E+1. Describe all ordered sets of four real numbers (a, b, c, d) for which the values $a+b, b+c, c+d, d+a$ are all non-zero and

$$\frac{a+2b+3c}{c+d} = \frac{b+2c+3d}{d+a} = \frac{c+2d+3a}{a+b} = \frac{d+2a+3b}{b+c}.$$

Problem proposed by András Imolay

First solution: Let the common value of the fractions in the problem be denoted by s. We are going to use the following useful claim:
Claim: if $s = \frac{x}{y} = \frac{z}{w}$, then $s = \frac{x+z}{y+w}$ if $y+w \neq 0$, otherwise $x+z = y+w = 0$.
Proof of claim: If $y+w \neq 0$, then $\frac{x}{y} = \frac{x+z}{y+w} \iff x(y+w) = y(x+z) \iff xw = yz$, which follows from the condition $\frac{x}{y} = \frac{z}{w}$. And if $y+w = 0$, then $y = -w$, so $\frac{x}{y} = \frac{z}{w} \implies x = -z$, thus $x+z = 0$. $\square$

Let's use the claim: if $a+b+c+d \neq 0$, then on one hand

$$s = \frac{a+2b+3c}{c+d} = \frac{c+2d+3a}{a+b} = \frac{(a+2b+3c)+(c+2d+3a)}{(c+d)+(a+b)} = \frac{4(a+c)+2(b+d)}{a+b+c+d},$$

on the other hand we have

$$s = \frac{b+2c+3d}{d+a} = \frac{d+2a+3b}{b+c} = \frac{(b+2c+3d)+(d+2a+3b)}{(d+a)+(b+c)} = \frac{2(a+c)+4(b+d)}{a+b+c+d}.$$

This can only hold if the two numerators are equal, that is $a+c = b+d$. Using this, we obtain

$$s = \frac{4(a+c)+2(b+d)}{a+b+c+d} = \frac{6(a+c)}{2(a+c)} = 3.$$

Now the fractions in the problem can be rewritten as equations: for example, $\frac{a+2b+3c}{c+d} = 3 \iff a+2b+3c = 3c+3d \iff a+2b = 3d$, and we can write this for the other fractions as well (changing the letters cyclically). We are going to show that all four numbers are equal. That is because if d is the largest among the four numbers, then $3d = a+2b \leq d+2d = 3d$, but the equality only holds if $a = b = d$. Finally, because of this we have $3c = d+2a = 3d$, so $c = d$.

What happens if $a+b+c+d = 0$? By the claim, the numerators of the two fractions are also equal (more precisely, both are 0), so $a+c = b+d$, and since their sum is 0, the common value is also the same. By substituting $c = -a$ and $d = -b$ into the equation in the problem, we get $\frac{2(b-a)}{-(a+b)} = \frac{-2(a+b)}{a-b}$, from which it would follow that $(\frac{b-a}{-(a+b)})^2 = -1$, contradiction.

In summary, $a = b = c = d = \lambda$ must hold, but $\lambda \neq 0$ (otherwise the denominators would also be 0). If, on the other hand, $a = b = c = d = \lambda \neq 0$, then indeed the value of every fraction is 3.

Second solution: We will present an alternative proof for $s = 3$. Let us rewrite the fractions as equations (like we did at the end of the previous solution), we just don't know their common value yet. For example, $s = \frac{a+2b+3c}{c+d} \iff a+2b+3c = sc+sd \iff a+2b+(3-s)c-sd = 0$, thus we obtain a system of homogeneous linear equations consisting of four equations with four unknowns (for s).

$$\begin{array}{rcrcrcr}
a & + & 2b & + & (3-s)c & - & sd = 0 \\
-sa & + & b & + & 2c & + & (3-s)d = 0 \\
(3-s)a & - & sb & + & c & + & 2d = 0 \\
2a & + & (3-s)b & - & sc & + & d = 0
\end{array}$$

If this has a solution other than the trivial $(0,0,0,0)$, then the equations (as vectors) are linearly dependent, so the determinant of the associated matrix $\begin{pmatrix} 1 & 2 & 3-s & -s \\ 2 & 3-s & -s & 1 \\ 3-s & -s & 1 & 2 \\ -s & 1 & 2 & 3-s \end{pmatrix}$ has to be zero. Let us compute the determinant:

