XVIII.

Dürer Competition

First round (22. 11. 2024.)

Solutions

E category

E1. In a forensic laboratory, we have a twin-pan balance and eight weights labelled with their masses, weighing $1, 2, \ldots, 8$ kg. During an investigation, a piece of gold was found with a mass equal to one of the known weights. In one weighing, a twin-pan balance is used to compare the piece of gold to one of the eight weights. The cost of such a measurement is equal to the weight of the weight used, measured in Dürer dollars. What is the minimum number of Dürer dollars required to determine the mass of the piece of gold with certainty?

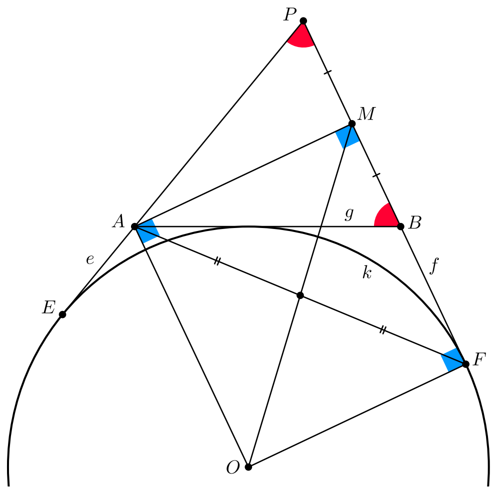
For example, if the piece of gold is compared to the weight of 2 *kg, the cost of this measurement is* 2 *Dürer dollars. The measurements may depend on the results of previous measurements.*

Solution: 12 is the minimum. Our first step is to compare the piece of gold with 5 kg. If the mass of the gold is greater than this, we can determine its value by comparing it to the weight of 7 kg. If it is less than 5 kg, we can determine its value by comparing it to the weight of 3 kg and then to the weight of 1 kg. This will cost up to 12 in total.

We prove that it cannot be cheaper. If we compare the gold block with 6 kg in the first measurement and it turns out that the block is heavier, we still have to compare it with either the 7 kg or the 8 kg block to determine the weight, but this will cost at least 13. If we start with a weight of at least 7 kg and get the answer that the gold is lighter, then if the gold is 5 or 6 kg, we have to compare it with one of the two to decide which it is, again costing at least 12. On the other hand, if the first measurement is to compare it with a weight of at most 4 kg, then to distinguish between the two cases of gold being 5 kg or 6 kg and the two cases of the gold being 7 kg or 8 kg, you need to compare the weight with one of the pairs of numbers, so you need at least two measurements totalling at least $5 + 7$, for a total of at least $1 + 5 + 7 = 13$. This completes the proof.

E2. Let k be a circle with centre O, and let P be a point outside the circle. The lines e and f pass through P and are tangent to k, touching the circle at points E and F, respectively. Let A be an interior point of segment PE. The two lines through A that are tangent to circle k are e and g. Denote the intersection of lines f and g by B. Suppose that $\angle EPF$ is an acute angle and $\angle PBA = \angle APB$. Prove that the midpoints of segments PB and AF are collinear with O.

Solution: Let M denote the midpoint of segment PB. From the condition of the problem, triangle ABP is isosceles. It follows that in this triangle the median AM is also an altitude and an internal angle bisector. Line PF is tangent to circle k, so $OF \perp PF$. Further, M lies on line PF, so $OF \perp MF$. Also, notice that the line AO bisects $\angle EAB$, as lines AE, AB are tangent to the circle k. Thus, AO is the external bisector of $\angle PAB$, which is known to be perpendicular to the internal angle bisector AM. From the above, in quadrilateral $AMFO$ the angles at vertices A, M and F are all right angles, implying that it is a rectangle. Since the diagonals of a rectangle bisect each other, the midpoint of segment AF coincides with the midpoint of segment OM. The statement follows.



E3. The infinite sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ consist of positive integers, and the following conditions hold for all $i \geq 1$:

- if $\gcd(a_i, b_i) > 1$ then $a_{i+1} = \dfrac{a_i}{\gcd(a_i, b_i)}$ and $b_{i+1} = \dfrac{b_i}{\gcd(a_i, b_i)}$,
- and if $\gcd(a_i, b_i) = 1$ then $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 2$.

Determine all pairs of positive integers (a_1, b_1) for which there exists a pair in the infinite sequence $(a_1, b_1), (a_2, b_2), \dots$ that appears infinitely many times.
Here, gcd(p, q) denotes the greatest common divisor of p and q.

