E+1. The infinite sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ consist of positive integers, and the following conditions hold for all $i \geq 1$:

- if $\gcd(a_i, b_i) > 1$ then $a_{i+1} = \dfrac{a_i}{\gcd(a_i, b_i)}$ and $b_{i+1} = \dfrac{b_i}{\gcd(a_i, b_i)}$,
- and if $\gcd(a_i, b_i) = 1$ then $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 2$.

Determine all pairs of positive integers (a_1, b_1) for which there exists a pair in the infinite sequence $(a_1, b_1), (a_2, b_2), \dots$ that appears infinitely many times.
Here, $\gcd(p, q)$ denotes the greatest common divisor of p and q.

Solution: Let $b_1 = 2a_1 + c_1$, where c_1 is an integer. Furthermore, for each i, write the elements of the sequence b_i in the form $b_i = 2a_i + c_i$ in a similar way. For some j, if a_j and b_j are relative primes, then the rules are $a_{j+1} = a_j + 1$ and $b_{j+1} = b_j + 2$. If $b_j = 2a_j + c_j$, then $b_{j+1} = b_j + 2 = 2a_j + c_j + 2 = 2(a_j + 1) + c_j = 2a_{j+1} + c_j$. So at these steps $c_{j+1} = c_j$. And for some k, if $\gcd(a_k, b_k) > 1$, then the next members of the series are obtained by dividing by their greatest common divisor. Then, for $b_k = 2a_k + c_k$, $a_{k+1} = \frac{a_k}{\gcd(a_k, b_k)}$ and $b_{k+1} = \frac{b_k}{\gcd(a_k, b_k)}$, so that $b_{k+1} = \frac{2a_k + c_k}{\gcd(a_k, b_k)} = 2a_{k+1} + \frac{c_k}{\gcd(a_k, b_k)}$, i. e. $c_{k+1} = \frac{c_k}{\gcd(a_k, b_k)}$. Since $\gcd(a_k, b_k) > 1$, after this step $0 < |c_{k+1}| < |c_k|$ (unless $c_k = 0$, in which case $c_{k+1} = 0$), but we know that c_{k+1} is still an integer, since b_{k+1} and $2a_{k+1}$ are both integers

If c_1 is nonzero, then we can only divide it by integers greater than 1 a finite number of times, otherwise the c_i series would become non-integer after a while. Then there is a final division, i.e. a final step before which $\gcd(a_i, b_i) > 1$ is satisfied. After that there is always $\gcd(a_i, b_i) = 1$, i.e. from now on the sequences continue according to the rules $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 2$. In this case, there can be no pair of numbers that occur infinitely often, since both sequences grow strictly monotonically after a finite length. And if $c_1 = 0$, then $b_1 = 2a_1$, i.e. either $a_1 = 1$ and $b_1 = 2$, or $a_2 = 1$ and $b_2 = 2$. From then on, the pairs $(1, 2)$ and $(2, 4)$ will follow each other in a cyclic sequence, i.e. they will occur an infinite number of times in the sequence. So, if and only if $b_1 = 2a_1$, there will be a pair of numbers that occur an infinite number of times.

Second solution: Consider a coordinate system and let the x axis denote the values of the a_i series and the y axis denote the values of the b_i series. For each integer i in the coordinate system, denote the point (a_i, b_i). Since the elements of the series are positive integers, all the points marked are in the first quadrant of the plane. Let us examine the distance of the marked points from the line $y = 2x$. If, for some k, $\gcd(a_k, b_k) > 1$, then there is still a grid point on the segment between the origin and the point (a_k, b_k), so in the next step we will get the closest of these grid points to the origin, since then $(a_{k+1}, b_{k+1}) = \left(\frac{a_k}{\gcd(a_k, b_k)}, \frac{b_k}{\gcd(a_k, b_k)}\right)$. If the point (a_k, b_k) falls on the line $y = 2x$, then so does (a_{k+1}, b_{k+1}), but if it does not, then it will not fall on the line $y = 2x$ either, but we are strictly closer to it (the distance is at least halved). And if for some j we have $\gcd(a_j, b_j) = 1$, then $(a_{j+1}, b_{j+1}) = (a_j + 1, b_j + 2)$, so we get the next point (a_j, b_j) by adding the vector $(1, 2)$. This vector is parallel to the line $y = 2x$, so the distance from the line does not change during this process (if it was 0 before, it remains 0 now).