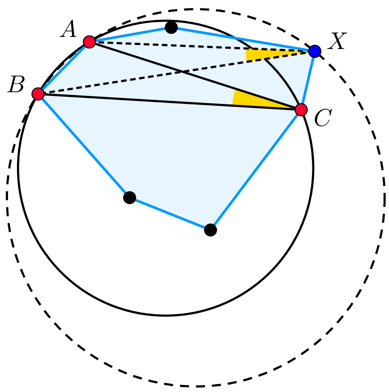
$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3-s & -s \\ 2 & 3-s & -s & 1 \\ 3-s & -s & 1 & 2 \\ -s & 1 & 2 & 3-s \end{vmatrix} &= 1 \cdot \begin{vmatrix} 3-s & -s & 1 \\ -s & 1 & 2 \\ 1 & 2 & 3-s \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 3-s & -s \\ -s & 1 & 2 \\ 1 & 2 & 3-s \end{vmatrix} \\
&+ (3-s) \cdot \begin{vmatrix} 2 & 3-s & -s \\ 3-s & -s & 1 \\ 1 & 2 & 3-s \end{vmatrix} + s \cdot \begin{vmatrix} 2 & 3-s & -s \\ 3-s & -s & 1 \\ -s & 1 & 2 \end{vmatrix} \\
&= \Big((3-s)^2 - 2s - 2s - 1 - s^2(3-s) - 4(3-s)\Big) \\
&- 2\Big(2(3-s) + 2(3-s) + 2s^2 + s + s(3-s)^2 - 8\Big) \\
&+ (3-s)\Big(-2s(3-s) + (3-s) - 2s(3-s) - s^2 - (3-s)^3 - 4\Big) \\
&+ s\Big(-4s - s(3-s) - s(3-s) + s^3 - 2(3-s)^2 - 2\Big) \\
&= 8s^3 - 24s^2 + 32s - 96 = 8(s-3)(s^2+4)
\end{aligned}$$

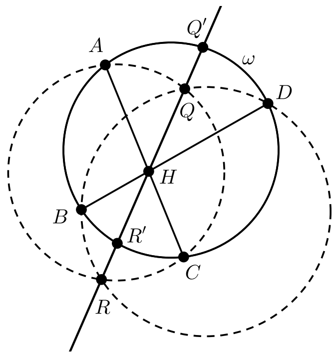
This polynomial can only take the value of 0 if $s = 3$ (because $s^2 + 4 > 0$).

E+2. For every subset P of the plane let $S(P)$ denote the set of circles and lines that intersect P in at least three points. Find all sets P consisting of 2024 points such that for any two distinct elements of $S(P)$, their intersection points all belong to P.
Problem proposed by Csongor Beke

Solution: We will show that these sets P are exactly the ones where the points lie on a circle or on a line. It is clear that these sets satisfy the condition of the problem. Now suppose that there is such a set P that also satisfies the problem but not all of the points are on a circle or on a line. Then the convex hull of P will have at least 3 sides.

Now let us regard the circles determined by the vertices of the convex hull (these all belong to $S(P)$), and let (one of the) largest one be ω. Notice that every point of P is contained inside or on the boundary of ω. Since suppose that the point $X \in P$ falls outside of ω. Let $A, B, C \in P$ be three points on ω. Then since A, B, C, X are all points on the convex hull, we can assume that the quadrilateral $ABCX$ is convex. Since the triangle ABC has an acute angle at either A or C, let us assume that $ACB\sphericalangle$ is acute. Since C and X fall onto the same side of line AB, this means that the directed angle $AXB\sphericalangle$ is smaller than the directed angle $ACB\sphericalangle$, since the radius of circle (AXB) is smaller than that of (ACB) (since the sine of the angle belonging to the same chord is smaller), which is a contradiction.





