XVII. DÜRER COMPETITION E+ CATEGORY

SOLUTIONS

DATE: 24TH NOVEMBER 2023.

E+1. There are 100 merchants selling salmon for Dürer dollars around the circular shore of the island of Dürerland. Since the beginning of times good and bad years have been alternating on the island. (So after a good year, the next year is bad; and after a bad year, the next year is good.) In every good year all merchants set their price as the maximum value between their own selling price from the year before and the selling price of their left-hand neighbour from the year before. In turn, in every bad year they sell it for the minimum between their own price from the year before and their left-hand neighbour's price from the year before. Paul and Pauline are two merchants on the island. This year Paul is selling salmon for 17 Dürer dollars a kilogram. Prove that there will come a year when Pauline will sell salmon for 17 Dürer dollars a kilogram.
The merchants are immortal, they have been selling salmon on the island for thousands of years and will continue to do so until the end of time.

Solution: During the proof, indices are always used modulo 100. Number the merchants from 1 to 100, starting from some merchant and proceeding to the left. Let a_1, a_2, ..., a_{100} denote the salmon prices in one specific year, called year A, and let b_1, b_2, ..., b_{100} denote the salmon prices in the year immediately following year A, called year B. Finally let $c_1, c_2, ..., c_{100}$ denote the prices in the year C immediately following year B. The conditions of the problem can be reworded to say that for every $1 \leq i \leq 100$, $b_i = \max(a_i, a_{i+1})$ if year B is good, and $b_i = \min(a_i, a_{i+1})$ if year B is bad; and the transition between years B and C can be described similarly.

Suppose that year B is good. Observe that in this case there cannot be an index i such that $b_{i-1} < b_i$ and $b_{i+1} < b_i$, since

$$\max(b_{i-1}, b_{i+1}) = \max(a_{i-1}, a_i, a_{i+1}, a_{i+2}) \geq \max(a_i, a_{i+1}) = b_i.$$

So in a good year, no merchant can have a price strictly greater than both neighbours' prices, and similarly in a bad year, no merchant can have a price strictly lower than both neighbours' prices.

If we use this observation for bad year A, this means that for every i, at least one of $b_i = \max(a_i, a_{i+1})$ and $b_{i+1} = \max(a_{i+1}, a_{i+2})$ is equal to a_{i+1}. Furthermore it is clear that both quantities are at least a_{i+1}, so

$$c_i = \min(b_i, b_{i+1}) = a_{i+1}.$$

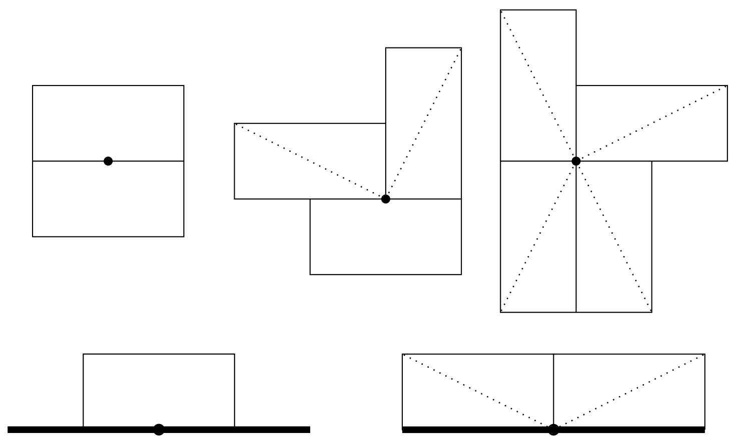
So in year C, all merchants will use exactly the same prices as what their left-hand neighbour used two years ago. Clearly this justification is valid for any year: in particular, a similar reasoning can be used if A and C are good years and B is bad.

So if Pauline is k spaces to the right of Paul along the shore of the island, then after $2k$ years Pauline will sell salmon for 17 Dürer dollars.

