XVII. DÜRER COMPETITION E+ CATEGORY

PROBLEMS – TRADITIONAL ROUND

FINAL ROUND – 09–11. 02. 2024.

---

1. Describe all ordered sets of four real numbers (a, b, c, d) for which the values $a+b, b+c, c+d, d+a$ are all non-zero and

$$\frac{a+2b+3c}{c+d} = \frac{b+2c+3d}{d+a} = \frac{c+2d+3a}{a+b} = \frac{d+2a+3b}{b+c}.$$

2. For every subset P of the plane let $S(P)$ denote the set of circles and lines that intersect P in at least three points. Find all sets P consisting of 2024 points such that for any two distinct elements of $S(P)$, their intersection points all belong to P.

3. On the island of Dürerland, the grand final of the ever popular gameshow, *Merchant of the island* has just arrived! To determine a winner, the contenders, *Paul* and *Pauline* have to first divide a salmon of size $2n$ equally amongst themselves (where n is a positive integer). They have a machine which upon receiving a piece of fish of size k, cuts it into two pieces with positive integer sizes, but the distribution cannot be predicted beforehand (k is an integer bigger than 1). What is the minimum number of cuts, after which Paul and Pauline can distribute the pieces, such that the sum of the sizes of the pieces they both receive is equal to n (no matter how the machine makes the cuts)?
The machine might not cut pieces of equal size the same way every time. After each cut, the sizes of the resulting pieces are measured right away.

4. Let $\mathcal{H}$ be the set of all lines in the plane. Call a function $f : \mathbb{R}^2 \to \mathcal{H}$ from the points of the plane *polarising*, if for any points $P, Q \in \mathbb{R}^2$, $P \in f(Q)$ implies $Q \in f(P)$.
a) Show that there is no surjective polarising function.
b) Give an example of an injective polarising function.
c) Prove that for every injective polarising function there exists a point P on the plane for which $P \in f(P)$.
A function $f : A \to B$ is surjective, if for all $b \in B$, there is an $a \in A$ such that $f(a) = b$. f is injective, if for any two distinct $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$.

5. Let p be a fixed prime number.
a) How many 3-tuples (a_1, a_2, a_3) exist, for which all three numbers are non-negative integers less than p and $p \mid a_1^2 + a_2^2 + a_3^2$ holds?
b) Let now k be a fixed positive odd number. Determine the number of k-tuples where all k numbers are non-negative integers less than p and $p \mid a_1^2 + a_2^2 + \ldots + a_k^2$.

6. Game: On a $1 \times n$ board there are $n-1$ *separating edges* between neighbouring cells. Initially none of the edges contain matchsticks. During a move of size $0 < k < n$, a player chooses a $1 \times k$ sub-board which contains no matches inside, and places a matchstick on all of the separating edges bordering the sub-board that don't already have one. A move is considered legal if at least one match can be placed and if either $k = 1$ or k is divisible by 4. The two players take turns making moves, the player in turn must choose one of the available legal moves of the largest size $0 < k < n$ and play it. If someone does not have a legal move, the game ends and that player loses.
Beat the organisers twice in a row in this game! First the organisers determine the value of n, then you get to choose whether you want to play as the first or the second player.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. For a substantially different second solution or generalization, up to 2 extra points per problem might be awarded. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVII. Dürer Competition