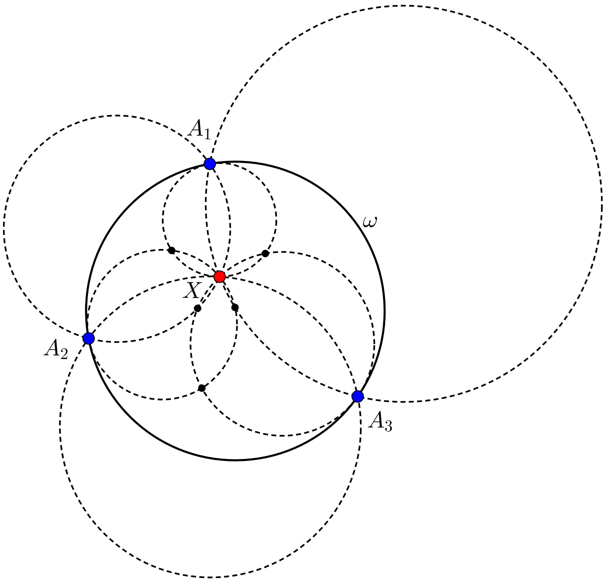
Now we will show that ω cannot have more than three points of P on its boundary. Suppose indirectly that $A, B, C, D \in \omega$ in this order (meaning that $ABCD$ is a convex quadrilateral). Firstly we will show that there is more than one point inside ω. Indeed, if M was the only such point, then circles (ABM) and (CDM) cannot have another point of intersection, therefore they are tangent. But then there has to be a point $E \in P$ on the circle $ABCD$, and circles (ABM) and (CEM) cannot be tangent, which is a contradiction.

Similarly circles $(BCM), (DAM)$ are also tangent. This means that the quadrilateral $ABCD$ is a parallelogram (and since convex, a rectangle). Since if there are 2023 points on the circle, then we can choose A, B, C, D in a way that they do not form a parallelogram (rectangle). Therefore that there are at least two points inside ω. This means that there is a point Q inside ω that is not the intersection of lines AC and BD. Then circles $(ACQ), (BDQ)$ exist and they have a second intersection $R \neq Q$ since they cannot be tangent as they have a common interior point (for example $AC \cap BD$). By the definition of ω, R is also in the interior of ω. Let the line QR intersect ω in points Q' and R'. The pairwise radical axes of circles $\omega, (ACQ), (BDQ)$ are lines AC, BD, QR, therefore they are concurrent in point H, the radical centre. But then

$$HQ' \cdot HR' = HA \cdot HC = HQ \cdot HR < HQ' \cdot HR',$$

which is impossible, we have reached a contradiction.

Therefore there are exactly three points of P on ω, let these be A_1, A_2 and A_3. Let X be a fixed interior point and Y any interior point different from X. Then the circle (XYA_i) cannot intersect ω in a fourth point (as there are exactly three points on ω). This leaves two possibilities: either it is tangent to ω or passes through another point A_j. Now look at circles (A_iA_jX) $(1 \leq i, j \leq 3)$ where (A_iA_iX) is the circle passing through A_i, X and tangent to ω. These six circles have at most $\binom{6}{2}$ intersection points inside ω that are different from X, but every point Y is a point of this kind. Since there are 2020 such points Y inside ω, we have reached a contradiction again and therefore concluded the proof.


A_1
ω
X
A_2
A_3

E+3. On the island of Dürerland, the grand final of the ever popular gameshow, *Merchant of the island* has just arrived! To determine a winner, the contenders, *Paul* and *Pauline* have to first divide a salmon of size $2n$ equally amongst themselves (where n is a positive integer). They have a machine which upon receiving a piece of fish of size k, cuts it into two pieces with positive integer sizes, but the distribution cannot be predicted beforehand (k is an integer bigger than 1). What is the minimum number of cuts, after which Paul and Pauline can distribute the pieces, such that the sum of the sizes of the pieces they both receive is equal to n (no matter how the machine makes the cuts)?
The machine might not cut pieces of equal size the same way every time. After each cut, the sizes of the resulting pieces are measured right away.
Problem proposed by Csongor Beke

Solution: We claim that they have the use the machine at least n times in order to halve the salmon. Firstly we show a case where n steps (of using the machine) are needed. If the machine always cuts off a piece of size 1, then we have no choice, as we always have pieces of size 1 and a big piece (and we cannot put the small ones into the machine). Therefore n steps are needed, since that is when the size of the big piece reaches n.

Now let us show that n steps are always sufficient. Let us proceed the following way: if there is a piece larger than n, then put one into the machine that is not the largest (and not size 1). If there is no such, then put the biggest piece. We stop the process when we cut the largest and it gets cut to two pieces both at most size n.

This way we have a piece of size k, another one of size l and $2n-(k+l)$ pieces of size 1, where $k \leq n$, $l \leq n$ and $k+l > n$. Since in every step the number of pieces increases by exactly one, therefore we made at most $(2n-(k+l)+2-1) \leq n$ steps. And indeed they can divide the salmon equally as one of them takes the piece of size k, the other one the one of size l, and each take pieces of size 1 to reach n in total.

