XVII. DÜRER COMPETITION E+ CATEGORY

PROBLEMS

DATE: 24TH NOVEMBER 2023.

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1. There are 100 merchants selling salmon for Dürer dollars around the circular shore of the island of Dürerland. Since the beginning of times good and bad years have been alternating on the island. (So after a good year, the next year is bad; and after a bad year, the next year is good.) In every good year all merchants set their price as the maximum value between their own selling price from the year before and the selling price of their left-hand neighbour from the year before. In turn, in every bad year they sell it for the minimum between their own price from the year before and their left-hand neighbour's price from the year before. Paul and Pauline are two merchants on the island. This year Paul is selling salmon for 17 Dürer dollars a kilogram. Prove that there will come a year when Pauline will sell salmon for 17 Dürer dollars a kilogram.
The merchants are immortal, they have been selling salmon on the island for thousands of years and will continue to do so until the end of time.

2. One quadrant of the Cartesian coordinate system is tiled by dominoes of size 1 cm $\times$ 2 cm. The dominoes don't overlap with each other, they cover the entire quadrant and they all fit in the quadrant. Farringdon, the flea is sitting at the origin in the beginning and is allowed to jump from one corner of a domino to the opposite corner any number of times. Is it possible that the dominoes are arranged in a way that Farringdon is unable to move more than 2023 cm away from the origin?
A quadrant *is one quarter of the plane with its boundaries being two perpendicular rays from the origin. An example of a quadrant is* $\{(x, y) : x, y \geq 0\}$.

3. A round table is surrounded by $n \geq 2$ people, each assigned one of the integers $0, 1, \ldots, n-1$ such that no two people received the same number. In each round, everyone adds their number to their right neighbour's number, and their new number becomes the remainder of the sum when divided by n. We call an initial configuration of the integers *glorious* if everyone's number remains the same after some finite number of rounds, never changing again.
a) For which integers $n \geq 2$ is every initial configuration glorious?
b) For which integers $n \geq 2$ is there no glorious initial configuration at all?

4. In the game of *Calculabyrinth* two players control an adventurer in an underwater dungeon. The adventurer starts with h hit points, where h is an integer greater than one. The dungeon consists of several chambers. There are some passageways in the dungeon, each leading from a chamber to a chamber. These passageways are one-way, and a passageway may return to its starting chamber. Every chamber can be exited through at least one passageway. There are 5 types of chambers:

- Entrance: the adventurer starts here, no passageway comes in here;
- Hollow: nothing happens;
- Spike: the adventurer loses a hit point;
- Trap: the adventurer gets shot by an arrow;
- Catacomb: the adventurer loses hit points equal to the total number of times they have been hit by an arrow.

The two players take turns controlling the character, always moving them through one passageway. A player loses if the adventurer's hit points fall below zero due to their action (at 0 hit points, the character stays alive). Show an example of a dungeon map, which consists of at most 20 chambers and contains exactly one Entrance, with the following condition: the first player has a winning strategy if h is a prime, and the second player has a winning strategy if h is composite.
If the game doesn't end after a finite number of moves, neither player wins.

5. For a given triangle $A_1A_2A_3$ and a point X inside of it we denote by X_i the intersection of line A_iX with the side opposite to A_i for all $1 \leq i \leq 3$.
Let P and Q be distinct points inside the triangle. We then say that the two points are *isotomic* (or we say they form an *isotomic pair*) if for all i the points P_i and Q_i are symmetric with respect to the midpoint of the side opposite to A_i.

Augustus wants to construct isotomic pairs with his favourite app, GeoZebra. In fact, he already constructed the vertices and sidelines of a non-isosceles acute triangle when suddenly his computer got infected with a virus. Most tools became unavailable, only a few are usable, some of which even require a fee:

Name of tool	**Description**	**Fee (per use)**
Point	select an arbitrary point (with respect to the position of the mouse) on the plane or on a figure (circle or line)	free
Intersection	intersection points of two figures (where each figure is a circle or a line)	free
Line	line through two points	5 Dürer dollars
Perpendicular	perpendicular from a point to an already constructed line	50 Dürer dollars
Circumcircle	circle through three points	10 Dürer dollars

a) Agatha selected a point P inside the triangle, which is not the centroid of the triangle. Show that Augustus can construct a point Q at a cost of at most 1000 Dürer dollars such that P and Q are isotomic.
b) Prove that for all positive integers n Augustus can construct n different isotomic pairs at a cost of at most $200 + 10n$ Dürer dollars.
*In both parts, partial points may be awarded for constructions exceeding Augustus's budget. The parts are unrelated, that is Augustus can't use his constructions from part **a)** in part **b)** .*

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVII. Dürer Competition