E+2. One quadrant of the Cartesian coordinate system is tiled by dominoes of size 1 cm $\times$ 2 cm. The dominoes don't overlap with each other, they cover the entire quadrant and they all fit in the quadrant. Farringdon, the flea is sitting at the origin in the beginning and is allowed to jump from one corner of a domino to the opposite corner any number of times. Is it possible that the dominoes are arranged in a way that Farringdon is unable to move more than 2023 cm away from the origin?

A quadrant *is one quarter of the plane with its boundaries being two perpendicular rays from the origin. An example of a quadrant is* $\{(x, y) : x, y \geq 0\}$.

Solution: Let us consider the graph where the vertices are the corners of the dominoes, and two vertices are joined by an edge if and only if they are opposite corners of the same domino. This is equivalent to saying that Farringdon can directly jump from one vertex to the other. We can see that for each grid point of the quadrant, there are an even number of edges meeting there, except at the origin where there is only one (the figures show each distinct case). So if the set of vertices Farringdon can reach only consisted of finitely many vertices, then in this subgraph the sum of the degrees of the vertices would be odd, which is impossible. So the subgraph reachable by Farringdon cannot be finite, so Farringdon can reach points arbitrarily far from the origin.



E+3. A round table is surrounded by $n \geq 2$ people, each assigned one of the integers $0, 1, \ldots, n-1$ such that no two people received the same number. In each round, everyone adds their number to their right neighbour's number, and their new number becomes the remainder of the sum when divided by n. We call an initial configuration of the integers *glorious* if everyone's number remains the same after some finite number of rounds, never changing again.
a) For which integers $n \geq 2$ is every initial configuration glorious?
b) For which integers $n \geq 2$ is there no glorious initial configuration at all?

Solution: We will show that if n is odd then there is no glorious initial configuration; if n is a power of 2 then all initial configurations are glorious, and if n is even but not a power of 2 then there exist both glorious and non-glorious initial configurations.

A configuration is glorious if and only if it reaches the all-0 state, since if one person's number does not change, then their right neighbour had to have 0, therefore if no one's number changes, then it means that everyone had 0.

Firstly we show that if n is odd, then there is no way of reaching all zeros. Suppose the opposite. Then before the all zero state let the number of the ith person be k_i. We know that $k_i + k_{i+1} \equiv 0 \ (mod\ n)$ for all $i\ (mod\ n)$, meaning that $k_i \equiv -k_{i+1}\ (mod\ n)$, therefore $k_i \equiv k_{i+2}\ (mod\ n)$. Continuing this we get that $k_i \equiv k_{i+2} \equiv \ldots \equiv k_{i+2n-2}\ (mod\ n)$, this includes everyone since 2 is coprime to n. Therefore $k_i \equiv k_{i+1}\ (mod\ n)$ holds as well, but since $k_i \equiv -k_{i+1}\ (mod\ n)$, this means that $k_{i+1} \equiv 0\ (mod\ n)$, therefore all k_i are zero. Therefore before the all zeros state there had to be all zeros as well, therefore this position cannot be reached from any starting position if n is odd.

Now we show that if n is even but not a power of 2, then there are both glorious and non-glorious configurations. Let the numbers of the $n = 2l$ people be $0, 1, -2, 3, -4, \ldots, -(2l-2), 2l-1$ in this order. (These are $2l$ numbers and are all different modulo n.) Then after the first step the numbers are $1, -1, 1, -1, \ldots, 1, -1$, meaning that they reach all zeros after the second step. However if the initial configuration is $0, 1, \ldots, 2l-2, 2l-1$, then it is not glorious: let $n = 2^a b$ where a is a positive integer and $b > 1$ is odd. After the first step the numbers are $1, 3, 5, \ldots, 4l-5, 4l-3, 2l-1$, meaning that the difference between neighbours is always 2. This means that in the next step the difference will be 4 everywhere and after the a steps the difference will be 2^a. Since 2^a is coprime to b, this means that after a steps the first b numbers will be different modulo b and the $b+1$th one will be the same as the first one modulo b. Moreover modulo b the numbers form 2^a cycles, each cycle containing all remainders. Therefore from here on each cycle will behave as if there were only b people, and since b is odd, the all zero state can never be reached. Therefore if n is even but not a power of 2, then both types of configurations exist.