Second solution: We show another way of proving that n cuts are sufficient. Imagine the salmon as a circular cake, where there are $2n$ radii from the centre to the edge, and this is where we cut. The first time the cut is made along two radii, and afterwards on one of the radii. Since after using the machine n times there will be $n+1$ cuts, we can choose two of them that make up a diameter and the salmon can be halved along it.

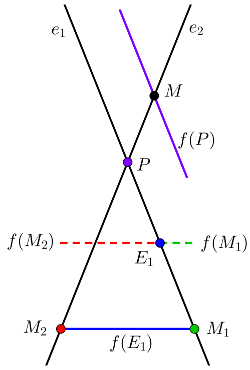
E+4. Let $\mathcal{H}$ be the set of all lines in the plane. Call a function $f : \mathbb{R}^2 \to \mathcal{H}$ from the points of the plane *polarising*, if for any points $P, Q \in \mathbb{R}^2$, $P \in f(Q)$ implies $Q \in f(P)$.
a) Show that there is no surjective polarising function.
b) Give an example of an injective polarising function.
c) Prove that for every injective polarising function there exists a point P on the plane for which $P \in f(P)$. *A function $f : A \to B$ is surjective, if for all $b \in B$, there is an $a \in A$ such that $f(a) = b$. f is injective, if for any two distinct $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$.*
Problem proposed by Áron Bán-Szabó

Solution: a) Suppose indirectly that f is a surjective polarising function. Let us take two parallel lines, let these be l_1 and l_2. Then there are two (disctint) points L_1, L_2 for which $f(L_1) = l_1$ and $f(L_2) = l_2$. But since the line L_1L_2 has a preimage, let K be such a point (meaning that $f(K) = L_1L_2$). Now by the condition since $L_1, L_2 \in f(K)$, $K \in f(L_1), f(L_2)$. But $f(L_1) \cap f(L_2) = l_1 \cap l_2 = \emptyset$, which is a contradiction.

b) Let f be the function that maps point (a, b) to the line defined by the equation $x = by - a$. It is clear that this function is injective, let us now show that it is polarising. Suppose that point (c, d) is on line $f((a, b))$, meaning that $c = bd - a$. By rearranging this we get that $a = db - c$, therefore (a, b) is also on $f((c, d))$.

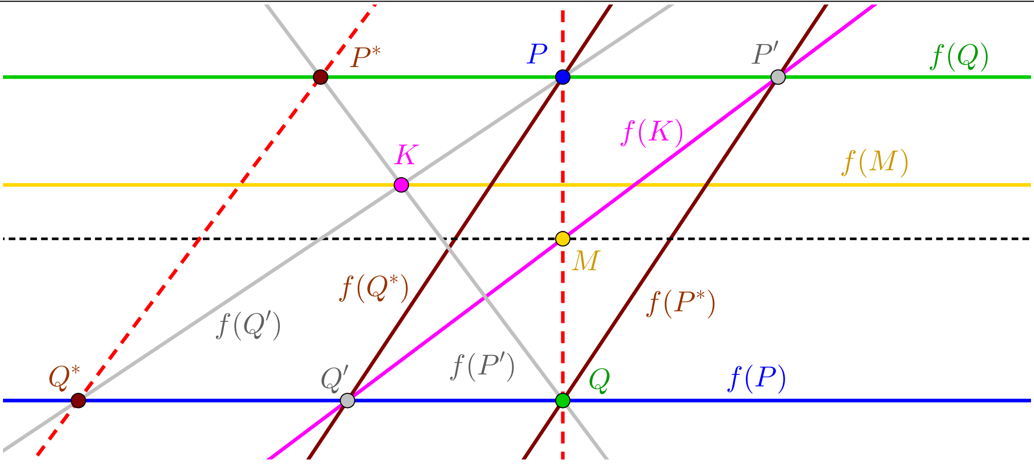
c) Now suppose indirectly that there is no such point. The key will be that f cannot be far from being surjective. Let us call the line e *lonely* if it has no preimage, meaning that it is not the image of any point. Then if $E_1, E_2 \in e$, then lines $f(E_1), f(E_2)$ cannot intersect, since if they did and $M = f(E_1) \cap f(E_2)$, then by the conditions of polarising $E_1, E_2 \in f(M)$, meaning that $f(M) = e$ which is impossible. Therefore the images of the points of a lonely line are different (by injectivity) and pairwise parallel lines.

