XVII. DÜRER COMPETITION

Problems and Solutions

Contents

Introduction – About the Dürer Competition **3**

1 Problems – grammar school categories **6**
1.1 Regional round . . . 6
1.1.1 Category A . . . 6
1.1.2 Category B . . . 12
1.2 Final round – day 1 . . . 18
1.2.1 Category A . . . 18
1.2.2 Category B . . . 21
1.3 Final round – day 2 . . . 24
1.3.1 Category A . . . 24
1.3.2 Category B . . . 30

2 Problems – high school categories **37**
2.1 Online round . . . 37
2.1.1 Category C . . . 37
2.1.2 Category D . . . 42
2.1.3 Category E . . . 46
2.2 Regional round . . . 49
2.2.1 Category C . . . 49
2.2.2 Category D . . . 52
2.2.3 Category E . . . 56
2.2.4 Category E^+ . . . 58
2.3 Final round – day 1 . . . 61
2.3.1 Category C . . . 61
2.3.2 Category D . . . 64
2.3.3 Category E . . . 67
2.3.4 Category E^+ . . . 69
2.4 Final round – day 2 . . . 71
2.4.1 Category C . . . 71
2.4.2 Category D . . . 76
2.4.3 Category E . . . 82
2.4.4 Category E^+ . . . 87

3 Solutions – grammar school categories **93**
3.1 Regional round . . . 93
3.1.1 Tables - Category A . . . 93
3.1.2 Tables - Category B . . . 94
3.2 Final round – day 1 . . . 94
3.2.1 Category A . . . 94
3.2.2 Category B . . . 99
3.3 Final round – day 2 . . . 102
3.3.1 Tables . . . 102
3.3.2 Category A . . . 103

3.3.3 Category B . . . 109

4 Solutions – high school categories **112**

4.1 Online round . . . 112

4.1.1 Tables . . . 112

4.1.2 Category C . . . 113

4.1.3 Category D . . . 116

4.1.4 Category E . . . 119

4.2 Regional round . . . 121

4.2.1 Category C . . . 121

4.2.2 Category D . . . 125

4.2.3 Category E . . . 129

4.2.4 Category E^+ . . . 131

4.3 Final round – day 1 . . . 138

4.3.1 Category C . . . 138

4.3.2 Category D . . . 144

4.3.3 Category E . . . 149

4.3.4 Category E^+ . . . 152

4.4 Final round – day 2 . . . 160

4.4.1 Tables . . . 160

4.4.2 Category C . . . 161

4.4.3 Category D . . . 167

4.4.4 Category E . . . 171

4.4.5 Category E^+ . . . 175

Introduction – About the Dürer Competition

Hungary has a rich tradition of hosting mathematics contests catering to participants of all ages and qualifications, from primary schoolers to university students. While most contests are individual-based, the Dürer Competition stands out as a unique experience, where teams of three work together to solve problems, fostering the advantages of cooperative thinking. Notably, students tend to be happier and more at ease during this collaborative setting.

One of the fundamental goals of the Dürer Competition is to present intriguing problems that showcase the elegance of mathematics and the joy of thinking to a wide array of students. The organizers strive to include as many original problems as possible, usually introducing around 150 problems each year. Although some problems may not be entirely original, the majority of the more challenging ones are thoughtfully crafted by the team.

The theme of the 17th competition were the seas and sailing.

An essential aspect worth mentioning is the youthful and dynamic composition of the organizing team. Traditionally comprising young individuals, mostly university students with a passion for mathematics, many of them are former competitors who now contribute as organizers. This community, ranging from 30 to 70 people, plays a pivotal role in the competition's success. Some have been devoted organizers for as long as 16 years, actively participating even while balancing full-time jobs, while others step into significant responsibilities as first-year undergraduates. Remarkably, several organizers have also served as coordinators at prestigious international events such as EGMO 2022 and IMO 2022.

Maintaining this spirit, the Dürer Competition has been successfully organized for 16 years, consistently drawing a growing number of students and schools each year. In the academic year 2023-24, over 1000 Hungarian high school students participated in the mathematics categories, along with more than 650 students from primary schools, demonstrating the competition's increasing popularity and impact on the mathematics community.

This was the fifth year that we opened our two hardest categories for international competitors. The contest was held online with a record breaking over 400 students from 15 countries taking part, with new countries Azerbaijan, Japan, Kyrgyzstan, Serbia, Thailand and Uzbekistan. The final round was also held online with the 5 best international teams participating, where the team from Romania topped the leaderboard.

Primary school students can take part in our competition in the following two categories:

- **Category A** is open to 5th and 6th grade students.
- **Category B** is open to 7th and 8th grade students.

In these two categories the contest is regional: the first round is organised in 6 cities in north-eastern Hungary, but is open to anyone provided that they travel to one of the locations.

Four categories are available for high school students:

- **Category C** is open to 9^{th} and 10^{th} graders who have never previously qualified for the final of any national math contest.
- **Category D** is open to 9^{th} to 12^{th} graders who are a bit more experienced, but do not come from a school that is outstanding in handling mathematical talents.
- **Category E** is open to 9^{th} to 12^{th} graders who already have good results from other contests, or come from a school outstanding in maths.
- **Category E$^+$** is designed for competitors who actively take part in olympiad training. In this category, most teams include some student who has taken part at an international olympiad (IMO, MEMO, EGMO, RMM, IMSC), or is about to qualify for one in the same academic year.

We also organise the contest in physics (*category F* and *F$^+$*) and chemistry (*categories K, K$^+$* and *L*), but these are omitted from this booklet.

High school students participating in categories C, D, and E face an initial **online relay round** comprising 9 problems. Each question requires an integer answer between 0 and 9999. Teams start with the first question and have three attempts to submit an answer. A correct response grants them a set number of points, allowing them to advance to the next question. However, each incorrect attempt reduces the potential score by 1. If a team fails to answer a question correctly after three attempts, they must move on to the next question without scoring. Additionally, an **online game** is included in this round.

In the second round for categories E and E$^+$, a traditional **olympiad-style contest** takes place, where teams must provide detailed proofs for 5 problems within a 3-hour timeframe. For categories C and D, the second round involves four short-answer and three olympiad-style questions, also lasting 3 hours. Notably, proof is not required for the short-answer questions, making the round more accessible to students less accustomed to competing. This round is organized in approximately 25 locations across the country.

The final round is hosted in Miskolc. For high schoolers (categories C, D, E, E$^+$, F, F$^+$ K, K$^+$, L), it spans a weekend in early February, from Thursday to Sunday. Friday serves as the first competition day, where students tackle five **olympiad-style problems** and participate in a **game**. If a team believes they have discovered a winning strategy for the game, they can challenge the organizers. By defeating the organizers twice in a row, the team earns the maximum score for the problem. However, if they lose, they still have two more attempts to challenge for a partial score. Saturday holds a **relay round** comprising 16 questions, following similar rules to the online round. The rankings are determined based on a combined score from both competition days.

Throughout the final weekend, students and teachers have the opportunity to engage in various educational and recreational activities, including lectures, games, and discussions about universities.

The primary school student competition follows a separate but similar format. Their first round consists of a **relay round** with 15 questions, and the final round mirrors the structure of the high school students' final round.

The competition is expanded year after year which contributes to the establishment of the Dürer Competition as a well-known and renowned contest which is one of our main objectives.

A subset of the organizers of the 17$^{\text{th}}$ Dürer Competition

Editor: Tímea Csahók

The compilers of the problem sets:
Anita Páhán, Eszter Szabó, Kartal Nagy, Máté Jánosik, Dániel Hegedűs, József Osztényi, Áron Jánosik, Áron Bán-Szabó, Csongor Beke, Áron Horváth, Benedek Kovács

Translation:
Tímea Csahók, Izabella Farkas, Tamás Gábriel, Áron Horváth, Péter Juhász, Leila Nagy, Panna Ottó, Gabriella Sztranyák, Tamás Takács, Damján Tárkányi

Publisher: Gábor Szűcs, The Joy of Thinking Foundation

If you have any questions or notes, you may contact: maths@durerinfo.com

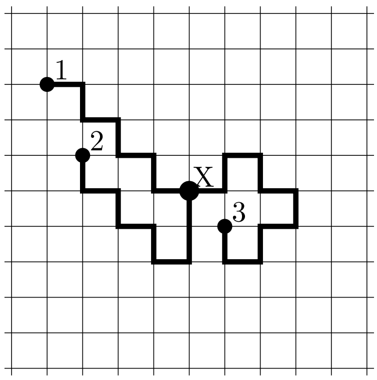
1 Problems – grammar school categories

1.1 Regional round

1.1.1 Category A

1. Three ships want to reach point X where the treasure is located, using the routes shown on the diagram. Which ship arrives there first if they start simultaneously and travel at the same speed?

(Proposed by Lili Mohay)



(Solutions)

2. Szása is a centipede who lives with their mother, father, and two siblings. Every morning this week the family members put on clean socks for each of their feet except Szása, who does not wear socks on Wednesday and Saturday. How many socks does Szása need to sew so that everyone has enough socks for a week?

A centipede has one hundred legs.

(Proposed by Zsombor Várkonyi)

(Solutions)

3. Hanga's morning walk to school takes 7 minutes, during which she likes to listen to music. She doesn't like to interrupt songs, so she always chooses her playlist so that the music ends exactly when she arrives at school. Hanga has four favorite songs, which are 1 minute, 2 minutes, 3 minutes, and 4 minutes long. How many different ways can Hanga compile her playlist for the trip if the order of the songs matters?

She starts the first song when she begins her trip, does not take any breaks between songs, and does not want to listen to any song more than once during her trip.

