XVIII.
Dürer Competition

First round (22. 11. 2024.)

Problems

1. The infinite sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ consist of positive integers, and the following conditions hold for all $i \geq 1$:

- if $\gcd(a_i, b_i) > 1$ then $a_{i+1} = \dfrac{a_i}{\gcd(a_i, b_i)}$ and $b_{i+1} = \dfrac{b_i}{\gcd(a_i, b_i)}$,
- and if $\gcd(a_i, b_i) = 1$ then $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 2$.

Determine all pairs of positive integers (a_1, b_1) for which there exists a pair in the infinite sequence $(a_1, b_1), (a_2, b_2), \dots$ that appears infinitely many times!
Here, $\gcd(p, q)$ denotes the greatest common divisor of p and q.

2. A positive integer n and a real number $c > 1$ are given. The underground Albrecht Bank has just been robbed, and the n robbers are fleeing the scene. Before the heist, each criminal hid a scooter at a different point on the surface. The robbers have now emerged at various exits onto the surface. We can observe that if the robbers' positions were scaled by a factor of c from the main surface entrance of the bank, each would be precisely at their own scooter.

The robbers wish to escape using the scooters (not necessarily their own), but each scooter can only carry one person. The police are on their way, so each robber must run to their chosen scooter on the shortest path possible. Prove that the total distance travelled by the robbers to reach the scooters cannot be less than if they all choose their own scooter.
The main entrance of the bank, the robbers, and the scooters are considered as points, and the terrain is completely flat.

3. Let $\mathbb{P}$ denote the set of real polynomials in the variable x. Determine all functions $F\colon \mathbb{P} \to \mathbb{P}$, such that all polynomials $p, q \in \mathbb{P}$ satisfy

$$F(p+q) = F(p) + F(q) \qquad \text{and} \qquad F(p(q)) = (F(p))(q) \cdot F(q).$$

Here $p(q)$ denotes the polynomial we get when we substitute q as the variable of p. Similarly, $(F(p))(q)$ denotes the polynomial we obtain by substituting q as the variable for polynomial $F(p)$.

4. Positive integers n and k are given with $n \geq k$. We wrote an integer into each cell of an $n \times n$ table, such that each row and column contains at most k distinct integers. What is the maximum number of distinct integers that can appear in the entire table?
Please provide the answer in terms of n and k.

5. We tiled the plane with congruent polygons such that if two of them share a point, it is on the boundary of both of them. The boundaries of the polygons are coloured black, while their interiors are coloured white. Let us denote this colouring by $\mathcal{S}$, and the polygon used in the tiling by $\mathcal{P}$.

We call an isometry of the plane *criminal*, if it doesn't change the colouring, that is it maps $\mathcal{S}$ to itself. It is known that there exist angles $0° < \alpha, \beta, \gamma < 360°$ and distinct points X, Y, Z for which triangle XYZ has a $30°$ interior angle, and the rotations around X by angle α, around Y by angle β and around Z by angle γ are all criminal.
a) Is it possible for both $\mathcal{S}$ and $\mathcal{P}$ to have neither reflectional nor central symmetry?
b) Considering all possible $\mathcal{S}$ colourings, determine the possible values of $\alpha + \beta + \gamma$.
c) Provide an example of $\mathcal{S}$, for which there exists a criminal glide reflection, but no line reflection alone is criminal. Prove that $\alpha = \beta = \gamma$ holds for all such examples!
A glide reflection is the composition of a translation and a reflection, where the vector of translation is parallel to the line of reflection. By definition, rotations are considered counterclockwise.

Please write all the solutions on separate sheets. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVIII. Dürer Competition