Solution: Let $b_1 = 2a_1 + c_1$, where c_1 is an integer. Furthermore, for each i, write the elements of the sequence b_i in the form $b_i = 2a_i + c_i$ in a similar way. For some j, if a_j and b_j are relative primes, then the rules are $a_{j+1} = a_j + 1$ and $b_{j+1} = b_j + 2$. If $b_j = 2a_j + c_j$, then $b_{j+1} = b_j + 2 = 2a_j + c_j + 2 = 2(a_j + 1) + c_j = 2a_{j+1} + c_j$. So at these steps $c_{j+1} = c_j$. And for some k, if $\gcd(a_k, b_k) > 1$, then the next members of the series are obtained by dividing by their greatest common divisor. Then, for $b_k = 2a_k + c_k$, $a_{k+1} = \frac{a_k}{\gcd(a_k,b_k)}$ and $b_{k+1} = \frac{b_k}{\gcd(a_k,b_k)}$, so that $b_{k+1} = \frac{2a_k+c_k}{\gcd(a_k,b_k)} = 2a_{k+1} + \frac{c_k}{\gcd(a_k,b_k)}$, i. e. $c_{k+1} = \frac{c_k}{\gcd(a_k,b_k)}$. Since $\gcd(a_k, b_k) > 1$, after this step $0 < |c_{k+1}| < |c_k|$ (unless $c_k = 0$, in which case $c_{k+1} = 0$), but we know that c_{k+1} is still an integer, since b_{k+1} and $2a_{k+1}$ are both integers

If c_1 is nonzero, then we can only divide it by integers greater than 1 a finite number of times, otherwise the c_i series would become non-integer after a while. Then there is a final division, i.e. a final step before which $\gcd(a_i, b_i) > 1$ is satisfied. After that there is always $\gcd(a_i, b_i) = 1$, i.e. from now on the sequences continue according to the rules $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 2$. In this case, there can be no pair of numbers that occur infinitely often, since both sequences grow strictly monotonically after a finite length. And if $c_1 = 0$, then $b_1 = 2a_1$, i.e. either $a_1 = 1$ and $b_1 = 2$, or $a_2 = 1$ and $b_2 = 2$. From then on, the pairs $(1, 2)$ and $(2, 4)$ will follow each other in a cyclic sequence, i.e. they will occur an infinite number of times in the sequence. So, if and only if $b_1 = 2a_1$, there will be a pair of numbers that occur an infinite number of times.

Second solution: Consider a coordinate system and let the x axis denote the values of the a_i series and the y axis denote the values of the b_i series. For each integer i in the coordinate system, denote the point (a_i, b_i). Since the elements of the series are positive integers, all the points marked are in the first quadrant of the plane. Let us examine the distance of the marked points from the line $y = 2x$. If, for some k, $\gcd(a_k, b_k) > 1$, then there is still a grid point on the segment between the origin and the point (a_k, b_k), so in the next step we will get the closest of these grid points to the origin, since then $(a_{k+1}, b_{k+1}) = \left(\frac{a_k}{\gcd(a_k,b_k)}, \frac{b_k}{\gcd(a_k,b_k)}\right)$. If the point (a_k, b_k) falls on the line $y = 2x$, then so does (a_{k+1}, b_{k+1}), but if it does not, then it will not fall on the line $y = 2x$ either, but we are strictly closer to it (the distance is at least halved). And if for some j we have $\gcd(a_j, b_j) = 1$, then $(a_{j+1}, b_{j+1}) = (a_j + 1, b_j + 2)$, so we get the next point (a_j, b_j) by adding the vector $(1, 2)$. This vector is parallel to the line $y = 2x$, so the distance from the line does not change during this process (if it was 0 before, it remains 0 now).

So if the first point is not on the line $y = 2x$, there won't be any points on that line. Since the line $y = 2x$ has a rational slope, there is a grid point closest to it. So our distance from the line can only decrease a finite number of times (since each decrease at least halves it), i.e. after a while we can only walk parallel to this line. In this case, there will be no point that is touched an infinite number of times. If, on the other hand, the first point falls on the line $y = 2x$, then, as in the previous solution, from the second pair of numbers $(1, 2)$ and $(2, 4)$ will be repeated alternately, i.e. there will be a pair of numbers in the sequence that is touched infinitely. Again we see that if and only if $b_1 = 2a_1$ there will be a pair of numbers that occur infinitely often.