(Proposed by Dorka Mezey)

(Solutions)

4. As a young child, Anita went to the beach one day to collect seashells. She collected two types of shells: grey ones and pink ones. After a successful adventure, she wrote down on a piece of paper how many of each type of shell she had collected separately, and also the total number of shells found. Several years later, Anita found the paper again, on which some digits had already blurred, so only this is legible:

Grey seashells: 2■
Pink seashells: ■■3
Total: 171

How many more pink shells did she collect than grey ones?

(Solutions)

5. Zsuzsi wants to make a necklace for Ludmilla using 3 orange and 7 yellow beads. How many different necklaces can Zsuzsi make for Ludmilla using all the beads?

The beads of the necklace are strung on a cord, the ends of which are knotted together. Two necklaces are not considered different if they can be laid out side by side on a table such that the beads are arranged in exactly the same way on both necklaces. The position of the knot on the cord does not matter.

(Proposed by Dorka Mezey)

(Solutions)

6. Five children want to play a sailor-themed game, but they cannot agree on who should play which role. So they line up in a row where each can only see those ahead of them. Then, an adult places a hat on each child's head from the following: 1 captain's hat, 1 first mate's hat, 1 helmsman's hat, and 2 petty officer's hats. After this, starting from the front of the line and moving backwards, each child says a sentence about what they know about their role:

1. I'm sorry, I don't know anything about it.
2. I can fill all four types of positions.
3. I am not the captain.
4. I am not a petty officer.
5. I am the helmsman.

What is the product of the first mate's and the captain's positions?

We assume that they all reasoned correctly and told the truth.

(Proposed by Lili Mohay)

(Solutions)

7. After their daily fishing, the fishermen would like to distribute the catch among themselves. They found that if they release back five fish, they can evenly divide the remaining catch among the 10 crew members. If they release six fish, the catch can be evenly divided among the 7 officers on board. How many fish were caught that day, knowing that they caught fewer than 100 fish?

(Proposed by Dorka Mezey)

(Solutions)

8. Liza's birthday cake has as many candles as the number of her age. She blew out the candles with 4 blows, ensuring each blow extinguished at least one candle. However, during the first three blows, she managed to extinguish less than half of the burning candles each time. On the fourth blow, she extinguished 2 candles. **The answer is the sum of all possible ages of Liza.**

(Proposed by András Bognár)

(Solutions)

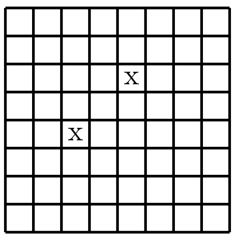
9. Blackbeard is currently 160 centimeters tall and grows 2 centimeters every year. His beard is now 8 centimeters long and it grows 14 centimeters each year. In how many years will Blackbeard's beard reach the ground, given that his head remains 20 centimeters tall throughout his life and his beard grows from the bottom of his head?

(Proposed by Áron Horváth)

(Solutions)

10. Two pieces were placed on a square of a large 8×8 board. In each move, the pieces stepped to an adjacent square. Both pieces made three moves without stepping on any square they had previously visited. Finally, they ended up on the two squares shown in the diagram. Based on this, from how many squares could they have started on the board?

(Proposed by Dorka Mezey)



(Solutions)

11. Five sailors — Ali, Bod, Csed, Doma, and Ede — climbed up to the mast basket one after another. We know that Ali stepped on every second rung, Bod every third, Csed every fourth, Doma every fifth, and Ede every sixth rung during their climb. How many rungs did none of them step on, knowing that only the first and last rungs were stepped on by all of them? (The ladder has more than one rung.)

(Proposed by Sára Szepessy)

(Solutions)

12. Marvin wrote down all pairs of numbers composing of two integers smaller than 100 where one of the numbers can be written using only the digits of the other number in the pair (digits can be used multiple times) and the difference of the two numbers is 1. How many different numbers did Marvin write down?

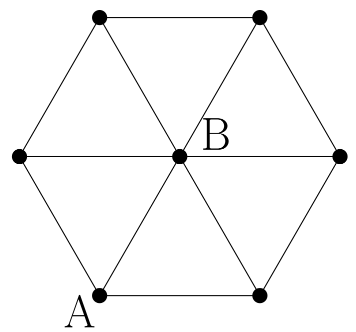
(Proposed by Erik Füredi)

(Solutions)

13. A siren spends every night on one of the 7 cliffs shown in the picture in such way that the cliff she chose on a given night was adjacent to the cliff from the previous night. (The adjacent cliffs are connected in the picture.) The siren was spotted on cliff A on Monday night and cliff B on Friday night by the crew of a ship that was passing by. In how many different ways could the siren have chosen her dwelling on the three nights in between?

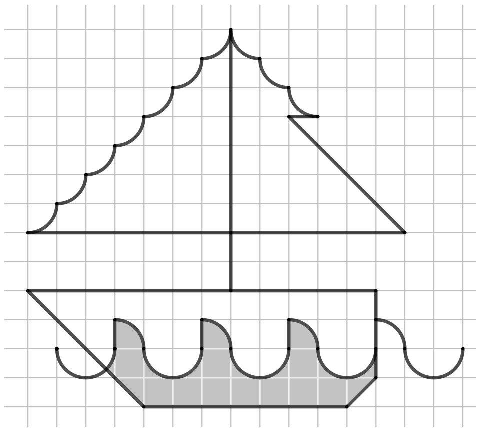
Two cases are different if there is a night which she spent on different cliffs in the two cases.

(Proposed by Eszter Szabó)



(Solutions)

14. The sail area of the ship shown in the diagram is 700 m^2. What is the area in square meters of the gray region, if in the diagram all arcs belong to circles of the same size?



(Proposed by Anita Páhán)

(Solutions)

15. Anchorless Jack, the pirate buried his treasure on a rectangular-shaped island. The northern shore of the island is 8 kilometers long, and the eastern shore is 10 kilometers long. He disclosed that the treasure is at least 4 kilometers away from the northern shore, at least 3 kilometers away from the southern shore, at most 6 kilometers away from the eastern shore, and at most 5 kilometers away from the western shore. Based on this information, what is the area in square kilometers where the treasure could be hidden?

(Proposed by Áron Horváth)

(Solutions)

1.1.2 Category B

1. Szása is a centipede who lives with their mother, father, and two siblings. Every morning this week the family members put on clean socks for each of their feet except Szása, who does not wear socks on Wednesday and Saturday. How many socks does Szása need to sew so that everyone has enough socks for a week?

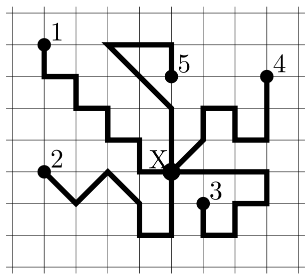
A centipede has one hundred legs.

(Proposed by Zsombor Várkonyi)

(Solutions)

2. Five ships want to reach point X, where the treasure is located, using the routes shown in the diagram. Which ship arrives there first if they start simultaneously and travel at the same speed?

(Proposed by Lili Mohay)



(Solutions)

3. As a young child, Anita went to the beach one day to collect seashells. She collected two types of shells: grey ones and pink ones. After a successful adventure, she wrote down on a piece of paper how many of each type of shell she had collected separately, and also the total number of shells found. Several years later, Anita found the paper again, on which some digits had already blurred, so only this is legible:

Grey seashells: 2■
Pink seashells: ■■3
Total: 171

How many more pink shells did she collect than grey ones?

(Solutions)

4. Hanga's morning walk to school takes 7 minutes, during which she likes to listen to music. She doesn't like to interrupt songs, so she always chooses her playlist so that the music ends exactly when she arrives at school. Hanga has five favorite songs, which are 1 minute, 2 minutes, 3 minutes, 3 minutes, and 4 minutes long. How many different ways can Hanga compile her playlist for the trip if the order of the songs matters?

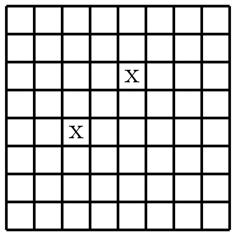
She starts the first song when she begins her trip, does not take any breaks between songs, and does not want to listen to any song more than once during her trip.

(Proposed by Dorka Mezey)

(Solutions)

5. Two pieces were placed on a square of a large 8×8 board. In each move, the pieces stepped to an adjacent square. Both pieces made three moves without stepping on any square they had previously visited. Finally, they ended up on the two squares shown in the diagram. Based on this, from how many squares could they have started on the board?

(Proposed by Dorka Mezey)



(Solutions)

6. Five children want to play a sailor-themed game, but they cannot agree on who should play which role. So they line up in a row where each can only see those ahead of them. Then, an adult places a hat on each child's head from the following: 1 captain's hat, 1 first mate's hat, 1 helmsman's hat, and 2 petty officer's hats. After this, starting from the front of the line and moving backwards, each child says a sentence about what they know about their role:

1. I'm sorry, I don't know anything about it.
2. I can fill all four types of positions.
3. I am not the captain.
4. I am not a petty officer.
5. I am the helmsman.

What is the product of the first mate's and the captain's positions?

We assume that they all reasoned correctly and told the truth.

(Proposed by Lili Mohay)

(Solutions)

7. Blackbeard is currently 160 centimeters tall and grows 2 centimeters every year. His beard is now 8 centimeters long and it grows 14 centimeters each year. In how many years will Blackbeard's beard reach the ground, given that his head remains 20 centimeters tall throughout his life and his beard grows from the bottom of his head?

(Proposed by Áron Horváth)

(Solutions)

8. Five sailors — Ali, Bod, Csed, Doma, and Ede — climbed up to the mast basket one after another. We know that Ali stepped on every second rung, Bod every third, Csed every fourth, Doma every fifth, and Ede every sixth rung during their climb. How many rungs did none of them step on, knowing that only the first and last rungs were stepped on by all of them? (The ladder has more than one rung.)

(Proposed by Sára Szepessy)

(Solutions)

9. Marvin wrote down all pairs of numbers composing of two integers smaller than 100 where one of the numbers can be written using only the digits of the other number in the pair (digits can be used multiple times) and the difference of the two numbers is 1. How many different numbers did Marvin write down?