Finally we show that if n is a power of 2, then all configurations are glorious. Let $n = 2^a$ and the initial numbers in order be $k_1, k_2, \ldots, k_{2^a}$. Since in every step everyone adds the right hand neighbour's number to theirs, after m steps the ith person will have

$$\binom{m}{0}k_i + \binom{m}{1}k_{i+1} + \binom{m}{2}k_{i+2} + \cdots + \binom{m}{m}k_{i+m}$$

where the index of k is considered modulo n. Let us observe the case where $m = 2^{a+1}-1$. We know that $\binom{2^{a+1}-1}{j}$ is odd for all $0 \leq j \leq 2^{a+1}-1$, therefore after $2^{a+1}-1$ steps the number of the ith person is

$$\binom{2^{a+1}-1}{0}k_i + \binom{2^{a+1}-1}{1}k_{i+1} + \cdots + \binom{2^{a+1}-1}{2^{a+1}-1}k_{i+2^{a+1}-1},$$

where there are 2^{a+1} terms, meaning that every k_j appears in exactly two of the terms. Since every coefficient of k_j is of the form $\binom{2^{a+1}-1}{j}$, which is odd, therefore after summing these we get that the coefficient of k_j in the whole sum is even. This means that after $2^{a+1}-1$ steps everyone will have an even number. After performing $2^{a+1}-1$ more steps, everyone's number will have another factor of 2, so will be divisible by 4. Therefore we can reach a state where every number is divisible by 2^a, meaning that modulo 2^a we reached all zeros, therefore we have proven that in this case all initial configurations are glorious.

E+4. In the game of *Calculabyrinth* two players control an adventurer in an underwater dungeon. The adventurer starts with h hit points, where h is an integer greater than one. The dungeon consists of several chambers. There are some passageways in the dungeon, each leading from a chamber to a chamber. These passageways are one-way, and a passageway may return to its starting chamber. Every chamber can be exited through at least one passageway. There are 5 types of chambers:

- Entrance: the adventurer starts here, no passageway comes in here;
- Hollow: nothing happens;
- Spike: the adventurer loses a hit point;
- Trap: the adventurer gets shot by an arrow;
- Catacomb: the adventurer loses hit points equal to the total number of times they have been hit by an arrow.

The two players take turns controlling the character, always moving them through one passageway. A player loses if the adventurer's hit points fall below zero due to their action (at 0 hit points, the character stays alive). Show an example of a dungeon map, which consists of at most 20 chambers and contains exactly one Entrance, with the following condition: the first player has a winning strategy if h is a prime, and the second player has a winning strategy if h is composite.
If the game doesn't end after a finite number of moves, neither player wins.

Solution: Let us denote the first player with A and the second player with B. The idea is the following: we are constructing a series of chambers where firstly B can decide how many times the the adventurer gets hit by an arrow (but at least twice), then again B decides how many catacombs the player goes through (again at least twice). During this the adventurer loses hit points, which can be any composite number of B's choice. Finally we take A into a spike, who loses if the adventurer had 0 lives, otherwise B loses by getting into an infinite series of spikes.

Now we will detail the more rigorous proof. Consider the map below where E denotes the entrance, T is a trap, C is a catacomb, S is a spike, and the arrows denote passageways. The shape of the chambers and the rooms **X**, **Y**, **Z** are only for demonstration purposes.

Firstly observe that A can only visit the square-shaped rooms and B can only visit the circular ones. Moreover, players only have a choice in rooms **X**, **Y** and **Z**, everywhere else the next step is determined. Now we are going to consider two cases.