So if the first point is not on the line $y = 2x$, there won't be any points on that line. Since the line $y = 2x$ has a rational slope, there is a grid point closest to it. So our distance from the line can only decrease a finite number of times (since each decrease at least halves it), i.e. after a while we can only walk parallel to this line. In this case, there will be no point that is touched an infinite number of times. If, on the other hand, the first point falls on the line $y = 2x$, then, as in the previous solution, from the second pair of numbers $(1, 2)$ and $(2, 4)$ will be repeated alternately, i.e. there will be a pair of numbers in the sequence that is touched infinitely. Again we see that if and only if $b_1 = 2a_1$ there will be a pair of numbers that occur infinitely often.

Note: If we replace 1 and 2 by arbitrary integers c, d, the statement remains true, i.e. there will be a pair of numbers that occur infinitely often if and only if $b_1 = \frac{d}{c}a_1$.

E+2. A positive integer n and a real number $c > 1$ are given. The underground Albrecht Bank has just been robbed, and the n robbers are fleeing the scene. Before the heist, each criminal hid a scooter at a different point on the surface. The robbers have now emerged at various exits onto the surface. We can observe that if the robbers' positions were scaled by a factor of c from the main surface entrance of the bank, each would be precisely at their own scooter.

The robbers wish to escape using the scooters (not necessarily their own), but each scooter can only carry one person. The police are on their way, so each robber must run to their chosen scooter on the shortest path possible. Prove that the total distance travelled by the robbers to reach the scooters cannot be less than if they all choose their own scooter.
The main entrance of the bank, the robbers, and the scooters are considered as points, and the terrain is completely flat.

Solution: Let us denote the robbers' positions by A and the scooters' positions by B. The task is to minimize the sum of distances in a perfect matching between A and B, where a perfect matching means that each point in A is paired with exactly one point from B and any two different points from A have different pairs in B.

Instead of perfect matchings between A and B, we will refer to the corresponding bijections $A \to B$ throughout the proof. We denote the distance of a bijection π between two sets by $c(\pi)$. Let π_{id} denote the bijection $A \to B$ taking each point to its scaled image.

Case 1. Suppose that c is so large that we can draw a circle centered at the origin such that A is completely inside and B is completely outside. Let f be a function on all points, defined as the distance of the given point and the circle. For any $u \in A, v \in B$, $d(u,v) \geq f(u) + f(v)$. This is because if we draw the segment uv, then it has to intersect the circle, and the distance it travels inside is at least $f(u)$, and the distance it travels outside is at least $f(v)$. So for any bijection π:

$$\sum_{a\in A} d(a,\pi(a)) \geq \sum_{a\in A} \left(f(a) + f(\pi(a))\right) = \sum_{x \in A\cup B} f(x)$$

It is easy to see that equality is reached if every point in A is matched to its image.

Case 2. If c is not large enough. We prove the statement by contradiction: suppose there exists a cheaper bijection $\pi : A \to B$. Let C be a sufficiently large scaling of A, so that there exists a circle centered at the origin with all points of A and B inside, and all points of C outside. Let $\rho_{\mathrm{id}} : B \to C$ and $\sigma_{\mathrm{id}} : A \to C$ assign to each point its scaled image, then $\rho_{\mathrm{id}} \circ \pi : A \to C$ is a bijection with smaller total cost than $\sigma_{\mathrm{id}} : A \to C$, since by the triangle inequality,

$$c(\rho_{\mathrm{id}} \circ \pi) = \sum_{a\in A} d(a, \rho_{\mathrm{id}}(\pi(a))) \leq$$

$$\leq \sum_{a\in A} \left(d(a,\pi(a)) + d(\pi(a), \rho_{\mathrm{id}}(\pi(a)))\right) = c(\pi) + c(\rho_{\mathrm{id}}) < c(\pi_{\mathrm{id}}) + c(\rho_{\mathrm{id}}) = c(\sigma_{\mathrm{id}}).$$

But this contradicts the minimality of σ_{id} from *Case 1.*

Second solution: We reduce the problem to the case where each point lies on a single ray $[0,\infty)$. For each point P, let $d_0(P)$ denote its distance from the origin O. For a bijection $\pi : A \to B$, let $c'(\pi) = \sum_{a\in A} |d_0(\pi(a)) - d_0(a)|$.