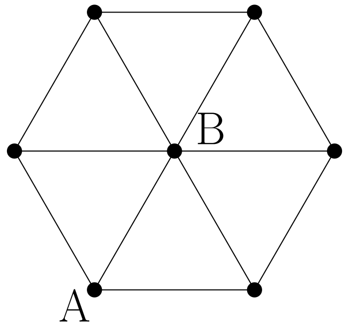
(Proposed by Erik Füredi)

(Solutions)

10. A siren spends every night on one of the 7 cliffs shown in the picture in such way that the cliff she chose on a given night was adjacent to the cliff from the previous night. (The adjacent cliffs are connected in the picture.) The siren was spotted on cliff A on Monday night and cliff B on Friday night by the crew of a ship that was passing by. In how many different ways could the siren have chosen her dwelling on the three nights in between?

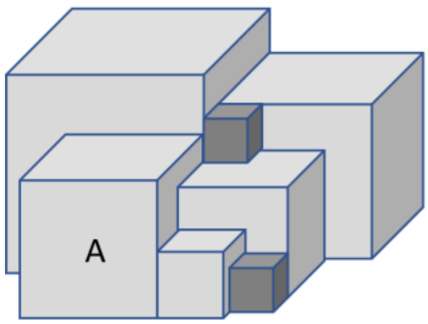
Two cases are different if there is a night which she spent on different cliffs in the two cases.

(Proposed by Eszter Szabó)



(Solutions)

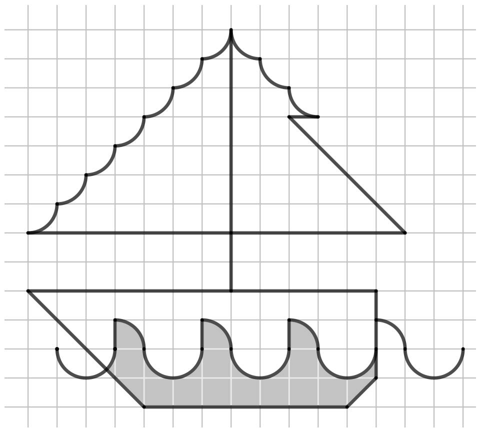
11. How long are the edges of cube A shown in the diagram? The edges of the largest cube are 57 centimeters long and the edges of the dark gray cubes are 13 centimeters long.



(Proposed by Kartal Nagy)

(Solutions)

12. The sail area of the ship shown in the diagram is 700 m^2. What is the area in square meters of the gray region, if in the diagram all arcs belong to circles of the same size?



(Proposed by Anita Páhán)

(Solutions)

13. Jack the pirate stumbled upon a treasure chest on a remote island. The following table shows the name, volume and value of the eight pieces of treasure. Alas, Jack's boat is not large enough to carry the whole contents of the chest. At most how many coins worth of treasure can he take away with him if he can fit 15 gallons of treasure in the boat alongside him?

Name of object	Crown	Sceptre	Chalice	Horn	Sword	Golden bowl	Pitcher	Shield
Volume (gallons)	2	2	2	3	3	4	5	8
Value (gold coins)	190	200	220	250	280	380	460	700

(Proposed by Sára Szepessy)

(Solutions)

14. As the three-masted sailing ship named Victoria arrived at the port of Seville, a parrot perched on the top of each of its masts. Marco, Pedro, and Rokko sat respectively on the foremast, the mainmast 5 meters behind it, and the mizzenmast 4 meters behind that. They said the following:

Pedro: "Both of my mates are exactly the same distance from me."

Rokko: "If Pedro's mast was 20 centimeters shorter, then I would be exactly $\frac{3}{5}$ as high as he is."

How tall is the mast on which Marco is sitting, if the mainmast is 12.2 meters high, making it the tallest of the three masts?

(Proposed by Anita Páhán)

(Solutions)

15. Zsuzsi wants to make a necklace for Ludmilla using 4 orange and 7 yellow beads. How many different necklaces can Zsuzsi make for Ludmilla using all the beads?

The beads of the necklace are strung on a cord, the ends of which are knotted together. Two necklaces are not considered different if they can be laid out side by side on a table such that the beads are arranged in exactly the same way on both necklaces. The position of the knot on the cord does not matter.

(Proposed by Dorka Mezey)

(Solutions)

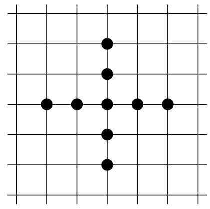
1.2 Final round – day 1

1.2.1 Category A

1. There are one thousand docks numbered from 1 to 1000 in the port of Rotterdam. Erik, a sailor on the ship named Mars, forgot which dock his ship is anchored in. He remembers that the dock number and the sum of its digits are both divisible by 5, but the digit 5 does not appear in the dock number. Based on this, which docks could the Mars be in? List all possible dock numbers.

(Solution)

2. Hanga marked 9 points in her grid notebook. Tomi wants to draw a polygon with a perimeter of 16 cm, where the sides follow the grid lines and the marked points lie on its perimeter. Show as many different polygons as possible that meet these conditions.
You don't need to justify why there are no more solutions. If two polygons can be transformed into each other by rotation or reflection, they are not considered different. In the diagram, the side length of a small square is 1 cm.



(Proposed by Gábor Szűcs)

(Solution)

3. Eight children participated in a mathematics team competition in four pairs. The height order of the eight children in descending order is: Anita Benedek, Csenge, Dani, Eszter, Ferenc, Gábor, and Hanga. At the end of the competition, the competitors said the following:

- Anita: We preceded Dani's team.
- Benedek: Csenge was my partner.
- Csenge: Ferenc's team finished exactly two places ahead of us.
- Dani: We preceded every team that had someone taller than me.
- Eszter: One member of the pair that finished directly behind us is taller than me, and the other member is shorter than me.
- Ferenc: We didn't come in second.
- Gábor: We preceded Hanga's team.
- Hanga: Anita and Gábor were not partners.

We know that exactly one child lied. Who was paired with whom, and which pair finished in which place?

Find the unique solution and justify why there are no other solutions. The children have different heights, and there were no ties in any position.

(Proposed by Áron Horváth)

(Solution)

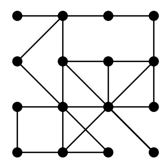
4. A mutiny broke out on a pirate ship. After abandoning the previous captain on Oxys Island, the crew elected a new captain from among themselves. Each crew member voted for two different crew members, and no one voted for themselves. A total of four people received votes: Arisztid, Bruno, Ciprian, and Délibáb. Arisztid received 1 vote, Bruno received 26 votes, and Ciprian received 3 votes. How many votes could have Délibáb received?

List the possible values and explain why these are the only possible values.

(Proposed by Erik Füredi)

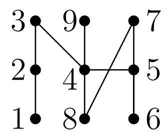
(Solution)

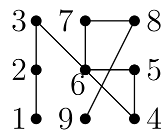
5. **a)** Sohaország consists of 16 cities arranged in a 4×4 grid. Dani, the postman wants to deliver packages to all cities in one day, visiting each city exactly once. The previous evening, he meticulously plans his route, numbering the cities from 1 to 16. The next day, he starts from city 1 and travels in numerical order through all cities, always moving in straight lines between cities and spending the night in city 16. It may happen that he visits cities he has previously been to, but Dani plans his route so that he never passes through a city he should visit later according to the numbering sequence. Is it possible for Dani's route to look like the diagram on the right?



If possible, show an appropriate numbering; if not possible, justify why it is not possible.

In the example on the left, a suitable numbering is shown, but the numbering in the example on the right is not suitable because the route from city 3 to city 4 would pass through city 6, which Dani has not visited before.





b) Dani has to work even on his birthday, and on this day he numbers and visits all the cities according to the rules described above. He celebrates by stopping at an inn for a jug of raspberry soda every time he visits a city, whether it's the first time or subsequent times. Dani wants to drink as many jugs of raspberry soda as possible in this manner. Show a valid numbering so that Dani can drink the most jugs of raspberry soda possible.

You don't need to justify why he can't drink more raspberry soda.

(Solution)

6. (Játék) The city of Óxisz is surrounded by a regular octagonal wall, where adjacent vertices are 10 km apart. The city's architect plans to build a watchtower at each of the eight vertices over four days, but bandits aim to thwart this effort. The architect constructs the towers such that if he passes by a vertex during his journey (either at the beginning or the end), where there is no tower yet, he will build one there.

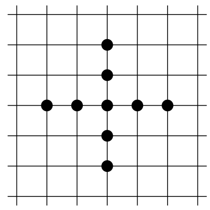
The architect starts from vertex A, moves only along the castle walls, can cover up to 40 km each day, and must spend the night at one of the vertices of the castle wall. After that, during the night, the bandits can choose a vertex and demolish the tower there. The architect wins if by sunset on the fourth day there is a tower standing at each of the eight vertices; otherwise, the bandits win.

Defeat the organizers twice in a row in this game! At the beginning of the game, you can decide whether you want to take on the role of the architect or the bandits.

(Solution)

1.2.2 Category B

1. Hanga marked 9 points in her grid notebook. Tomi wants to draw a polygon with a perimeter of 16 cm, where the sides follow the grid lines and the marked points lie on its perimeter. Show as many different polygons as possible that meet these conditions.
You don't need to justify why there are no more solutions. If two polygons can be transformed into each other by rotation or reflection, they are not considered different. In the diagram, the side length of a small square is 1 cm.



(Proposed by Gábor Szűcs)

(Solution)

2. Eight children participated in a mathematics team competition in four pairs. The height order of the eight children in descending order is: Anita Benedek, Csenge, Dani, Eszter, Ferenc, Gábor, and Hanga. At the end of the competition, the competitors said the following:

- Anita: We preceded Dani's team.
- Benedek: Csenge was my partner.
- Csenge: Ferenc's team finished exactly two places ahead of us.
- Dani: We preceded every team that had someone taller than me.
- Eszter: One member of the pair that finished directly behind us is taller than me, and the other member is shorter than me.
- Ferenc: We didn't come in second.
- Gábor: We preceded Hanga's team.
- Hanga: Anita and Gábor were not partners.