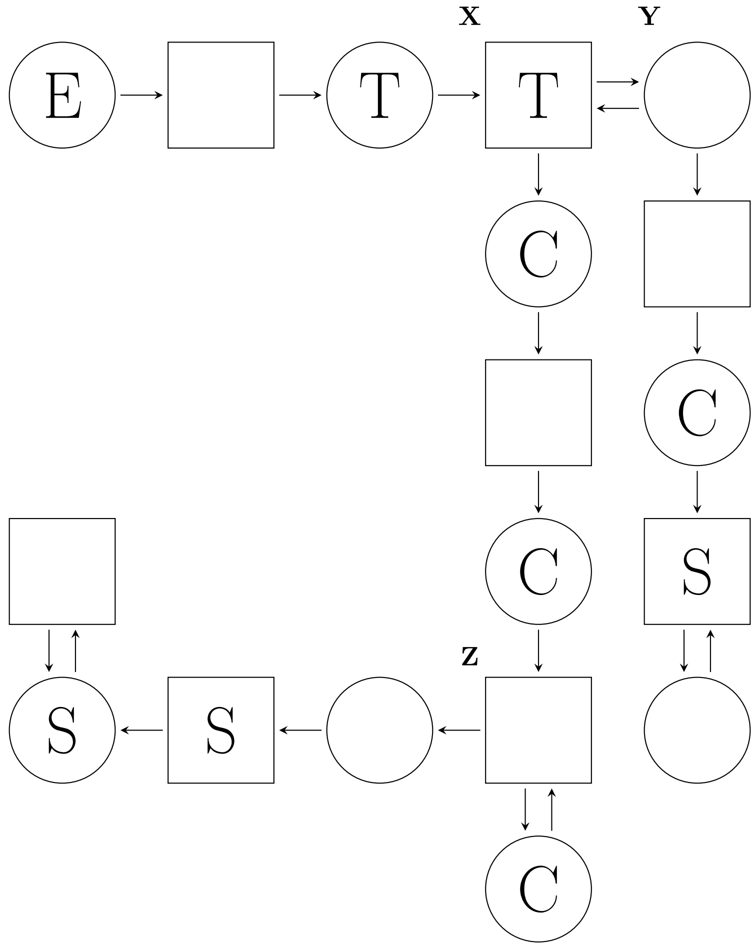
Case 1: h is a prime.

We show that in this case A has a winning strategy. Let the strategy be the following. If we are in room **Y** and the adventurer has been hit by more than h arrows, then move down, otherwise move left. This is a strategy for A, since they do not have a choice in the other rooms. Now let us investigate what B can do. If B moves right the first h times at **X**, then the first $h-1$ times A moves back to **X** from **Y**, and when the player has been hit by $h+1$ arrows, A moves down and B loses in the catacomb. If after some $0 \leq c < h$ moves to **Y** B moves downwards, then the player has been hit by $c+1$ arrows. Then if the adventurer is still alive, B has a choice in room **Z**. Observe that until B only moves down, A cannot lose. Assume that B moved down from **Z** $k \geq 0$ times, therefore the adventurer lost $(c+2)(k+2)$ hit points. Since we assumed that the player is still alive and h prime (therefore $h-(c+2)(k+2) > 0$), the adventurer has a positive number of hit points. This means that after moving left, A will not lose, but then in the infinite series of spikes B will. Therefore we can conclude that this is a winning strategy for A.

Case 2: h is composite.

Let $h = (c+2)(k+2)$ where $c, k \geq 0$. The strategy of B will be as follows. In room **X** move right the first c times, then down. In room **Z** move down the first k times and then to the left. Player A only has a choice in room **Y**. According to B's strategy the player visits room **Y** at most c times. Therefore if A decides to move downward from **Y**, the play will be hit by at most $c+1 < h$ arrows, therefore B does not die in the catacomb, but A will in the infinite series of spikes. If A goes back to **X** every time, then after the $(c+1)$st time in **X**, B will move down and A will not have any more choice. After moving down from **Z** k times, B moves to the left. At this point the player has been hit by $(c+2)$ arrows and visited $(k+2)$ catacombs, therefore has exactly $h-(c+2)(k+2) = 0$ hit points remaining. This means that when A enters the spike, they die, proving that this is indeed a winning strategy for B, and this concludes the proof.