Claim. For every pair of points x, y, we have $d(x,y) \geq |d_0(x) - d_0(y)|$.
Proof of claim. Suppose that $d_0(x) \geq d_0(y)$ without loss of generality. Then the statement is equivalent to $d(O,x) - d(O,y) \leq d(x,y)$, which is true by the triangle inequality.

Let $A' = \{d_0(a) : a \in A\}$ and $B' = \{d_0(b) : b \in B\}$. Then $c'(\pi)$ corresponds to the cost c of the bijection corresponding to π between A' and B'. Listing the points of these two sets as $0 \leq a_1 \leq ... \leq a_n$ and $0 \leq b_1 \leq ... \leq b_n$, the bijection taking each point to its scaled image corresponds to $a_1 \mapsto b_1$, ..., $a_n \mapsto b_n$. We will show that this bijection is optimal for this one-dimensional version of the problem. Then this means that π_ι is optimal for the original problem, because for any $\pi : A \to B$, $c(\pi) \geq c'(\pi) \geq c'(\pi_\iota)$ by summing over all $a \in A$ and using the Claim.

Now if all points lie on the ray $[0,\infty)$, then the statement of the problem is easy to establish: supposing there are points $a < a'$ whose matched pairs are b and b' respectively, then we must have $b \leq b'$. Supposing otherwise, if $b > b'$ then $d(a,b) + d(a',b')$ could be reduced by swapping the two pairs. So the order of the points is indeed the same as the order of their pairs in the matching.

E+3. Let $\mathbb{P}$ denote the set of real polynomials in the variable x. Determine all functions $F\colon \mathbb{P} \to \mathbb{P}$, such that all polynomials $p, q \in \mathbb{P}$ satisfy

$$F(p+q) = F(p) + F(q) \qquad \text{and} \qquad F(p(q)) = (F(p))(q) \cdot F(q).$$

Here $p(q)$ denotes the polynomial we get when we substitute q as the variable of p. Similarly, $(F(p))(q)$ denotes the polynomial we obtain by substituting q as the variable for polynomial $F(p)$.

Solution: We will prove that only the functions $F \equiv 0$ and $F(p) = p'$ satisfy the problem's conditions, where p' denotes the *derivative* of the polynomial p. That is, if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then $F(p) = p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$.

Let $\deg(p)$ denote the degree of each polynomial $p \in \mathbb{P}$. Let $\mathbf{I}(p,q)$ represent the condition $F(p)+F(q) = F(p+q)$, and let $\mathbf{II}(p,q)$ represent the condition $F(p(q)) = F(p)(q) \cdot F(q)$.

Now let $p(x) \equiv c$, where $c \in \mathbb{R}$, and let $\deg(q) > 1$. Consider $\mathbf{II}(p,q)$. Then $p(q(x)) = c$, so $F(p) = F(p)(q) \cdot F(q)$. We will prove that $F(p)$ is constant. Assume indirectly that $\deg(F(p)) > 0$. Then $\deg(F(p)(q)) > \deg(F(p))$, thus $F(q) \equiv 0$, since $\deg(F(p)) = \deg(F(p)(q) \cdot F(q)) = \deg(F(p)(q)) + \deg(F(q))$. Hence, $F(p) = F(p)(q) \cdot 0 \equiv 0$, so the degree of $F(p)$ cannot be positive, which is a contradiction. Thus, $F(p) \equiv c^*$ for some $c^* \in \mathbb{R}$. Now let q be an arbitrary polynomial, even constant. According to $\mathbf{II}(p,q)$, $c^* = F(p) = F(p)(q)F(q) = c^*F(q)$, so $c^* = 0$, or $F(q) \equiv 1$ for all $q \in \mathbb{P}$. If $F(q) \equiv 1$ for all $q \in \mathbb{P}$, then choosing q to be the constant zero polynomial, $\mathbf{I}(q,q)$ gives $2 = F(q \equiv 0) + F(q \equiv 0) = F(q \equiv 0) = 1$, which is a contradiction. Thus $F(p) \equiv 0$ if p is constant.