Now let us regard a point P and its pencil of lines $\mathcal{S}$ (the set of lines passing through P). We will show that there is at most one line from $\mathcal{S}$ that does not have a preimage. Suppose that lines e_1 and e_2 are both like this. Then the image of every point on e_1 is a line in the direction $\mathbf{u}_1$, while the image of every point on e_2 is a line in the direction $\mathbf{u}_2$. But since $P \in e_1, e_2$, it implies that $\mathbf{u}_1 \parallel \mathbf{u}_2 \parallel f(P)$. Suppose that $e_1 \parallel f(P)$, then the point $M = e_2 \cap f(P)$ exists. Since $M \in f(P)$, therefore $P \in f(M)$. But since $M \in e_2$, it is also true that $f(M) \parallel f(P)$, meaning that $f(M) = e_2$, but this is a contradiction as we supposed that e_2 has no preimage. Therefore neither of e_1 or e_2 are parallel to $f(P)$. Let $E_1 \in e_1$ be a point for which $P \notin f(E_1)$ (such point exists by injectivity and by parallelity). Then points $M_1 = f(E_1) \cap e_1$, $M_2 = f(E_1) \cap e_2$ exist and are distict. But these two intersection points are on $f(E_1)$, therefore by the conditions of polarising-ness $E_1 \in f(M_1), f(M_2)$. But since both intersection points lie on $f(E_1)$, it means that $E_1 \in f(M_1), f(M_2)$. But since these two lines are both parallel to $f(P)$ and pass through E_1, it follows that $f(M_1) = f(M_2)$ but this contradicts the injectivity.



With this f is indeed very close to being surjective: in every pencil of lines through a point P there is at most one lonely line. Now we will show that in every pencil of lines there is exactly one lonely line. If P is not such a point, then the line through P parallel to $f(P)$ would have a preimage Q which would lie on $f(P)$, but then line PQ is not lonely as it passes through P, but its preimage would have to lie both on $f(P)$ and $f(Q)$ but it is impossible as these lines are parallel.

Now we know that in every pencil of lines there is exactly one lonely line. Observe furthermore that the lonely lines must be parallel since if two of them had an intersection, then their intersection would have two lonely lines in its pencil. Let us now take an arbitrary point P and consider the line through P that is parallel to $f(P)$ (which is different from $f(P)$ as we supposed indirectly). This line cannot be lonely as otherwise for any point $Q \in f(P)$ the lonely line through would be parallel to $f(P)$, therefore would be the same as $f(P)$, which is not lonely. Therefore there is a point Q for which $f(Q)$ is the line passing through P and parallel to $f(P)$. And as we have seen, line PQ has to be lonely.



Therefore all lonely lines are parallel to PQ. Now we will show that if M is the midpoint of segment PQ, then $f(M)$ passes through M. Clearly $f(M) \parallel f(P), f(Q)$ as the line PMQ is lonely. Now let us take a point K on $f(M)$ that is not on line PQ. Then $f(K)$ passes through M and intersects lines $f(P)$ and $f(Q)$ in points Q' and P' respectively. Since M is a midpoint, the quadrilateral $PP'QQ'$ is a parallelogram. Let $Q^* = PK \cap f(P)$ and $P^* = QK \cap f(Q)$. Notice that $f(P')$ will be the line QK and therefore $f(P^*)$ is the line $P'Q$. Similarly we can show that $f(Q^*)$ is the line PQ'. But since $P'Q \parallel PQ'$, therefore the line P^*Q^* has to be lonely since otherwise its preimage would lie on both lines (and this cannot happen as they are parallel). Therefore $P^*Q^* \parallel PQ$, which implies that PQQ^*P^* is a rectangle, therefore K, the intersection of diagonals lies on the midsegment, meaning that $M = f(M)$ and this is a contradiction.

E+5. Let p be a fixed prime number.
a) How many 3-tuples (a_1, a_2, a_3) exist, for which all three numbers are non-negative integers less than p and $p \mid a_1^2+a_2^2+a_3^2$ holds?
b) Let now k be a fixed positive odd number. Determine the number of k-tuples where all k numbers are non-negative integers less than p and $p \mid a_1^2 + a_2^2 + \ldots + a_k^2$.
Problem proposed by Csongor Beke

Solution: Part **a)** can be proven on its own, but here we are presenting a solution that works for both cases. Throughout the solutions the equations will be written modulo p. We will show that there are p^{k-1} ordered k-tuples. This can be conjectured by looking at small values of k.