Note: We believe that any proof will be based on a similar idea, but there are many different such maps, including ones consisting of fewer chambers. In this proof the aim was to provide a solution that is easy to follow, not necessarily to include a minimal example. Luckily this is still under the 20 chamber limit.



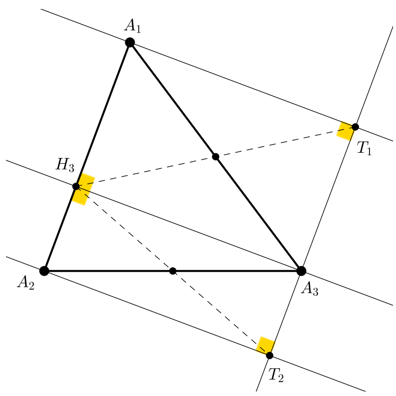
E+5. For a given triangle $A_1A_2A_3$ and a point X inside of it we denote by X_i the intersection of line A_iX with the side opposite to A_i for all $1 \le i \le 3$.
Let P and Q be distinct points inside the triangle. We then say that the two points are *isotomic* (or we say they form an *isotomic pair*) if for all i the points P_i and Q_i are symmetric with respect to the midpoint of the side opposite to A_i.

Augustus wants to construct isotomic pairs with his favourite app, GeoZebra. In fact, he already constructed the vertices and sidelines of a non-isosceles acute triangle when suddenly his computer got infected with a virus. Most tools became unavailable, only a few are usable, some of which even require a fee:

Name of tool	Description	Fee (per use)
Point	select an arbitrary point (with respect to the position of the mouse) on the plane or on a figure (circle or line)	free
Intersection	intersection points of two figures (where each figure is a circle or a line)	free
Line	line through two points	5 Dürer dollars
Perpendicular	perpendicular from a point to an already constructed line	50 Dürer dollars
Circumcircle	circle through three points	10 Dürer dollars

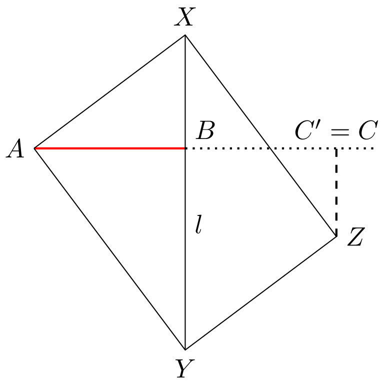
a) Agatha selected a point P inside the triangle, which is not the centroid of the triangle. Show that Augustus can construct a point Q at a cost of at most 1000 Dürer dollars such that P and Q are isotomic.
b) Prove that for all positive integers n Augustus can construct n different isotomic pairs at a cost of at most $200 + 10n$ Dürer dollars.
In both parts, partial points may be awarded for constructions exceeding Augustus's budget. The parts are unrelated, that is Augustus can't use his constructions from part ***a)*** *in part* ***b)*** .

Solution: In both parts of the problem, D\$ represents Dürer dollars. Let the vertices of our triangle be A_1, A_2, A_3. We will frequently use the notation X_i as mentioned in the problem.
a) We will construct Q straight from the definition. For this, we need the midpoints of two sides of the triangle. We demonstrate a relatively simple but less cost-effective construction (a more economical solution will be presented in part **b)**). Draw a perpendicular from A_3 to line A_1A_2 (50 D\$), denote the point of intersection as H_3. Additionally, draw a perpendicular to line A_3H_3 at A_3 (50 D\$), then draw perpendiculars from points A_1 and A_2 to this new line ($2 \cdot 50$ D\$ $= 100$ D\$); let T_1 and T_2 denote the resulting intersections. Notice that due to the right angles, quadrilaterals $A_1T_1A_3H_3$ and $A_2H_3A_3T_2$ are rectangles, so their diagonals bisect each other. Therefore, if we draw lines T_1H_3 and T_2H_3 ($2 \cdot 5$ D\$ $= 10$ D\$), they intersect sides A_1A_3 and A_2A_3 at their midpoints.