Now consider $\mathbf{II}(p, q(x) = x)$. We obtain $F(p) = F(p) \cdot F(q)$, so if there exists p for which $F(p) \neq 0$, then $F(q) \equiv 1$; otherwise, $F(q) \equiv 0$ for all $q \in \mathbb{R}$, which can easily be verified as a solution.

Henceforth, we may assume $F(x) \equiv 1$. Now consider the condition $\mathbf{II}\left(p(x) = ax, q(x) = \frac{1}{a}x\right)$. According to this, $1 = F(x) = F(p(q)) = F(p)(q)F(q)$, so $F(q)$ is constant, and q can be any linear polynomial whose constant term is 0.

Let $p(x) = ax$, $q(x) = bx$, and consider the statements $\mathbf{I}(p,q)$ and $\mathbf{II}(p,q)$:

$$F(p) + F(q) = F(p+q = (a+b)x),$$

$$F(abx) = F(p(q)) = F(p) \cdot F(q).$$

Since $F(p)$ is constant, let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(a) = F(p(x) = ax)(0)$. Based on the equations above, it follows that:

$$g(a) + g(b) = g(a+b),$$

and

$$g(ab) = g(a)g(b),$$

since $F(p)$ and $F(q)$ are constant. It is known that such a function is either $g(a) = a$ or $g(a) = 0$ for all a. Thus, $F(p(x) = ax) \equiv a$, or $F(p(x) = ax) \equiv 0$. However, since $F(p(x) = x) = x$, it follows that $F(p(x) = ax) \equiv a$.

For any $b \in \mathbb{R}$, we then have:

$$F(p(x) = ax + b) = F(p_1(x) = ax) + F(p_2(x) = b) = F(p_1(x) = ax) \equiv a.$$

Now, consider the relationship between $F(p(x) = ax)$ and an arbitrary polynomial $q(x)$ from $\mathbf{II}(p,q)$. The statement gives:

$$F(aq) = aF(q).$$

We will prove by induction on the degree of the polynomial that $F(p) = p'$, where p' denotes the derivative of the polynomial p. First, the base Case: If $\deg(p) < 2$, then by the arguments above, $F(p) = p'$. Now, the inductive Step: Assume that if $\deg(p) \leq n-1$, then $F(p) = p'$. We will prove that $F(p) = p'$ for $\deg(p) = n$. Let $p(x) = x^n$, $q(x) = x+1$, and consider $\mathbf{II}(p,q)$. According to this:

$$F((x+1)^n) = F(x^n)(x+1).$$

Using the binomial theorem, $(x+1)^n = x^n + nx^{n-1} + \ldots + 1$, so:

$$F((x+1)^n) = F(x^n) + F((x+1)^n - x^n).$$

Since the highest degree term in $(x+1)^n - x^n$ is less than n, by the induction hypothesis:

$$F((x+1)^n - x^n) = ((x+1)^n - x^n)' = n(x+1)^{n-1} - nx^{n-1}.$$

Thus:

$$F(x^n)(x+1) = F(x^n) + n(x+1)^{n-1} - nx^{n-1}.$$

Let $p(x) = F(x^n) - nx^{n-1}$. Then:

$$p(x+1) + n(x+1)^{n-1} = F(x^n)(x+1) = F(x^n) + n(x+1)^{n-1} - nx^{n-1}.$$

Hence:

$$p(x+1) = p(x),$$

which implies that $p(x)$ is periodic. Since $p(x)$ is a polynomial, it must be constant, so $p(x) \equiv c$ for some $c \in \mathbb{R}$. This means:

$$F(x^n) = nx^{n-1} + c.$$

Let $p(x) = x^n$, $q_1(x) = ax$, and $q_2(x) = \sqrt[n]{a}x$, where $a \in \mathbb{R}$. Consider $\mathbf{II}(q_1, p)$ and $\mathbf{II}(p, q_2)$:

$$a(nx^{n-1} + c) = F(q_1(p)) = F(p(q_2)) = \left(n\left(\sqrt[n]{a}x\right)^{n-1} + c\right) \cdot \sqrt[n]{a}.$$

From this:

$$a(nx^{n-1} + c) = anx^{n-1} + c\sqrt[n]{a}.$$

This holds only if $c = 0$. Thus:

$$F(x^n) = nx^{n-1}.$$

We have shown that if $\deg(p) = n$ and $p(x) = \sum_{i=0}^{n} a_i x^i$, then:

$$F(p) = \sum_{i=0}^{n} a_i F(x^i) = \sum_{i=0}^{n} a_i (ix^{i-1}) = p'(x).$$

This completes the proof.