The case $p = 2$ is easy as the first $k-1$ terms can be chosen arbitrarily and there is only one choice for a_k, meaning that there are 2^{k-1} solutions. From now on suppose that p is an odd prime.

The idea is the following: we will regard all series $(a_1, a_2, ..., a_{k-1})$ and observe how many suitable a_k exist.

We can observe the following:

- It is known that is q is a nonzero quadratic residue class module p, then there are exactly two numbers c and $-c$ that satisfy $x^2 = q$. Therefore if there is a suitable a_k for a series $(a_1, ..., a_{k-1})$, then there are exactly two of them which are multiples by -1 modulo p. This happens exactly when $p - \sum_{i=1}^{k-1} a_i^2$ is a nonzero quadratic residue. If $p - \sum_{i=1}^{k-1} a_i^2$ is a quadratic non-residue, then there is no suitable a_k and if $p - \sum_{i=1}^{k-1} a_i^2 = 0 \pmod p$ then there is only $a_k = 0$.
- It is also known that if l is a quadratic non-residue modulo p, then for any $p \nmid k$ exactly one of k and $l \cdot k$ will be a quadratic residue.

We will find a bijection ϕ on

$$S = \{(a_1, a_2, ..., a_{k-1}) \mid \forall i \ : \ a_i \in \{0, 1, ..., p-1\}\}$$

for which the following holds: for all $\mathbf{x} \in S$ either in both of $\mathbf{x}$ és $\phi(\mathbf{x})$ the sum of the squares of the elements is 0, meaning that there is only one suitable a_k, or only one of them has suitable a_k, but that one has 2. Then for all pairs of $(\mathbf{x}, \phi(\mathbf{x}))$ there are exactly two solutions, meaning that in total there are p^{k-1} solutions.

Clearly there is a quadratic non-residue l for which $l-1$ is a quadratic residue. Let n be a number for which $n^2 = l-1$. Since $l \neq 1$, $n \neq 0$.

Now we are defining the bijection:

$$\phi : S \to S$$

$$\phi\left((a_1, a_2, ..., a_{k-1})\right) = (na_1 + a_2, a_1 - na_2, na_3 + a_4, a_3 - na_4, ..., na_{k-2} + a_{k-1}, a_{k-2} - na_{k-1})$$

meaning that if $\mathbf{a} = (a_1, a_2, ..., a_{k-1})$ and $\phi(\mathbf{a}) = \mathbf{b} = (b_1, b_2, ..., b_{k-1})$ then $b_{2i+1} = na_{2i+1} + a_{2i+2}$ és $b_{2i+2} = a_{2i+1} - na_{2i+2}$.

Firstly let us show that this indeed a $S \to S$ bijection. For this what is needed is that for any element $\mathbf{b} = (b_1, b_2, \ldots b_{k-1}) \in S$ there is exactly one $\mathbf{a} \in S$ for which $\phi(\mathbf{a}) = \mathbf{b}$. To show this we need that there is only one choice of a_{2i+1}, a_{2i+2} for which $b_{2i+1} = na_{2i+1} + a_{2i+2}$ and $b_{2i+2} = a_{2i+1} - na_{2i+2}$. It is enough to show this for $i = 0$ only, as it follows for all other i. By multiplying the first equation by n and summing with the second one we get that $nb_1 + b_2 = (n^2+1)a_1 = la_1$, therefore the only choice is $a_1 = \frac{nb_1 + b_2}{l}$. From this $a_2 = b_1 - na_1$. The solution is indeed unique, therefore ϕ is a bijection.

Then

$$(na_{2i+1} + a_{2i+2})^2 + (a_{2i+1} - na_{2i+2})^2 =$$
$$= n^2 a_{2i+1}^2 + 2na_{2i+1}a_{2i+2} + a_{2i+2}^2 + a_{2i+1}^2 - 2na_{2i+1}a_{2i+2} + n^2 a_{2i+2}^2 =$$
$$= (n^2+1)(a_{2i+1}^2 + a_{2i+2}^2) = l(a_{2i+1}^2 + a_{2i+2}^2),$$

meaning that $\sum_{i=1}^{k-1} b_i^2 = l \sum_{i=1}^{k-1} a_i^2$. From this we get that $p - \sum_{i=1}^{k-1} b_i^2 = l\left(p - \sum_{i=1}^{k-1} a_i^2\right)$, therefore either both of them are zero or (by observation 2) only one of them is a quadratic residue, meaning that ϕ is as we desired. Therefore the number of solutions is p^{k-1}.