It remains to reflect points P_1 and P_2 in the midpoints. For this, we will use the following construction lemma: we can reflect a point in another point.

Construction: Suppose that we wish to reflect point A in point B. Construct the line through B perpendicular to AB, and call it l. Let $X \neq B$ be an arbitrary point of line l. Draw a perpendicular to line AX at A, and let it intersect l at point Y. Finally, let the perpendicular to AX at X and the perpendicular to AY at Y meet at Z. Drop a perpendicular from Z to AB, with its foot being C'. Then in triangle $AC'Z$, line l is parallel to the base $C'Z$, and bisects segment AZ, since the diagonals of a rectangle bisect each other. So l is a midsegment of $AC'Z$, therefore B is the midpoint of AC'. So the requested point is $C' = C$.



We can check that since we already had line AB constructed (as a sideline of ABC), the construction above costs us $5 \cdot 50$ D\$$+5$ D\$ $= 255$ D\$. To summarize: we take point P, construct points P_1 and P_2 ($2 \cdot 5$ D\$ $= 10$ D\$), then we reflect them in the midpoints of the corresponding segments ($2 \cdot 255$ D\$ $= 510$ D\$), hence getting points Q_1 and Q_2. The intersection of lines A_1Q_1 and A_2Q_2 is Q ($2 \cdot 5$ D\$ $= 10$ D\$). We prove that Q and P form an isotomic pair indeed. Clearly, by the method of construction of Q, it suffices to show that points P_3 and Q_3 are symmetrical with respect to the midpoint of side A_1A_2. Write down Ceva's theorem for points P and Q:

$$\frac{A_1P_3}{P_3A_2} \cdot \frac{A_2P_1}{P_1A_3} \cdot \frac{A_3P_2}{P_2A_1} = 1 = \frac{A_1Q_3}{Q_3A_2} \cdot \frac{A_2Q_1}{Q_1A_3} \cdot \frac{A_3Q_2}{Q_2A_1}.$$

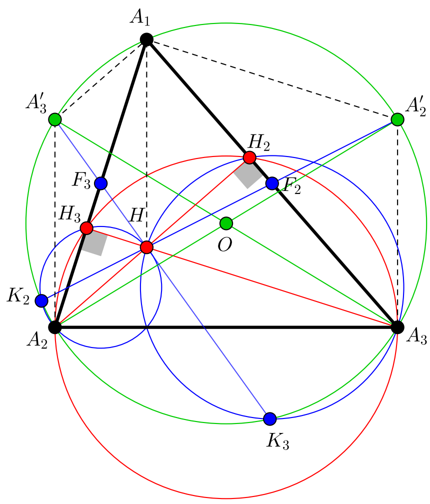
Since $A_2P_1 = Q_1A_3$, $P_1A_3 = A_2Q_1$, $A_3P_2 = Q_2A_1$, $P_2A_1 = A_3Q_2$, we see that $\frac{A_1P_3}{P_3A_2} = \frac{Q_3A_2}{A_1Q_3}$. However, as a point X moves along side A_1A_2, the ratio $\frac{A_1X}{XA_2}$ takes every positive real number precisely once, so the previous equality can only hold if $A_1P_3 = Q_3A_2$ and $P_3A_2 = A_1Q_3$, that is, if points P_3, Q_3 are indeed symmetrical in the midpoint.