Note: If we replace 1 and 2 by arbitrary integers c, d, the statement remains true, i.e. there will be a pair of numbers that occur infinitely often if and only if $b_1 = \frac{d}{c}a_1$.

E4. A positive integer n and a real number $c > 1$ are given. The underground Albrecht Bank has just been robbed, and the n robbers are fleeing the scene. Before the heist, each criminal hid a scooter at a different point on the surface. The robbers have now emerged at various exits onto the surface. We can observe that if the robbers' positions were scaled by a factor of c from the main surface entrance of the bank, each would be precisely at their own scooter.

The robbers wish to escape using the scooters (not necessarily their own), but each scooter can only carry one person. The police are on their way, so each robber must run to their chosen scooter on the shortest path possible. Prove that the total distance travelled by the robbers to reach the scooters cannot be less than if they all choose their own scooter. *The main entrance of the bank, the robbers, and the scooters are considered as points, and the terrain is completely flat.*

Solution: Let us denote the robbers' positions by A and the scooters' positions by B. The task is to minimize the sum of distances in a perfect matching between A and B, where a perfect matching means that each point in A is paired with exactly one point from B and any two different points from A have different pairs in B.

Instead of perfect matchings between A and B, we will refer to the corresponding bijections $A \to B$ throughout the proof. We denote the distance of a bijection π between two sets by $c(\pi)$. Let π_{id} denote the bijection $A \to B$ taking each point to its scaled image.

Case 1. Suppose that c is so large that we can draw a circle centered at the origin such that A is completely inside and B is completely outside. Let f be a function on all points, defined as the distance of the given point and the circle. For any $u \in A, v \in B$, $d(u,v) \geq f(u) + f(v)$. This is because if we draw the segment uv, then it has to intersect the circle, and the distance it travels inside is at least $f(u)$, and the distance it travels outside is at least $f(v)$. So for any bijection π:

$$\sum_{a \in A} d(a, \pi(a)) \geq \sum_{a \in A} (f(a) + f(\pi(a))) = \sum_{x \in A \cup B} f(x)$$

It is easy to see that equality is reached if every point in A is matched to its image.

Case 2. If c is not large enough. We prove the statement by contradiction: suppose there exists a cheaper bijection $\pi : A \to B$. Let C be a sufficiently large scaling of A, so that there exists a circle centered at the origin with all points of A and B inside, and all points of C outside. Let $\rho_{\text{id}} : B \to C$ and $\sigma_{\text{id}} : A \to C$ assign to each point its scaled image, then $\rho_{\text{id}} \circ \pi : A \to C$ is a bijection with smaller total cost than $\sigma_{\text{id}} : A \to C$, since by the triangle inequality,

$$c(\rho_{\text{id}} \circ \pi) = \sum_{a \in A} d(a, \rho_{\text{id}}(\pi(a))) \leq$$

$$\leq \sum_{a \in A} (d(a, \pi(a)) + d(\pi(a), \rho_{\text{id}}(\pi(a)))) = c(\pi) + c(\rho_{\text{id}}) < c(\pi_{\text{id}}) + c(\rho_{\text{id}}) = c(\sigma_{\text{id}}).$$

But this contradicts the minimality of σ_{id} from *Case 1.*

Second solution: We reduce the problem to the case where each point lies on a single ray $[0, \infty)$. For each point P, let $d_0(P)$ denote its distance from the origin O. For a bijection $\pi : A \to B$, let $c'(\pi) = \sum_{a \in A} |d_0(\pi(a)) - d_0(a)|$.

Claim. For every pair of points x, y, we have $d(x,y) \geq |d_0(x) - d_0(y)|$.
Proof of claim. Suppose that $d_0(x) \geq d_0(y)$ without loss of generality. Then the statement is equivalent to $d(O,x) - d(O,y) \leq d(x,y)$, which is true by the triangle inequality.

Let $A' = \{d_0(a) : a \in A\}$ and $B' = \{d_0(b) : b \in B\}$. Then $c'(\pi)$ corresponds to the cost c of the bijection corresponding to π between A' and B'. Listing the points of these two sets as $0 \leq a_1 \leq ... \leq a_n$ and $0 \leq b_1 \leq ... \leq b_n$, the bijection taking each point to its scaled image corresponds to $a_1 \mapsto b_1$, ..., $a_n \mapsto b_n$. We will show that this bijection is optimal for this one-dimensional version of the problem. Then this means that π_ι is optimal for the original problem, because for any $\pi : A \to B$, $c(\pi) \geq c'(\pi) \geq c'(\pi_\iota)$ by summing over all $a \in A$ and using the Claim.