E+6. Game: On a $1 \times n$ board there are $n-1$ *separating edges* between neighbouring cells. Initially none of the edges contain matchsticks. During a move of size $0 < k < n$, a player chooses a $1 \times k$ sub-board which contains no matches inside, and places a matchstick on all of the separating edges bordering the sub-board that don't already have one. A move is considered legal if at least one match can be placed and if either $k = 1$ or k is divisible by 4. The two players take turns making moves, the player in turn must choose one of the available legal moves of the largest size $0 < k < n$ and play it. If someone does not have a legal move, the game ends and that player loses.
Beat the organisers twice in a row in this game! First the organisers determine the value of n, then you get to choose whether you want to play as the first or the second player.
Problem proposed by Márton Németh

Solution: Answer: The second player wins if $n = 7$, or $n = 8m + 1$ or $n = 8m + 4$.

Restatement: Consider the following, different game: A board contains the number 4 initially. Two players aternate writing new numbers on the board, either a 4, a 3, or two instances of the number 2, except during the first turn, when First can choose from a set S, where S is fixed, and $S \in \{\{\emptyset\}, \{\emptyset, \{2\}\}, \{\{3\}, \{2\}\}, \{\{4\}, \{3\}, \{2, 2\}\}\}$.
After d such moves, the players have a new moveset:
-Delete a 4, and write a 3, or
-Delete a 4, and write a 2, or
-Delete a 3, and write a 2, or
-Delete a 3, or
-Delete a 2.
The player who can't make a move loses.

Claim: The new game is equivalent to the one in the problem, according to the following parametrisation:
-If $n = 4m + 1$, then $S = \{\emptyset\}$,
-If $n = 4m + 2$, then $S = \{\emptyset, \{2\}\}$,
-If $n = 4m + 3$, then $S = \{\{3\}, \{2\}\}$,
-If $n = 4m + 4$, then $S = \{\{4\}, \{3\}, \{2, 2\}\}$. Also let

$$d = \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Notice that the largest k for which there is a size k move decreases by 4 every time, and the sub-table a player chooses is always contained in the sub-table chosen on the previous turn. The numbers on the board then correspond to the sizes of the sub-tables generated with size at most 4.

2. phase: In the new game, call phase 2 the stage when only the new moveset is available. Let (x, y, z) be the state, where x denotes the number of 4s, y the number of 3s and z the number of 2s. Then $x \geq 0$, $y \geq 0$, $z \geq 0$ always holds. Then the legal moves are:

$$\begin{gathered}(x, y, z) \to (x-1, y+1, z)\\ (x, y, z) \to (x-1, y, z+1)\\ (x, y, z) \to (x, y-1, z+1)\\ (x, y, z) \to (x, y-1, z)\\ (x, y, z) \to (x, y, z-1)\end{gathered}$$

Let the starting state in phase 2 be (x_0, y_0, z_0).

Claim: The player who comes second in phase 2 wins, if y_0 ans z_0 are even, first wins otherwise.

This is easy to see: let (x, y, z) be a winning state, is y and z are even. From a non-winning state, one can always make a legal move, and in fact, make a move that leads to a winning state.

1. phase: Let's consider the cases where d is odd or even.

Even d: We'll show that if $n = 4m + 2$ or $n = 4m + 3$, then First wins, otherwise Second does. Notice that phase two is also started by First, so their aim is to make either y_0 or z_0 odd.
-$n = 4m + 1$: First is forced to write nothing. Second should then write a 4, and copy First after that.
-$n = 4m + 2$ or $n = 4m + 3$: First writes a 2. The parity of the number of 2s remains odd, so First wins.
-$n = 4m + 4$: Second should copy what First does.

Odd d: We'll show First always wins.
-$n = 4m + 1$: First writes nothing, then copies second.
-$n = 4m + 2$: First writes nothing. The number of 2s remains even, with their last move, First insures that the number of 3s is even as well.
-$n = 4m+3$ or $n = 4m+4$: First writes a 3. The number of 2s remains even, with their last move, First ensures, that the number of 3s is even as well. In case $d = 1$, First can't do so, and Second wins in this case.