Altogether we spent $50 + 50 + 100 + 10 + 10 + 510 + 10 = 740$ D\$, so we have nicely remained within our budget. It is easy to see that $P \neq Q$, otherwise P would be the centroid.

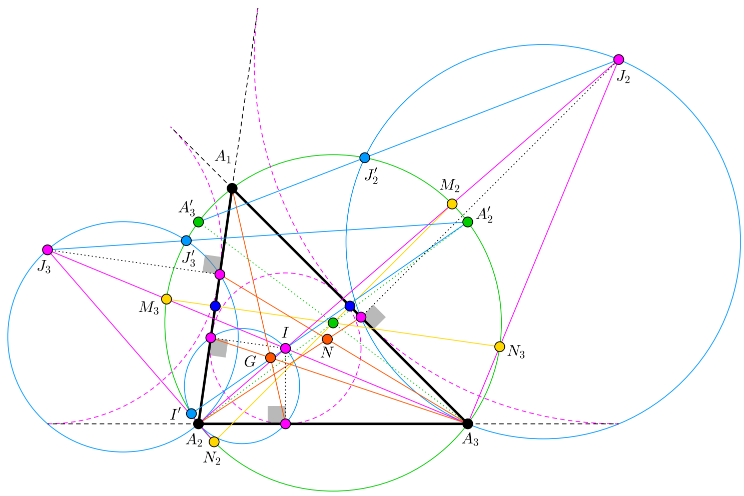
b) The task consists of two parts: first we show that we can construct one isotomic pair for 210 D\$, and then we show that from here we can always construct new pairs for 10 D\$ each.

As the centroid cannot be used because of the definition, we need to find another isotomic pair. We can recall that the touchpoints of the incircle and excircles lie symmetrically on each side. From Ceva's theorem, it is also clear that if we connect each vertex with the opposite touchpoint of the incircle, we get three concurrent lines (since if we write down the subdivision ratios on each side, each of three tangent lengths from A, B, C to the incircle will appear once as a numerator and once as a denominator). We can make the same observation about the touchpoints of the excircles on each side. The two common intersection points are called the Gergonne and Nagel points of the triangle. These are what we will construct. (Since the triangle is not equilateral, the two points differ.)

Drop a perpendicular from A_2 to line A_1A_3 (50 D\$), let its foot be H_2. Then construct circle $(A_2A_3H_2)$ (10 D\$). Since this will be the circle with diameter A_2A_3, the intersection of the circle and A_1A_2 will be the foot of the altitude belonging to A_3. If we also draw the line A_3H_3 (5 D\$), we will get the orthocentre H too. Now draw circles $(A_1A_2A_3)$, (A_3HH_2), and (A_2HH_3) too ($3 \cdot 10$ D\$ $= 30$ D\$). Denote the new intersection points by $(A_1A_2A_3) \cap (A_3HH_2) = K_3$ and $(A_1A_2A_3) \cap (A_2HH_3) = K_2$. Draw line K_2H (5 D\$), let it intersect side A_1A_3 at F_2 and let its second intersection with the circumcircle be A_2'. Then $A_2'K_2A_2\angle = 90°$, so A_2A_2' is a diameter of the circumcircle. Furthermore, $A_1HA_3A_2'$ is a parallelogram, since $A_1H, A_2'A_3 \perp A_2A_3$ és $A_1A_2', HA_3 \perp A_1A_2$. Since the diagonals of a paralellogram bisect each other, F_2 will be the midpoint of side A_1A_3. Similarly draw line K_3H (5 D\$), intersecting line A_1A_2 at its midpoint F_3, and having a second intersection A_3' (opposite to A_3) with the circumcircle. So by constructing lines A_2A_2' and A_3A_3', we also get the circumcentre O ($2 \cdot 5$ D\$ $= 10$ D\$).