E+4. Positive integers n and k are given with $n \geq k$. We wrote an integer into each cell of an $n \times n$ table, such that each row and column contains at most k distinct integers. What is the maximum number of distinct integers that can appear in the entire table?
Please provide the answer in terms of n and k.
Solution: The answer is $(k-1)n + \lfloor \frac{n}{n-k+1} \rfloor$, we first show a construction where we can reach this many numbers, and to do so we use an equivalent question: Each edge of the complete bipartite graph $K_{n,n}$ contains an integer, such that each vertex of the graph has at most k different integers on the edges adjacent to it. What is the maximum number of distinct integers that can appear in the entire table?

It is well known that for all $\ell \leq m$ positive integers there exists an ℓ-regular bipartite graph with m vertices in each class. Let $G_1 = K_{n-k+1,n-k+1}$ and G_2 be an $n-k+1$-regular bipartite graph on $m = n - \left(\lfloor \frac{n}{n-k+1} \rfloor - 1\right)(n-k+1) \geq n-k+1$ vertices. Then we can embed $\left(\lfloor \frac{n}{n-k+1} \rfloor - 1\right)$ copies of vertex disjoint copies of G_1 and a further vertex disjoint copy of G_2 to $K_{n,n}$. Then each vertex of $K_{n,n}$ appears in exactly one of these graphs. For each embedded graph we assign a unique integer, and write this on the edges of that graph. Then, for the rest of the edges of the $K_{n,n}$ we write a new, unique integer. Then each vertex has exactly k different numbers on the edges adjacent to it, and in total there are $(k-1)n + \lfloor \frac{n}{n-k+1} \rfloor$ different integers in the graph.

Next, we show that each numbering of the $n \times n$ table contains at most this many distinct integers. Take a numbering that achieves the maximal number of integers, say $n(k-1)+a$, so by the previous part $a \geq 0$. Then go through the rows from top to bottom, and for each row, note how many new integers the given row contains, so integers we haven't seen in the rows above. By the pigeonhole principle, there must be at least a rows that contain k new integers, as each row can contain at most k distinct integers. Let A be a set of such rows, with $|A| = a$. Then each row in A has a unique set of k integers in it, so if we intersect A with any column, the rest of that column can contain at most $k-a$ new integers. So in total, the number of different integers in the whole table is at most $a \cdot k + n \cdot (k-a)$, the first term corresponding to the integers in A, and the second is the new integers for each column. But we also know that the number of different integers in the whole table is $n(k-1)+a$, so we can get the following inequality.

$$a \cdot k + n(k-a) \geq n \cdot (k-1) + a$$
$$n \geq a \cdot (n-k+1)$$
$$\frac{n}{n-k+1} \geq a$$

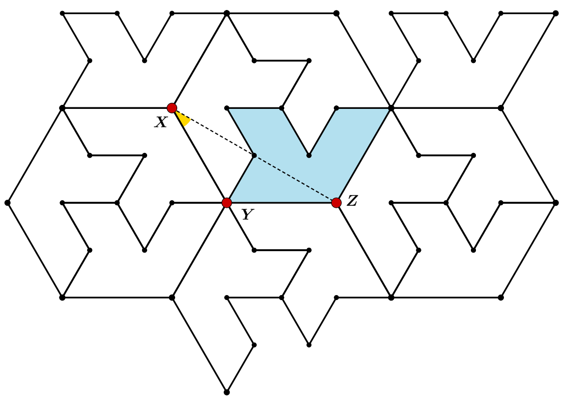
But a must be an integer, so we showed that the total number of different integers must be at most$(k-1)n + \lfloor \frac{n}{n-k+1} \rfloor$.

E+5. We tiled the plane with congruent polygons such that if two of them share a point, it is on the boundary of both of them. The boundaries of the polygons are coloured black, while their interiors are coloured white. Let us denote this colouring by $\mathcal{S}$, and the polygon used in the tiling by $\mathcal{P}$.