We know that exactly one child lied. Who was paired with whom, and which pair finished in which place?

Find the unique solution and justify why there are no other solutions. The children have different heights, and there were no ties in any position.

(Proposed by Áron Horváth)

(Solution)

3. A mutiny broke out on a pirate ship. After abandoning the previous captain on Oxys Island, the crew elected a new captain from among themselves. Each crew member voted for two different crew members, and no one voted for themselves. A total of four people received votes: Arisztid, Bruno, Ciprian, and Délibáb. Arisztid received 1 vote, Bruno received 26 votes, and Ciprian received 3 votes. How many votes could have Délibáb received?

List the possible values and explain why these are the only possible values.

(Proposed by Erik Füredi)

(Solution)

4. Gombóc Artúr keeps track of his chocolates. He categorizes them based on four criteria: size, shape, filling, and type. Chocolates can be small, medium, or large in size; round or square in shape; filled with orange cream or marzipan; and made of either milk or dark chocolate. One day, Picur visits Artúr, who wants to prepare chocolates in such a way that he can offer Picur any combination of two (different criteria) characteristics. (For example, Picur might ask for a medium marzipan-filled chocolate.) How many chocolates does Artúr need to prepare at least?

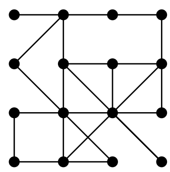
Show an example with as few chocolates as possible, and justify why fewer chocolates are not possible.

(Proposed by Kristóf Zólomy)

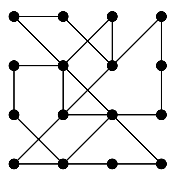
(Solution)

5. Sohaország consists of 16 cities arranged in a 4×4 grid. Dani, the postman, wants to deliver packages in one day by visiting all the cities. The evening before, he meticulously plans his route and numbers the cities from 1 to 16. The next day, he starts from city 1 and travels through all cities in numerical order, always moving in a straight line between cities, and finally spends the night in city 16. He may visit cities he has been to before, but Dani has planned his route so that he never passes through a city he should visit later according to the numbering sequence.

I.



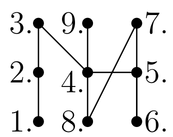
II.

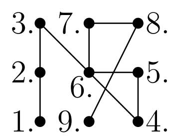


a) Decide separately for the **I.** and **II.** diagrams on the right whether Dani's route could look like that.

If possible, show a suitable numbering; if not possible, justify why it is not possible.

In the example on the left, a suitable numbering is shown, but the numbering in the example on the right is not suitable because the route from city 3 to city 4 would pass through city 6, where Dani has not visited before.





b) On Dani's birthday, he also has to work, and on this day he numbers and visits all the cities according to the rules described above. He celebrates by stopping at an inn for a jug of raspberry soda every time he visits a city, whether it's the first time or subsequent times. Dani wants to drink as many jugs of raspberry soda as possible in this manner. Show a valid numbering so that Dani can drink the most jugs of raspberry soda possible.

You don't need to justify why he can't drink more raspberry soda.

(Solution)

6. (Játék) The city of Óxisz is surrounded by a regular decagon wall, where adjacent vertices are 10 km apart. The city's architect plans to build a watchtower at each of the ten vertices over four days, but bandits aim to thwart this effort. The architect constructs the towers such that if he passes by a vertex during his journey (either at the beginning or the end), where there is no tower yet, he will build one there.

The architect starts from vertex A, moves only along the castle walls, can cover up to 50 km each day, and must spend the night at one of the vertices of the castle wall. After that, during the night, the bandits can choose a vertex and demolish the tower there. The architect wins if by sunset on the fourth day there is a tower standing at each of the ten vertices; otherwise, the bandits win.

Defeat the organizers twice in a row in this game! At the beginning of the game, you can decide whether you want to take on the role of the architect or the bandits.

(Solution)

1.3 Final round – day 2

1.3.1 Category A

1. In today's date (2024-01-13) every digit ranges from 0 to 4. How many days from now will the next date occur where every digit ranges from 0 to 5?

(Proposed by Kartal Nagy)

(Solution)

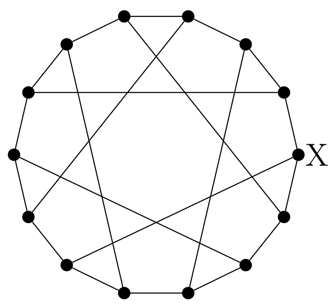
2. Bogi, Dorka, Andris, and Kristóf are going fishing. For fishing, they need to bring a bucket, a fishing rod, a bait box, and a net. Bogi can bring a bucket, a fishing rod, and a net, but not a bait box. Dorka can only bring a fishing rod. Andris can bring only a fishing rod and a net. Kristóf, however, can bring all four items. If they want everyone to bring exactly one item each, who should bring the net? **Respond with the number of letters in the given person's name.**

(Proposed by Lili Mohay)

(Solution)

3. On the map below a pirate hid his treasure in city X before the garrison began pursuing him. What is the minimum number of other cities he must visit before he can return to his treasure, if he can only travel on the roads marked on the map and can never backtrack due to his pursuers?

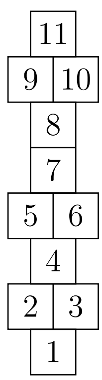
In the diagram, the points represent cities, and the segments represent roads. If the pirate sets off on a road, he can only change direction at the next city.



(Solution)

4. Jack, the one-legged pirate is playing on the hopscotch shown in the diagram. He jumps through it in a way that in every row with two squares, he jumps only on one of them, but touches every other square. Once, Jack jumped through (only in one direction, not back) so that the product of the numbers on the squares he skipped was exactly 120. What is the sum of the squares he jumped on?

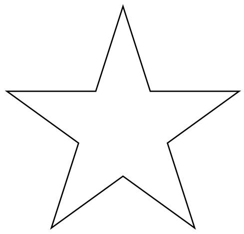
(Proposed by Kartal Nagy)



(Solution)

5. The star-shaped iceberg shown in the picture floated on the surface of the ocean, with penguins on it. Suddenly, the iceberg broke into triangular pieces, each piece containing exactly one penguin. What is the minimum number of penguins that could have been on the iceberg before it broke into pieces?
The star is regular, meaning its sides lie on pairs of straight lines.

(Proposed by Anita Páhán)



(Solution)

6. Anett is spending her 17-day winter break at Lake Balaton. She wants to go ice skating every day, but she can only go out on a given day if the average daily temperature was below -5°C on at least two of the previous three days. At least how many days during the break must have an average daily temperature below -5°C for Anett to be able to go ice skating every day?

(Proposed by Kristóf Zólomy)

(Solution)

7. A prisoner sent his mate the code for a safe, but knowing that the prison guards check all messages, he encrypted it. His encryption method is the following: he selects an integer between 0 and 9, then replaces each digit in the message with the last digit of the number that is that many greater than the original digit as his originally selected integer. For example, if he chooses 2, then he writes 013 instead of 891. The prison guards intercepted a phone call between the prisoner and his mate, during which the following (originally correct) addition was mentioned: $593 + 98 = 514$. What is the safe's code if in the intercepted message the guards read was 2763, and the prisoner used the same integer during the encryption both times?

(Proposed by Anita Páhán)

(Solution)

8. Andris, Balázs, and Kristóf went fishing on three consecutive days. They caught a total of 37 fish on the first day, 30 fish on the second day, and 26 fish on the third day. By the end of the third day, they counted that Andris caught 40 fish, while Balázs caught only 24 fish over the three days. They also noticed that Andris and Balázs together caught exactly 50 fish in the first two days. How many fish did Kristóf catch on the third day?

(Proposed by Anett Kocsis)

(Solution)

9. A few sailors bring a 24×54 meter rectangular sail into the ship's hold. Their captain instructed them to fold it first from the right, then from the bottom, then from the left, and finally from the top before placing it. However, they remembered the instructions incorrectly: they folded it correctly in the right sequence, but each time they folded only one third onto the remaining part. By how many square meters did the sail's area increase after these four folds compared to what the captain intended?

(Proposed by Bence Kovács)

(Solution)

10. Two large and five small octopuses are heading home from the coral reef. Only one of them knows the way back, so they cling together in a connected formation to avoid getting lost. Two octopuses can interlock tentacles to form this structure. Each large octopus can carry 3 pearls on each free tentacle, and each small octopus can carry 1 pearl on each free tentacle. They cannot carry pearls on the tentacles that are used to hold each other. What is the maximum number of pearls they can bring home if they decide how to interlock?
Each octopus has 8 tentacles.

(Proposed by Eszter Szabó)

(Solution)

11. In Zsombor's house, each room has exactly three doors. How many rooms are there in Zsombor's house if there are a total of 11 doors, and one of them is the entrance door?
In the house, every room is rectangular in shape.

(Proposed by Bence Kovács)

(Solution)

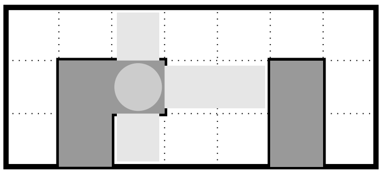
12. Csenge drew a rectangle in her grid notebook that is 30 units long and 20 units wide, with its sides aligned along the grid lines. How many rectangles are there within this drawn rectangle whose sides are 10 and 8 units long and also aligned along the grid lines?
The boundaries of the large and small rectangles may coincide.