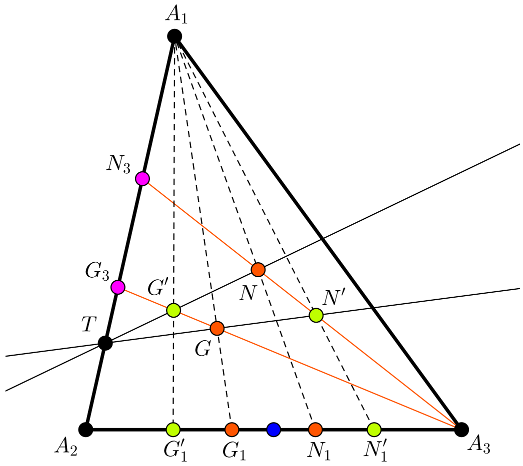


Construct the perpendicular bisectors OF_2 and OF_3 ($2 \cdot 5$ D\$ $= 10$ D\$). These intersect the circumcircle at the midpoints of arcs belonging to the sides A_1A_3 and A_1A_2. Let these be denoted by M_2, N_2, M_3, N_3. So if we draw the lines $A_2M_2, A_2N_2, A_3M_3, A_3N_3$, that is, the internal and external angle bisectors ($4 \cdot 5$ D\$ $= 20$ D\$) we will have constructed the incentre and the excentres. Denote these by I, J_1, J_2, J_3. Let line $A_2'I$ intersect the circumcircle for the second time at I' (5 D\$). Then circle (A_2II') will be the circle with diameter A_2I (since $90^\circ = A_2I'A_2'\angle = A_2I'I\angle$), so it intersects sides A_2A_1, A_2A_3 at the touchpoints of the incircle (10 D\$). If we connect these with the opposing vertices, we get the Gergonne point ($2 \cdot 5$ D\$ $= 10$ D\$). We proceed similarly for the excircles. Let line $A_3'J_2$ intersect the excircle at J_2' (5 D\$), then draw circle $(A_3J_2J_2')$ (10 D\$), which will actually be the circle of diameter A_3J_2, so it intersects side A_1A_3 at the projection of point J_2, that is, the touchpoint of the excircle for the second time. We can do the same for J_3 (5 D\$ + 10 D\$ = 15 D\$), and if we connect the two touchpoints with the opposing vertices, we get the Nagel point ($2 \cdot 5$ D\$ $= 10$ D\$).



Our total expense was $50 + 5 + 30 + 5 + 5 + 10 + 10 + 20 + 5 + 10 + 10 + 5 + 10 + 15 + 10 = 200$ D\$, so 10 D\$ remains. From half of the remaining 10 D\$, we buy an antivirus software, and the other half will be used later.

Now we will elaborate on how we can find a further isotomic pair. Take an arbitrary point G' on the part of line A_3G which is inside the triangle. Let line NG' meet the extended side A_1A_2 at point T, and let line TG meet line A_3N at N' ($2 \cdot 5$ D\$ $= 10$ D\$). We will prove that G' and N' are isotomic too.



For symmetry reasons, it suffices to prove that G_1' and N_1' are symmetric with respect to the midpoint of side A_2A_3. Observe that $(A_2, A_3; G_1, G_1') \overset{A_1}{=} (G_3, A_3; G, G') \overset{T}{=} (N_3, A_3; N'; N) \overset{A_1}{=} (A_2, A_3; N_1', N_1) = (A_3, A_2; N_1, N_1')$. If the reflection of G_1' to the midpoint of side A_2A_3 is denoted by G_1^*, then as reflection in a point preserves cross-ratios, $(A_2, A_3; G_1, G_1') = (A_3, A_2; N_1, G_1^*)$. So $(A_3, A_2; N_1, N_1') = (A_3, A_2; N_1, G_1^*)$, therefore $N_1' = G_1^*$, which is what we needed to prove. However we have to be careful as T needs to exist. We can ensure this by choosing T to lie inside the segment A_1A_2. The only problem can happen if we choose the intersection point $GN \cap A_1A_2$ exactly, but this can be avoided by constructing line GN (5 D$).