We call an isometry of the plane *criminal*, if it doesn't change the colouring, that is it maps $\mathcal{S}$ to itself. It is known that there exist angles $0^\circ < \alpha, \beta, \gamma < 360^\circ$ and distinct points X, Y, Z for which triangle XYZ has a 30° interior angle, and the rotations around X by angle α, around Y by angle β and around Z by angle γ are all criminal.
a) Is it possible for both $\mathcal{S}$ and $\mathcal{P}$ to have neither reflectional nor central symmetry?
b) Considering all possible $\mathcal{S}$ colourings, determine the possible values of $\alpha + \beta + \gamma$.
c) Provide an example of $\mathcal{S}$, for which there exists a criminal glide reflection, but no line reflection alone is criminal. Prove that $\alpha = \beta = \gamma$ holds for all such examples!
A glide reflection is the composition of a translation and a reflection, where the vector of translation is parallel to the line of reflection. By definition, rotations are considered counterclockwise.

Solution: Let's first introduce some notations: Let ρ_O^ϕ denote a rotation with center O and angle ϕ, $\tau_{\mathbf{v}}$ denote a translation by vector $\mathbf{v}$, and σ_e and σ_E denote reflections with respect to line e and point E respectively. We call point O a ϕ-center if ρ_O^ϕ is criminal. We can immediately observe that the composition of criminal transformations is also criminal; moreover, a criminal transformation maps another criminal transformation to a criminal one (for instance, if ρ_D^δ and ρ_E^ϵ are criminal, then $\rho^\epsilon_{\rho_D^\delta(E)}$ is also criminal). Additionally, we will use $\circ$ to denote the composition of transformations, and by $t_1 \circ t_2$, we mean that t_2 is applied first, followed by t_1.

a) It is possible. Consider the following example where we first cover the plane with regular hexagons and then decompose each one into three congruent, non-symmetric polygons:



A regular hexagon with three consecutive vertices forms a triangle with two angles of 30° each, and all three vertices are 120°-centers.

b) We will need a few lemmas.

Lemma 1. Given two distinct points D and E in the plane, and two angles δ and ϵ. Let $d = \rho_D^{-\delta/2}(DE)$ and $e = \rho_E^{\epsilon/2}(DE)$. Then:

- If $\delta + \epsilon \not\equiv 0 \pmod{2\pi}$, then $\rho_E^\epsilon \circ \rho_D^\delta = \rho_O^{\delta+\epsilon}$, where $O = d \cap e$.
- If $\delta + \epsilon \equiv 0 \pmod{2\pi}$, i.e., $d \parallel e$, then $\rho_E^\epsilon \circ \rho_D^\delta = \tau^2$, where τ is the translation that maps d to e.

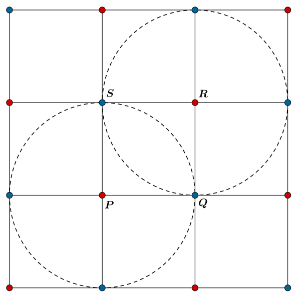
Proof. Using the representation of rotation as a composition of two reflections:

$$\rho_E^\epsilon \circ \rho_D^\delta = (\sigma_e \circ \sigma_{DE}) \circ (\sigma_{DE} \circ \sigma_f) = \sigma_e \circ \sigma_d.$$

Lemma 2. If O is a ϕ-center, then $\phi \in \{60°, 90°, 120°, 180°, 240°, 270°, 300°\}$.

Proof. Observe that ρ_X^α (or ρ_Y^β if $O = X$) maps O to another ϕ-center. If O' is the nearest ϕ-center to O (such a point exists since there are only finitely many ϕ-centers in a neighborhood of a point, as they map each polygon to another polygon, preserving vertices), then the rotation of O about O' must also be a ϕ-center and farther from O than O'. Since a larger angle implies a larger side, this means that no angle smaller than 60° can be generated by the sequence of multiples of ϕ modulo 360°. The lemma follows.

Lemma 3. None of α, β, γ is 90° or 270°.



