \geq 2$ is every initial configuration glorious?
b) For which integers $n \geq 2$ is there no glorious initial configuration at all?

Solution: We will show that if n is odd then there is no glorious initial configuration; if n is a power of 2 then all initial configurations are glorious, and if n is even but not a power of 2 then there exist both glorious and non-glorious initial configurations.

A configuration is glorious if and only if it reaches the all-0 state, since if one person's number does not change, then their right neighbour had to have 0, therefore if no one's number changes, then it means that everyone had 0.

Firstly we show that if n is odd, then there is no way of reaching all zeros. Suppose the opposite. Then before the all zero state let the number of the ith person be k_i. We know that $k_i + k_{i+1} \equiv 0 \ (mod\ n)$ for all $i \ (mod\ n)$, meaning that $k_i \equiv -k_{i+1} \ (mod\ n)$, therefore $k_i \equiv k_{i+2} \ (mod\ n)$. Continuing this we get that $k_i \equiv k_{i+2} \equiv \ldots \equiv k_{i+2n-2} \ (mod\ n)$, this includes everyone since 2 is coprime to n. Therefore $k_i \equiv k_{i+1} \ (mod\ n)$ holds as well, but since $k_i \equiv -k_{i+1} \ (mod\ n)$, this means that $k_{i+1} \equiv 0 \ (mod\ n)$, therefore all k_i are zero. Therefore before the all zeros state there had to be all zeros as well, therefore this position cannot be reached from any starting position if n is odd.

Now we show that if n is even but not a power of 2, then there are both glorious and non-glorious configurations. Let the numbers of the $n = 2l$ people be $0, 1, -2, 3, -4, \ldots, -(2l-2), 2l-1$ in this order. (These are $2l$ numbers and are all different modulo n.) Then after the first step the numbers are $1, -1, 1, -1, \ldots, 1, -1$, meaning that they reach all zeros after the second step. However if the initial configuration is $0, 1, \ldots, 2l-2, 2l-1$, then it is not glorious: let $n = 2^a b$ where a is a positive integer and $b > 1$ is odd. After the first step the numbers are $1, 3, 5, \ldots, 4l-5, 4l-3, 2l-1$, meaning that the difference between neighbours is always 2. This means that in the next step the difference will be 4 everywhere and after the a steps the difference will be 2^a. Since 2^a is coprime to b, this means that after a steps the first b numbers will be different modulo b and the $b+1$th one will be the same as the first one modulo b. Moreover modulo b the numbers form 2^a cycles, each cycle containing all remainders. Therefore from here on each cycle will behave as if there were only b people, and since b is odd, the all zero state can never be reached. Therefore if n is even but not a power of 2, then both types of configurations exist.

Finally we show that if n is a power of 2, then all configurations are glorious. Let $n = 2^a$ and the initial numbers in order be $k_1, k_2, \ldots, k_{2^a}$. Since in every step everyone adds the right hand neighbour's number to theirs, after m steps the ith person will have

$$\binom{m}{0} k_i + \binom{m}{1} k_{i+1} + \binom{m}{2} k_{i+2} + \cdots + \binom{m}{m} k_{i+m}$$

where the index of k is considered modulo n. Let us observe the case where $m = 2^{a+1} - 1$. We know that $\binom{2^{a+1}-1}{j}$ is odd for all $0 \leq j \leq 2^{a+1} - 1$, therefore after $2^{a+1} - 1$ steps the number of the ith person is

$$\binom{2^{a+1}-1}{0} k_i + \binom{2^{a+1}-1}{1} k_{i+1} + \cdots + \binom{2^{a+1}-1}{2^{a+1}-1} k_{i+2^{a+1}-1},$$

where there are 2^{a+1} terms, meaning that every k_j appears in exactly two of the terms. Since every coefficient of k_j is of the form $\binom{2^{a+1}-1}{j}$, which is odd, therefore after summing these we get that the coefficient of k_j in the whole sum is even. This means that after $2^{a+1} - 1$ steps everyone will have an even number. After performing $2^{a+1} - 1$ more steps, everyone's number will have another factor of 2, so will be divisible by 4. Therefore we can reach a state where every number is divisible by 2^a, meaning that modulo 2^a we reached all zeros, therefore we have proven that in this case all initial configurations are glorious.