XVIII. Dürer Competition

First round (22. 11. 2024.)

Problems

E category

1. In a forensic laboratory, we have a twin-pan balance and eight weights labelled with their masses, weighing $1, 2, \dots, 8$ kg. During an investigation, a piece of gold was found with a mass equal to one of the known weights. In one weighing, a twin-pan balance is used to compare the piece of gold to one of the eight weights. The cost of such a measurement is equal to the weight of the weight used, measured in Dürer dollars. What is the minimum number of Dürer dollars required to determine the mass of the piece of gold with certainty?

For example, if the piece of gold is compared to the weight of 2 *kg, the cost of this measurement is* 2 *Dürer dollars. The measurements may depend on the results of previous measurements.*

2. Let k be a circle with centre O, and let P be a point outside the circle. The lines e and f pass through P and are tangent to k, touching the circle at points E and F, respectively. Let A be an interior point of segment PE. The two lines through A that are tangent to circle k are e and g. Denote the intersection of lines f and g by B. Suppose that $\angle EPF$ is an acute angle and $\angle PBA = \angle APB$. Prove that the midpoints of segments PB and AF are collinear with O.

3. The infinite sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ consist of positive integers, and the following conditions hold for all $i \geq 1$:

- if $\gcd(a_i, b_i) > 1$ then $a_{i+1} = \dfrac{a_i}{\gcd(a_i, b_i)}$ and $b_{i+1} = \dfrac{b_i}{\gcd(a_i, b_i)}$,
- and if $\gcd(a_i, b_i) = 1$ then $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 2$.

Determine all pairs of positive integers (a_1, b_1) for which there exists a pair in the infinite sequence $(a_1, b_1), (a_2, b_2), \dots$ that appears infinitely many times!

Here, $\gcd(p, q)$ denotes the greatest common divisor of p and q.

4. A positive integer n and a real number $c > 1$ are given. The underground Albrecht Bank has just been robbed, and the n robbers are fleeing the scene. Before the heist, each criminal hid a scooter at a different point on the surface. The robbers have now emerged at various exits onto the surface. We can observe that if the robbers' positions were scaled by a factor of c from the main surface entrance of the bank, each would be precisely at their own scooter.

The robbers wish to escape using the scooters (not necessarily their own), but each scooter can only carry one person. The police are on their way, so each robber must run to their chosen scooter on the shortest path possible. Prove that the total distance travelled by the robbers to reach the scooters cannot be less than if they all choose their own scooter.

The main entrance of the bank, the robbers, and the scooters are considered as points, and the terrain is completely flat.

5. We call a pair of positive integers (a, b) *criminal*, if they have the same number of digits in base-10, and we can obtain the difference of their squares by writing one of them after the other one.

a) Find all criminal pairs of positive integers, for which a divides b.

b) Does there exist a criminal pair of positive integers, where a and b are coprime?

Please write all the solutions on separate sheets. Make sure to write the name of your team and the category on every paper.

Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVIII. Dürer Competition