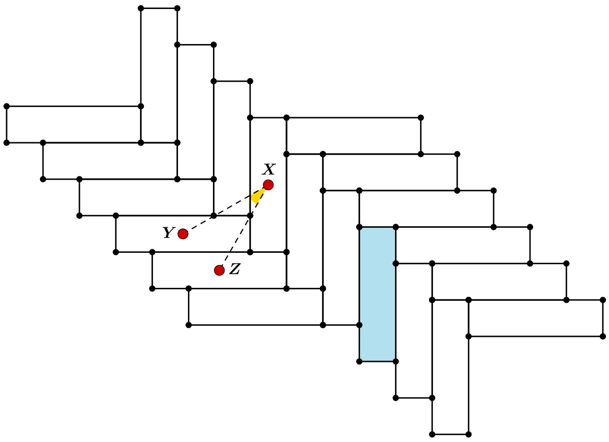
Proof. Since $3 \cdot 270° \equiv 90° \pmod{360°}$, it is sufficient to show that a 90°-center cannot exist in the plane. Suppose, for contradiction, that P is a 90°-center. Then, no 60°- or 120°-center can exist because their composition would result in a criminal rotation with an angle contradicting Lemma 2. Thus, all centers in the plane are either 90°- or 180°-centers (or both). Let Q denote the nearest center to P. (Such a point exists because there are only finitely many centers within any circle.) If Q is a 90°-center, then, by Lemma 1, $\rho_P^{90°} \circ \rho_Q^{90°}$ would yield a criminal 180° rotation with a center on the circle whose diameter is PQ, contradicting the minimality of $|PQ|$. Thus, Q must be a 180°-center. Rotating P around Q by 90°, 180°, and 270° produces more 180°-centers. Similarly, reflecting P through 180°-centers generates more 90°-centers. It is easy to see that this arrangement forms a square grid lattice, where every point is a 180°-center, and every alternate grid point is also a 90°-center. Can there be any other centers distinct from these grid points? No, because such a center would have to lie within one of the unit squares (shown as $PQRS$ on the diagram), but it must be farther from the corners than $|PQ|$, which is impossible because the circles centered at 90°-centers with radius $|PQ|$ cover the square. Now, consider that the lattice triangle XYZ has a 30° angle. This implies that the angle between two grid lines is 30°, meaning if the slopes of these lines are x and y, then

$$\tan(30°) = \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

However, this is impossible because $\tan x$ and $\tan y$ are rational (since they are grid lines), while $\tan(30°) = 1/\sqrt{3}$ is irrational.

To summarize, α, β, γ are all of the form $c \cdot 60°$ where $1 \le c \le 5$ is an integer. Consequently, $\alpha + \beta + \gamma$ can take values of the form $C \cdot 60°$ where $3 \le C \le 15$ is an integer. All such values are possible, for example, by covering the plane with equilateral triangles and selecting points X, Y, Z as vertices of a 30°-30°-120° triangle, where every vertex is a 60°-center.

c) Let's start with an example.

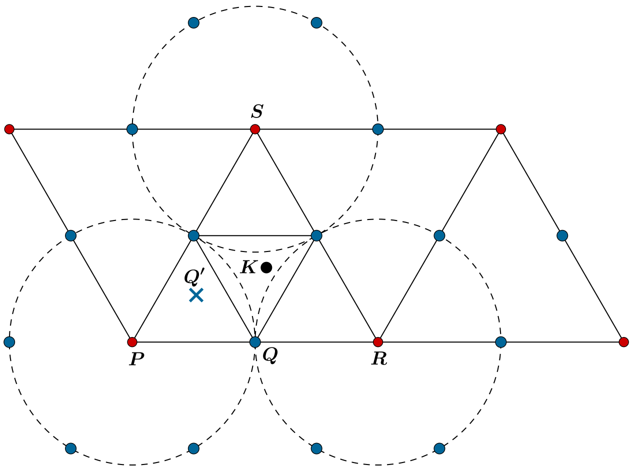


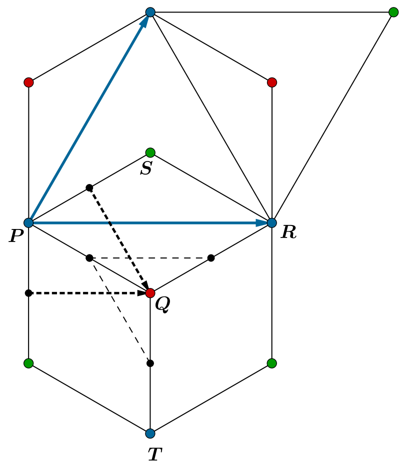
Here, X, Y, Z denote the centers of rectangles. Clearly, these are all 180°-centers, and by choosing the side ratios of the rectangles appropriately, we can achieve that the angle at X is 30°.