Now if all points lie on the ray $[0, \infty)$, then the statement of the problem is easy to establish: supposing there are points $a < a'$ whose matched pairs are b and b' respectively, then we must have $b \leq b'$. Supposing otherwise, if $b > b'$ then $d(a,b) + d(a',b')$ could be reduced by swapping the two pairs. So the order of the points is indeed the same as the order of their pairs in the matching.

E5. We call a pair of positive integers (a, b) *criminal*, if they have the same number of digits in base-10, and we can obtain the difference of their squares by writing one of them after the other one.
a) Find all criminal pairs of positive integers, for which a divides b.
b) Does there exist a criminal pair of positive integers, where a and b are coprime?

Solution:

a) Let the number of digits in a and b be k, without loss of generality suppose that $b \geq a$. Then $k \geq 1$ and $10^{k-1} \leq a, b \leq 10^k$. If we write b after a then we get $10^k \cdot a + b$ and if a after b we get $10^k \cdot b + a$.

Observe that $b^2 - a^2 < b^2 < 10^k \cdot b < 10^k \cdot b + a$, so it's not possible that $b^2 - a^2 = 10^k \cdot b + a$ (even if we neglect $a|b$).

So we only need to consider the case $b^2 - a^2 = 10^k \cdot a + b$. Since $a|b$, there exists some positive integer ℓ, such that $\ell \cdot a = b$. Since a and b have the same number of digits, we have $1 \leq \ell \leq 9$. Substitute $\ell \cdot a$ for b to get:

$$(\ell a)^2 - a^2 = a \cdot (10^k + \ell)$$

Rearranging:

$$a^2 \cdot (\ell^2 - 1) = a \cdot (10^k + \ell)$$

So

$$a \cdot (\ell^2 - 1) = 10^k + \ell$$

We distinguish the following cases

- If ℓ is odd, then the left hand side is even (since $\ell^2 - 1$ is even), while the right hand side is odd (the sum of an even and odd term), a contradiction. Suppose for the next cases that ℓ is even (so $2, 4, 6$, or 8).
- If $\ell = 2$, then $a = \frac{10^k+2}{2^2-1} = 333...334$ and $b = 2a = 666...668$ (where each number contains $k - 1$ copies of 3, and 6 respectively). The pairs of this form satisfy the conditions.
- If $\ell > 2$ and $k > 1$, then the left hand side is at least $15a$ (since $4^2 - 1 = 15$), and since $a \geq 10^{k-1}$ the left hand side is at least $15 \cdot 10^{k-1} = 10^k + 5 \cdot 10^{k-1}$. The right hand side is at most $10^k + 9$ (as $\ell \leq 9$). Furthermore as $k > 1$,we get $10^k + 5 \cdot 10^{k-1} \geq 10^k + 5 \cdot 10^1 > 10^k + 9 \geq 10^k + \ell$, contradiction again.
- If $\ell > 4$ and $k = 1$, the right hand side is at most 20, but the left hand side is at least $6^2 - 1 = 35$, so no solution.
- For $\ell = 4$ and $k = 1$ none of $(a, b) = (1, 4)$ or $(a, b) = (2, 8)$ work, so no such pair exists.

So the only pairs that work are $a = 333...334$, $b = 666...668$, where a and b have the same numbers of digits.

b) Suppose that a and b are coprime and (a, b) is criminal. We get again $b^2 - a^2 = 10^k \cdot a + b$, so rearranging gives: $b(b - 1) = a(10^k + a)$. Since a and b are coprimes, $a \mid b - 1$, let $b = a \cdot \ell + 1$. Since both a and b have k digits, $1 \leq \ell \leq 9$. Substituting $b = a \cdot \ell + 1$ to the equation and rearranging similar to **a)** we get:

$$a \cdot (\ell^2 - 1) = 10^k - \ell$$

Again, we distinguish a few cases:

- For ℓ odd, similarly to **a)** we get no solution, by parity reasoning.
- For $\ell = 2$ we get no solution, as $3 \nmid 10^k - 2$.
- For $\ell > 3$ similarly to **a)** , the left hand side is at least $15a \geq 15 \cdot 10^{k-1} > 10^k > 10^k - \ell$, contradicting the assumptions.

So there are no coprime criminal pair of integers.