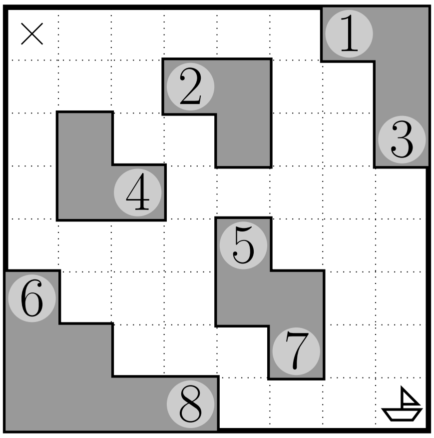
(Proposed by András Bognár)

(Solution)

13. Lili would like to sail with her ship overnight to the cell marked with ×. To make sure that she doesn't lose her way, she asked her friend Sára to turn on a few of the lighthouses. The lighthouses emit light in four directions (left, right, up, down) when turned on, all the way until they hit land (as shown on the diagram below). Sára turned on three lighthouses such that there is now a lit path connecting the ship to the cell marked with ×. Which lighthouses did Sára turn on? **The answer is the product of the numbers of the lighthouses that are turned on.**
The ship can only move to a cell that has a side common with its current cell. Both the start and end cells need to be lit too.

(Proposed by Lili Mohay and Kartal Nagy)





(Solution)

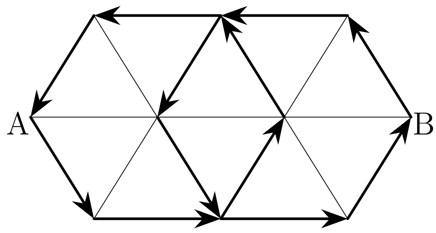
14. The captain of the Wind Queen hid 13 gold blocks in 11 chests, each of which can hold up to 2 blocks. A pirate hid on the ship, sneaking into the storage room every night. He either breaks into and steals from a new chest or opens and checks all previously opened chests. If he finds gold blocks, he steals exactly one. How many nights does the pirate need at least to ensure he steals 7 gold blocks, if the captain redistributes the remaining blocks among the 11 chests every morning?

(Proposed by Anita Páhán)

(Solution)

15. The boat wants to travel from A to B following the lines shown on the diagram, without visiting any point twice. How many different ways can it do this if, due to the current, it can only travel along the paths marked with arrows in the direction of the arrow?

(Proposed by Anita Páhán)



(Solution)

16. There are 180 colored balls in a bag. Each ball is single colored, and there are an equal number of balls of each color in the bag. What is the minimum number of balls that must be drawn from the bag to determine with certainty how many different colors of balls were originally in the bag?

(Proposed by András Imolay)

(Solution)

1.3.2 Category B

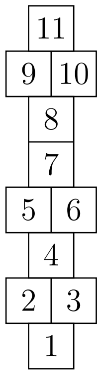
1. Bogi, Dorka, Andris, and Kristóf are going fishing. For fishing, they need to bring a bucket, a fishing rod, a bait box, and a net. Bogi can bring a bucket, a fishing rod, and a net, but not a bait box. Dorka can only bring a fishing rod. Andris can bring only a fishing rod and a net. Kristóf, however, can bring all four items. If they want everyone to bring exactly one item each, who should bring the net? **Respond with the number of letters in the given person's name.**

(Proposed by Lili Mohay)

(Solution)

2. Jack, the one-legged pirate is playing on the hopscotch shown in the diagram. He jumps through it in a way that in every row with two squares, he jumps only on one of them, but touches every other square. Once, Jack jumped through (only in one direction, not back) so that the product of the numbers on the squares he skipped was exactly 120. What is the sum of the squares he jumped on?

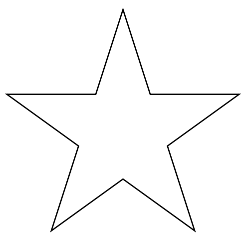
(Proposed by Kartal Nagy)



(Solution)

3. The star-shaped iceberg shown in the picture floated on the surface of the ocean, with penguins on it. Suddenly, the iceberg broke into triangular pieces, each piece containing exactly one penguin. What is the minimum number of penguins that could have been on the iceberg before it broke into pieces?
The star is regular, meaning its sides lie on pairs of straight lines.

(Proposed by Anita Páhán)



(Solution)

4. Anett is spending her 17-day winter break at Lake Balaton. She wants to go ice skating every day, but she can only go out on a given day if the average daily temperature was below -5°C on at least two of the previous three days. At least how many days during the break must have an average daily temperature below -5°C for Anett to be able to go ice skating every day?

(Proposed by Kristóf Zólomy)

(Solution)

5. A prisoner sent his mate the code for a safe, but knowing that the prison guards check all messages, he encrypted it. His encryption method is the following: he selects an integer between 0 and 9, then replaces each digit in the message with the last digit of the number that is that many greater than the original digit as his originally selected integer. For example, if he chooses 2, then he writes 013 instead of 891. The prison guards intercepted a phone call between the prisoner and his mate, during which the following (originally correct) addition was mentioned: $593 + 98 = 514$. What is the safe's code if in the intercepted message the guards read was 2763, and the prisoner used the same integer during the encryption both times?

(Proposed by Anita Páhán)

(Solution)

6. A few sailors bring a 24×54 meter rectangular sail into the ship's hold. Their captain instructed them to fold it first from the right, then from the bottom, then from the left, and finally from the top before placing it. However, they remembered the instructions incorrectly: they folded it correctly in the right sequence, but each time they folded only one third onto the remaining part. By how many square meters did the sail's area increase after these four folds compared to what the captain intended?

(Proposed by Bence Kovács)

(Solution)

7. Two large and five small octopuses are heading home from the coral reef. Only one of them knows the way back, so they cling together in a connected formation to avoid getting lost. Two octopuses can interlock tentacles to form this structure. Each large octopus can carry 3 pearls on each free tentacle, and each small octopus can carry 1 pearl on each free tentacle. They cannot carry pearls on the tentacles that are used to hold each other. What is the maximum number of pearls they can bring home if they decide how to interlock?
Each octopus has 8 tentacles.

(Proposed by Eszter Szabó)

(Solution)

8. In Zsombor's house, each room has exactly three doors. How many rooms are there in Zsombor's house if there are a total of 11 doors, and one of them is the entrance door?
In the house, every room is rectangular in shape.

(Proposed by Bence Kovács)

(Solution)

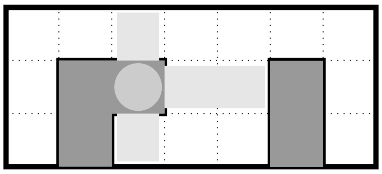
9. Csenge drew a rectangle in her grid notebook that is 30 units long and 20 units wide, with its sides aligned along the grid lines. How many rectangles are there within this drawn rectangle whose sides are 10 and 8 units long and also aligned along the grid lines?
The boundaries of the large and small rectangles may coincide.

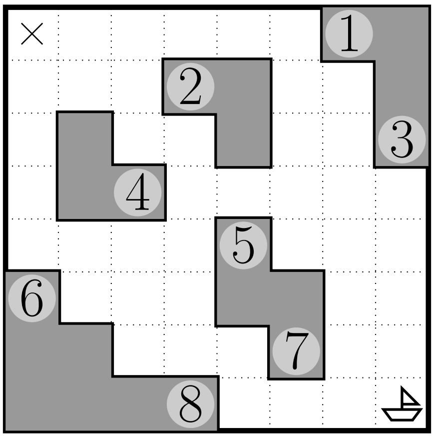
(Proposed by András Bognár)

(Solution)

10. Lili would like to sail with her ship overnight to the cell marked with $\times$. To make sure that she doesn't lose her way, she asked her friend Sára to turn on a few of the lighthouses. The lighthouses emit light in four directions (left, right, up, down) when turned on, all the way until they hit land (as shown on the diagram below). Sára turned on three lighthouses such that there is now a lit path connecting the ship to the cell marked with $\times$. Which lighthouses did Sára turn on? **The answer is the product of the numbers of the lighthouses that are turned on.**
The ship can only move to a cell that has a side common with its current cell. Both the start and end cells need to be lit too.

(Proposed by Lili Mohay and Kartal Nagy)





(Solution)

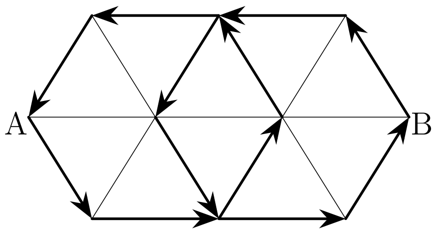
11. The captain of the Wind Queen hid 13 gold blocks in 11 chests, each of which can hold up to 2 blocks. A pirate hid on the ship, sneaking into the storage room every night. He either breaks into and steals from a new chest or opens and checks all previously opened chests. If he finds gold blocks, he steals exactly one. How many nights does the pirate need at least to ensure he steals 7 gold blocks, if the captain redistributes the remaining blocks among the 11 chests every morning?

(Proposed by Anita Páhán)

(Solution)

12. The boat wants to travel from A to B following the lines shown on the diagram, without visiting any point twice. How many different ways can it do this if, due to the current, it can only travel along the paths marked with arrows in the direction of the arrow?

(Proposed by Anita Páhán)



(Solution)

13. A flea wants to travel from the bottom-left corner to the top-right corner of a 5×5 grid, moving one square at a time either up, down, left, or right. It wants to achieve this without making three consecutive jumps in the same direction. What is the minimum number of steps needed to reach the destination?

(Proposed by András Imolay)

(Solution)

14. Which occurrence in the current millennium is today (13/01/2024) such that in the numerical representation of the day, month, or year, at least one of them contains the digit '3'?