Now, let us show that in this case, $\alpha = \beta = \gamma = 180°$. By Lemma 3, we only need to demonstrate that from the existence of a criminal glide reflection, a criminal reflection about a line necessarily follows:

Case 1: There exists a 60°-center. Let P denote a 60°-center and Q the nearest center to it. Q cannot be a 120°-center, because then, by Lemma 1, $\rho_P^{60^\circ} \circ \rho_Q^{120^\circ}$ would result in a criminal 180° rotation with its center lying on the circle whose diameter is PQ, contradicting the minimality of $|PQ|$. Thus, Q must be a 180°-center. By rotating Q around P by $60°, 120°, \ldots, 300°$, we obtain additional 180°-centers. Reflecting the 60°-centers across the 180°-centers yields more 60°-centers. This process generates a regular triangular lattice, where each vertex is a 180°-center, and every second vertex is also a 60°-center. We claim that no other 60°-centers exist outside this triangular lattice. If such a K existed, it would lie within one of the unit regular triangles, say PRS. K would then be in the medial triangle of PRS (but not anywhere arbitrarily within it) due to the minimality of $|PQ|$. Therefore, Q's rotation about K by $-60°$ would produce a new 180°-center Q', which is closer to P than Q, a contradiction.

Thus, the 60°-centers coincide precisely with the vertices of this triangular lattice. Notice that $\tau_{\overrightarrow{PR}}$ is also criminal, since it equals $\rho_Q^{180^\circ} \circ \rho_P^{180^\circ}$. Similarly, $\tau_{\overrightarrow{PS}}$ is criminal, so any two points in the triangular lattice can be mapped to each other by criminal translations. If a glide reflection is criminal, it maps this lattice onto itself. Assume that the image of P is P'. Composing the glide reflection with $\tau_{\overrightarrow{P'P}}$ results in a criminal isometry that reverses orientation and fixes P, which is therefore a reflection about a line. Hence, we have found a criminal reflection about a line.





Case 2: There is no 60°-center, but there is a 120°-center. In this scenario, every center is a 120°-center (since a 180°-center would imply the existence of a 60°-center). Let P be a 120°-center, and let Q denote the nearest center to P. The 120° rotations again produce a regular triangular lattice, with vertices corresponding to the 120°-centers. No other centers can exist since they would have to lie within a small unit triangle, contradicting minimality. However, not every translation is criminal in this arrangement. We observe that $\rho_P^{120^\circ} \circ \rho_Q^{-120^\circ}$ results in a translation whose vector corresponds to the longer diagonal of a $60^\circ - 120^\circ$ rhombus. Thus, we decompose the triangular lattice into three disjoint sublattices, where the edges themselves serve as criminal translation vectors. Alternatively, we can say that we partition the triangular lattice into three sublattices where points within the same sublattice can be mapped to each other by criminal translations.

Since the glide reflection maps the entire lattice onto itself (being criminal), it maps P to some other lattice point. If $PQRS$ is a minimal $60^\circ - 120^\circ$ rhombus, then P and R belong to the same sublattice, while Q and S belong to different sublattices. Composing the glide reflection with an appropriate translation results in a glide reflection that maps P either onto itself, Q, or S. If it maps P onto itself, we have found a reflection about a line. The other two cases are symmetric (by rotation), so we may assume that P is mapped to Q. Applying the glide reflection twice yields a translation whose vector is twice that of the glide reflection's translation vector. Therefore, Q must map to a point within the same sublattice as P (since a vector between different sublattice points cannot be a criminal translation; otherwise, combining it with an appropriate rotation would produce a new center). Furthermore, due to distance preservation, Q must map to one of the vertices of the small hexagon around it. Thus, by the previous reasoning, it maps to P, R, or T. If it maps to P, the glide reflection is a reflection about a line, completing the proof. If it maps to R or T, then the axis of the glide reflection is the perpendicular bisector of segment PQ or QT respectively, implying that the translation vector is half the length of the minimal vector in Q's sublattice, a contradiction.