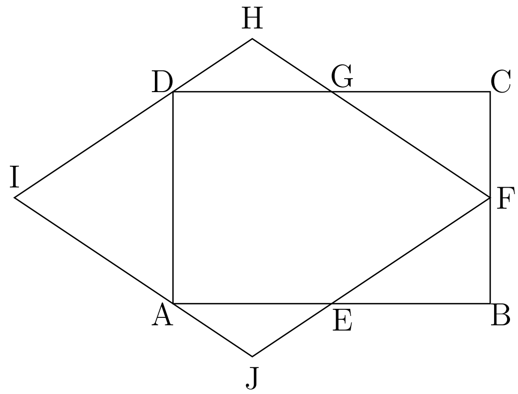
The first day of the current millennium was 1/1/2001.

(Proposed by Erik Füredi)

(Solution)

15. In rectangle $ABCD$ the midpoint of side AB is point E, the midpoint of side BC is F and the midpoint of side CD is point G. The quadrangle $FHIJ$ is a rhombus and its side HI passes through D and its side IJ through A. How many millimeters is the length of segment IH if the length of segment AB is 112 millimeters and the length of segment EF is 72 millimeters?

(Proposed by Anita Páhán)



(Solution)

16. There are 180 colored balls in a bag. Each ball is single colored, and there are an equal number of balls of each color in the bag. What is the minimum number of balls that must be drawn from the bag to determine with certainty how many different colors of balls were originally in the bag?

(Proposed by András Imolay)

(Solution)

2 Problems – high school categories

2.1 Online round

2.1.1 Category C

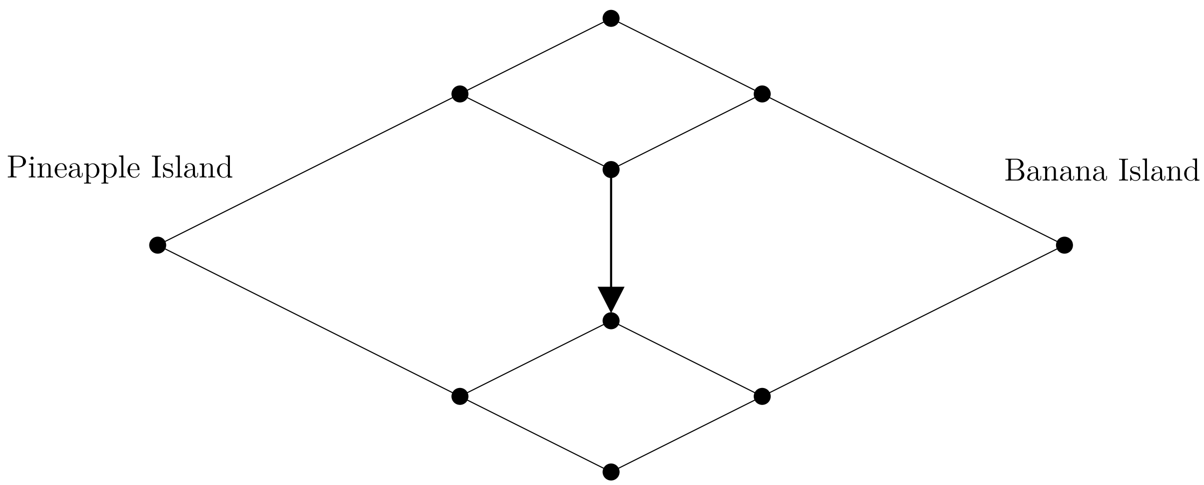
1. There is a lighthouse on each of the two each sides of a bay. One sends signals every 18 seconds and the other every 28 seconds. This second, both of them sent signals. In how many seconds will they send signals at the same time again?

(Proposed by Dániel Hegedűs)

(Solution)

2. A pirate crew wants to travel from Pineapple Island to Banana Island. Along the way, they decide to visit a few other islands. In how many different ways can they complete their journey if they can only travel along the paths marked in the diagram? On the path marked with an arrow, the current is so strong that they can only travel in the indicated direction, and they can visit each island at most once.

(Proposed by Tamás Gábriel)



(Solution)

3. The pirates left a chest filled with treasure on a deserted island, protected by a combination lock. They revealed that the code is the smallest positive integer that gives a remainder of 2 when divided by 3, a remainder of 1 when divided by 5, and a remainder of 7 when divided by 8. What is the code that can open the chest?

(Proposed by Ágoston Török)

(Solution)

4. The crew of a ship consists of pirates and monkeys. Among the pirates, there are some with one leg and some with two legs, while each of the monkeys has four legs. Originally, the average of the number of legs was 1,75. On a beautiful summer day, the crew welcomed a new (also four-legged) baby monkey. At this point, the average number of legs increased to 2. How many members were originally in the crew?

(Proposed by Lili Mohay and Sára Szepessy)

(Solution)

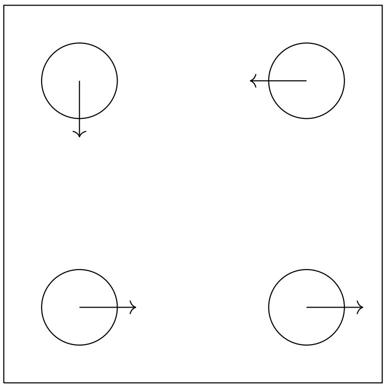
5. The sail factory orders pieces of cloth, sized $5\,\mathrm{m} \times 6\,\mathrm{m}$. Then they cut all the pieces in a way that each piece is cut into congruent rectangular sails with each side being an integer number of metres long. After that, they displayed exactly one of each different sail that could be made in this way. How many square metres is the total area of the sails on display? *If the sail remains intact, that also counts as a cutting. Two sails are different if they are not congruent. For instance, a* $1\,\mathrm{m} \times 2\,\mathrm{m}$ *sail is displayed, because a* $5\,\mathrm{m} \times 6\,\mathrm{m}$ *sail can be cut into rectangles sized* $1\,\mathrm{m} \times 2\,\mathrm{m}$.

(Solution)

6. In a square-shaped room, there is a statue in each corner, which initially faces in a direction parallel to one of the walls (up, down, right, or left, as shown in the picture). The statues' positions can be controlled by two buttons: one button rotates the two statues at the top simultaneously, in clockwise direction, by 90°, and the other button rotates the two statues on the right side simultaneously, in clockwise direction, by 90°. How many possible initial positions are there for the four statues from which it is possible to make all statues face an another statue that is facing them?

Two initial positions are considered different if there is a statue that faces different directions in the two positions.

(Proposed by Bence Kovács and Áron Horváth)



(Solution)

7. On the map seen below three letters A, B and C mark the islands where there's no wind, but on every other island, the wind blows in one of the four directions (in the picture it's either up, down, left or right). If a boat is in one of the squares, then it will drift in the direction of the wind, to an adjacent square. We know, that the boats starting from square X will drift to Island B, and the boats starting from square Y will drift to Island C, without leaving the map. If a boat, at some point, rounds up at an island, it won't be able to move away, because of the lack of wind. At most, how many squares can there be from which the wind will take the boat to Island A without leaving the map (including Island A itself)?

(Proposed by Anita Páhán)

					B
					A
Y					
		X			C

(Solution)

8. Timi received 27 identical white cubes for her birthday, which she immediately used to build a larger $3 \times 3 \times 3$ cube. Her brother, András, chose 3 faces of the large cube, which meet at one vertex and painted them red. After the paint dried, he disassembled the cube and reassembled the 27 small cubes into another $3 \times 3 \times 3$ cube. Their little sister, Nóri, then painted red three faces of the new large cube, which also meet at one vertex. What is the maximum number of small cubes that have at least 3 red faces after this process?

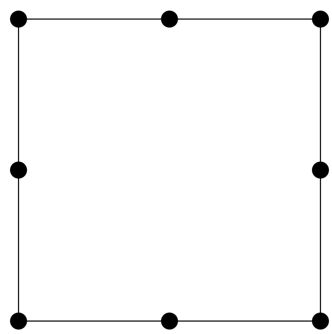
(Proposed by Alex Kempf)

(Solution)

9. How many such triangles are there that their vertices are among the vertices and midpoints of the sides of the square below, and there is at least one angle in the triangle that measures at least 90°?

Two triangles are considered different if they differ in at least one vertex.

(Proposed by Gyuri Győrffi)



(Solution)

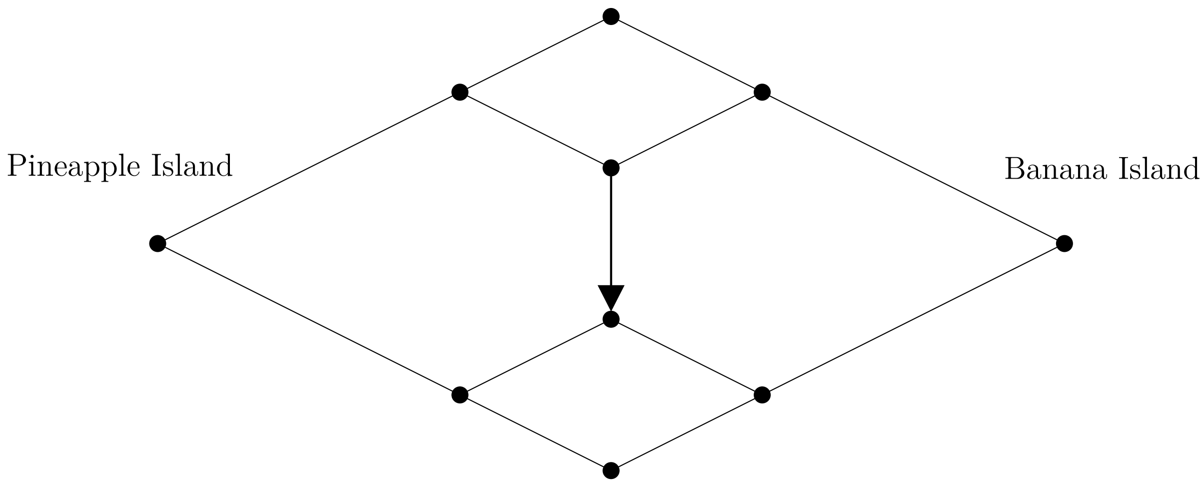
10. Game: Initially, there is a pile of 17 chips on the table. Two players take alternating turns. The current player can either take one chip from the pile or, if there is an even number of chips on the table, they can take half of the chips. The player who makes the move after which no chips are left on the table wins. *You can decide whether you want to play as the first or the second player.*

(Solution)

2.1.2 Category D

1. A pirate crew wants to travel from Pineapple Island to Banana Island. Along the way, they decide to visit a few other islands. In how many different ways can they complete their journey if they can only travel along the paths marked in the diagram? On the path marked with an arrow, the current is so strong that they can only travel in the indicated direction, and they can visit each island at most once.

(Proposed by Tamás Gábriel)



(Solution)

2. The pirates left a chest filled with treasure on a deserted island, protected by a combination lock. They revealed that the code is the smallest positive integer that gives a remainder of 2 when divided by 3, a remainder of 1 when divided by 5, and a remainder of 7 when divided by 8. What is the code that can open the chest?

(Proposed by Ágoston Török)

(Solution)

3. The crew of a ship consists of pirates and monkeys. Among the pirates, there are some with one leg and some with two legs, while each of the monkeys has four legs. Originally, the average of the number of legs was 1,75. On a beautiful summer day, the crew welcomed a new (also four-legged) baby monkey. At this point, the average number of legs increased to 2. How many members were originally in the crew?

(Proposed by Lili Mohay and Sára Szepessy)

(Solution)

4. Four sailors are stranded on a desert island. They need help, so they set up two stations, one on the north shore and one on the south shore. They can send smoke signals at each station, one sailor at a time. Yesterday, each sailor started sending smoke signals at a whole hour and continued to send it for two hours. Each of them sent signals once, and there wasn't any smoke before 8 in the morning and after 3 in the afternoon. In how many different ways could they have sent the signals? *Different means that there exists a sailor who sent a signal at a different time or place.*

(Solution)

5. Unlucky Ubul and Unfortunate Ulrik sail out to the Vast Sea to search for treasure. The Vast Sea is square-shaped and divided into a grid of 4×4 sectors. The two friends have a net, which when thrown into the water in any sector of the sea, can retrieve treasures from that sector and the sectors that share a side with it. How many sectors at minimum, contain treasures if they are guaranteed to find treasure on their first attempt, no matter where they throw their net?

(Proposed by Márton Móra and Áron Horváth)

(Solution)

6. On the map seen below three letters A, B and C mark the islands where there's no wind, but on every other island, the wind blows in one of the four directions (in the picture it's either up, down, left or right). If a boat is in one of the squares, then it will drift in the direction of the wind, to an adjacent square. We know, that the boats starting from square X will drift to Island B, and the boats starting from square Y will drift to Island C, without leaving the map. If a boat, at some point, rounds up at an island, it won't be able to move away, because of the lack of wind. At most, how many squares can there be from which the wind will take the boat to Island A without leaving the map (including Island A itself)?

(Proposed by Anita Páhán)

					B
					A
Y					
		X			C

(Solution)

7. Timi received 27 identical white cubes for her birthday, which she immediately used to build a larger $3 \times 3 \times 3$ cube. Her brother, András, chose 3 faces of the large cube, which meet at one vertex and painted them red. After the paint dried, he disassembled the cube and reassembled the 27 small cubes into another $3 \times 3 \times 3$ cube. Their little sister, Nóri, then painted red three faces of the new large cube, which also meet at one vertex. What is the maximum number of small cubes that have at least 3 red faces after this process?

(Proposed by Alex Kempf)

(Solution)

8. Two pirates, Zorka and Kristof want to make a deal. Zorka has one one-dollar bill and two three-dollar bills, and Kristof has three five-dollar bills. According to the Ancient Pirate Law, they can only perform 3 different operations:

- They exchange a three-dollar bill and a five-dollar bill;
- They exchange a one-dollar bill and a three-dollar bill;
- One of them gives the other a one-dollar bill.

Zorka writes down the amount of money she has after each exchange. At most how many different numbers can appear on Zorka's paper?

(Proposed by Anett Kocsis)

(Solution)

9. Let P be a point indside the square $ABCD$ so that $\sphericalangle CDP = 19°$ and $\sphericalangle PAB = 52°$. What is the measure of $\sphericalangle PBC$?

(Proposed by Ágoston Török)

(Solution)

10. **Game:** Initially, there is a pile of chips on the table, with the number of chips being at least 20 and at most 25. Two players take turns alternatingly. The current player can either take one chip from the pile or, if there is an even number of chips on the table, they can take half of the chips. The player who makes the move after which no chips are left on the table, wins. *Knowing the starting setup, you can decide whether you want to play as the first or the second player.*

(Solution)

2.1.3 Category E

1. The crew of a ship consists of pirates and monkeys. Among the pirates, there are some with one leg and some with two legs, while each of the monkeys has four legs. Originally, the average of the number of legs was 1,75. On a beautiful summer day, the crew welcomed a new (also four-legged) baby monkey. At this point, the average number of legs increased to 2. How many members were originally in the crew?

(Proposed by Lili Mohay and Sára Szepessy)

(Solution)

2. Four sailors are stranded on a desert island. They need help, so they set up two stations, one on the north shore and one on the south shore. They can send smoke signals at each station, one sailor at a time. Yesterday, each sailor started sending smoke signals at a whole hour and continued to send it for two hours. Each of them sent signals once, and there wasn't any smoke before 8 in the morning and after 3 in the afternoon. In how many different ways could they have sent the signals? *Different means that there exists a sailor who sent a signal at a different time or place.*

(Solution)

3. Unlucky Ubul and Unfortunate Ulrik sail out to the Vast Sea to search for treasure. The Vast Sea is square-shaped and divided into a grid of 4×4 sectors. The two friends have a net, which when thrown into the water in any sector of the sea, can retrieve treasures from that sector and the sectors that share a side with it. How many sectors at minimum, contain treasures if they are guaranteed to find treasure on their first attempt, no matter where they throw their net?

(Proposed by Márton Móra and Áron Horváth)

(Solution)

4. Captain Alex the pirate is preparing for another one of his raid journey and is getting his treasure chests ready. He has a total of six empty chests of different sizes and wants to pack them inside each other to save space. Each chest has two compartments, one golden and one silver. A chest can fit into any compartment of a larger chest. After packing, each compartment of every chest will contain at most one chest, but that chest may contain additional chests inside it. How many ways can Captain Alex pack the chests if, in the end, all chests are packed inside the largest one? *We consider two ways of packing different if there is a chest that ends up in a different chest or in the other compartment of the same chest.*

(Proposed by Áron Horváth)

(Solution)

5. On the map seen below three letters A, B and C mark the islands where there's no wind, but on every other island, the wind blows in one of the four directions (in the picture it's either up, down, left or right). If a boat is in one of the squares, then it will drift in the direction of the wind, to an adjacent square. We know, that the boats starting from square X will drift to Island B, and the boats starting from square Y will drift to Island C, without leaving the map. If a boat, at some point, rounds up at an island, it won't be able to move away, because of the lack of wind. At most, how many squares can there be from which the wind will take the boat to Island A without leaving the map (including Island A itself)?

(Proposed by Anita Páhán)

					B
					A
Y					
		X			C

(Solution)

6. Timi received 27 identical white cubes for her birthday, which she immediately used to build a larger $3 \times 3 \times 3$ cube. Her brother, András, chose 3 faces of the large cube, which meet at one vertex and painted them red. After the paint dried, he disassembled the cube and reassembled the 27 small cubes into another $3 \times 3 \times 3$ cube. Their little sister, Nóri, then painted red three faces of the new large cube, which also meet at one vertex. What is the maximum number of small cubes that have at least 3 red faces after this process?

(Proposed by Alex Kempf)

(Solution)

7. Two pirates, Zorka and Kristof want to make a deal. Zorka has one one-dollar bill and two three-dollar bills, and Kristof has three five-dollar bills. According to the Ancient Pirate Law, they can only perform 3 different operations:

- They exchange a three-dollar bill and a five-dollar bill;
- They exchange a one-dollar bill and a three-dollar bill;
- One of them gives the other a one-dollar bill.

Zorka writes down the amount of money she has after each exchange. At most how many different numbers can appear on Zorka's paper?

(Proposed by Anett Kocsis)

(Solution)

8. Let P be a point indside the square $ABCD$ so that $\sphericalangle CDP = 19°$ and $\sphericalangle PAB = 52°$. What is the measure of $\sphericalangle PBC$?

(Proposed by Ágoston Török)

(Solution)

9. The pirates left a chest filled with treasure on a deserted island, protected by a four-digit combination lock. They revealed that the code does not contain the digit 0, and if you sum up all the four-digit numbers that can be formed by rearranging the digits of the code, you get exactly 11 times the code itself. What is the code that opens the chest?

For example, for the number 1161*, the numbers that can be formed by rearranging its digits are* $1116, 1161, 1611, 6111$.

(Proposed by Csongor Beke)

(Solution)

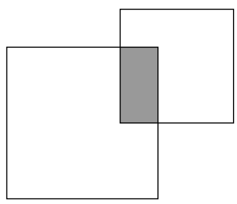
10. **Game:** Initially, there is a pile of chips on the table. The two players take turns alternatingly. The current player can either take one chip from the pile or, if there is an even number of chips on the table, they can take half of the chips. The player who makes the move after which no chips are left on the table, wins. *Knowing the starting setup, you can decide whether you want to play as the first or the second player.*

(Solution)

2.2 Regional round

2.2.1 Category C

1. Matthew drew two squares, and their intersection is a polygon.
a) How many sides can this polygon have? Draw examples for as many polygons with different numbers of sides as possible.
b) How many right angles can the common part have? Draw examples for as many different numbers of right angles as possible.
For example in the figure on the right, the common part of the two squares is the gray rectangle, which has 4 sides and 4 right angles.
The squares can be of different sizes and you can rotate them. You might have more places available than there are different cases.



(Proposed by Máté Jánosik)

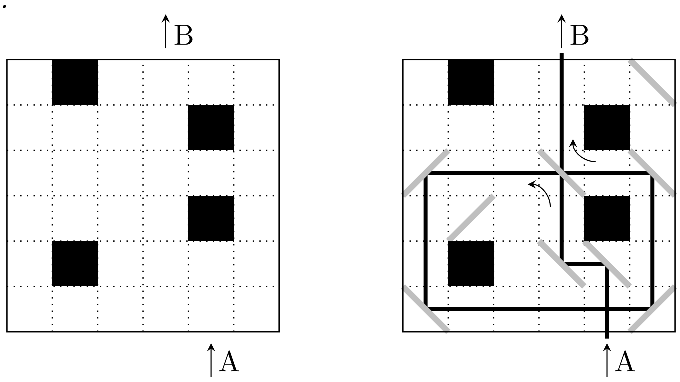
(Solution)

2. Mesi thought that she was eating too much Turo Rudi (Hungarian dessert), so she tried to go on a diet from Monday to Sunday. She called a day of her diet successful (starting on Tuesday) if she ate less Turo Rudi that day than the day before. We know that there was only one day that her diet was not successful. List all the possibilities of how many Turo Rudis she could've eaten on the days of the week, if Mesi ate either $0, 1, 2$ or 3 Turo Rudis each day.
For example, if Mesi ate $3, 2, 0, 3, 2, 1, 0$ *Turo Rudis on the days of the week, her diet was only unsuccessful on Thursday.*
Write your answers in each column on the Answer Sheet. You might have more columns than there are solutions.

(Proposed by Áron Horváth)

(Solution)

3. In the figure below you can see the floor plan of the mirror room of the pirate museum. There are 4 squares marked with black squares in the room, which notes the place of four pillars, while the remaining part is divided into 32 squares, on which mirrors can be placed diagonally. Both sides of the mirrors refract the path of the light that reaches it, at right angles. The mirrors in the room are currently arranged in such a way that if you shine a lamp along arrow A, the light leaves the room along arrow B. Show an example in which the light bounces off as many mirrors as possible. On the Answer Sheet, draw the mirrors and the path of light as in the example below.
The light beam can cross itself. The light can bounce off both sides of a mirror, but even then, we only count this mirror once. In the example shown in the second figure, the path of light is refracted on 7 mirrors.



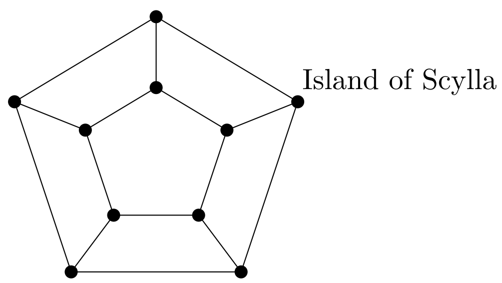
(Solution)

4. Find three different positive single-digit numbers in such a way that using the four basic operations and parentheses the most number of the numbers $1, 2, \ldots, 10$ can be produced from them. You can use all three digits exactly once in each equation. On the Answer Sheet, write down the three selected numbers and the equations producing as many numbers as possible from them. *For example, if you choose the digits* $2, 3, 7$, *then* $1 = -(2 \cdot 3) + 7$ *and* $5 = (7 + 3)/2$ *are allowed constructions of 1 and 5. Operations may only be performed with single-digit numbers, so for example* $9 = 27/3$ *is not allowed.*

(Solution)

5. The map below shows the ten islands of the Dürerenciás Sea, one of which hides a treasure. Two islands are connected in the diagram if and only if there is a direct ship route between them. All waterways have ships in both directions. Leila has a friend on each island, and she wants to get the treasure with their help. Before Leila leaves for the islands, she wants to make sure she finds the treasure, so she talks to some friends on the phone. If Leila calls a friend, they will only tell her the minimum number of boat trips from their island to the treasure island. At least how many friends must she call in order to be sure which island hides the treasure, if we know that she will first call Lili, who lives on the Island of Scylla?

(Proposed by Áron Horváth and Bence Kovács)

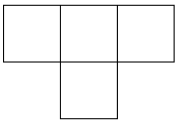


(Solution)

6. **a)** We wish to cover a board consisting of 8×8 squares using 16 pieces of the *T-tetrominos* shown in the figure below. Is this possible if the tetrominos cannot overlap and cannot hang off the board? If it is possible, provide such a covering, if not, explain why it is not possible.

b) We remove the 4 corners of the 8×8 board. Can the remaining board be covered in a similar manner with 15 tetrominos? If yes, provide such a covering, if not, explain why it is not possible. *The tetrominos can be rotated, but each of the squares of the tetrominos must fit onto the squares of the board.*

(Proposed by Gergő Halasi)



(Solution)

7. Let $ABCD$ be a parallelogram, and let E be the midpoint of side CD. Denote the intersection of segments AE and BD by F. Suppose that the angle AEB is a right angle and $EB = ED$. Calculate the angle AFB.

(Proposed by József Osztényi)

(Solution)

2.2.2 Category D

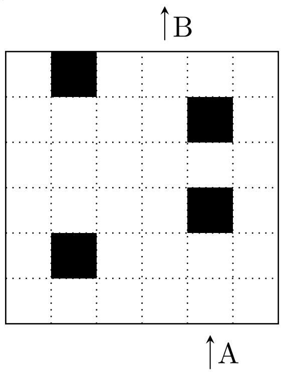
1. We call a filling of a 4×4 table *magical* if

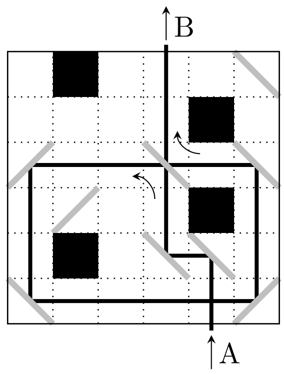
- each of its cells contains one of the digits $0, 1, \ldots, 9$,
- every digit is contained in at least one of the cells of the board,
- the sum of the digits is the same in every row and column.

Provide a magical filling where the sum of the digits written on the table is the smallest possible.

(Solution)

2. In the figure below you can see the floor plan of the mirror room of the pirate museum. There are 4 squares marked with black squares in the room, which notes the place of four pillars, while the remaining part is divided into 32 squares, on which mirrors can be placed diagonally. Both sides of the mirrors refract the path of the light that reaches it, at right angles. The mirrors in the room are currently arranged in such a way that if you shine a lamp along arrow A, the light leaves the room along arrow B. Show an example in which the light bounces off as many mirrors as possible. On the Answer Sheet, draw the mirrors and the path of light as in the example below.
The light beam can cross itself. The light can bounce off both sides of a mirror, but even then, we only count this mirror once. In the example shown in the second figure, the path of light is refracted on 7 mirrors.





(Solution)

3. Find three different positive single-digit numbers in such a way that using the four basic operations and parentheses the most number of the numbers $1, 2, \ldots, 10$ can be produced from them. You can use all three digits exactly once in each equation. On the Answer Sheet, write down the three selected numbers and the equations producing as many numbers as possible from them. *For example, if you choose the digits* $2, 3, 7$, *then* $1 = -(2 \cdot 3) + 7$ *and* $5 = (7 + 3)/2$ *are allowed constructions of 1 and 5. Operations may only be performed with single-digit numbers, so for example* $9 = 27/3$ *is not allowed.*

(Solution)

4. When Jóska entered the classroom he saw several positive integers on the board which were not necessarily all distinct and among which there were at least two different numbers whose greatest common divisor was not 1. Then, he wrote under each number how many numbers were originally on the board with which its greatest common divisor was 1. He noticed that he wrote under each number itself. Show an example with as few numbers as possible that Jóska could have seen on the board upon his arrival.

Write each number as many times as it originally appeared on the board.

(Proposed by Kartal Nagy)

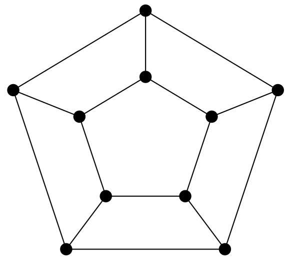
(Solution)

5. Let $ABCD$ be a parallelogram, and let E be the midpoint of side CD. Denote the intersection of segments AE and BD by F. Suppose that the angle AEB is a right angle and $EB = ED$. Calculate the angle AFB.

(Proposed by József Osztényi)

(Solution)

6. a) This is the map of the ten islands of the Düreric Ocean. There is a treasure hidden on one of the islands. Two islands are connected on the map if there is a direct ship connection between them. Leila, who has a friend on each of the islands, wants to find the treasure. Before she visits the archipelago, she wants to make sure she knows where to go, therefore she calls some of her friends on the phone. Any friend she calls can only tell her how many direct ship trips are needed to reach the treasure island from his/her own island. How many people does Leila need to call in order to be able to tell the location of the treasure with certainty, if first she calls Lily who lives on Scylla's Island?



b) This time Leila is visiting another archipelago made up of five islands and similarly to the previous part, one island holds a treasure. Leila managed to find out which islands are connected by direct ship connections. After a bit of thinking she discovered that she could definitely determine the location of the treasure by calling not more than two of her friends living on different islands. Based on this, what is the maximum number of direct ship connections between the islands?

The conditions are the same as in the first part: on each island, she has a friend whom she can call, and the friend will tell her how many direct ship trips are needed to reach the treasure island from his/her own island. Between two islands, there is at most one direct ship route, and ships travel in both directions. We also know that every island is reachable from every island via ship trips.

(Proposed by Áron Horváth and Bence Kovács)

(Solution)

7. Let $a_1, a_2, \ldots, a_{2023}$ be real numbers such that

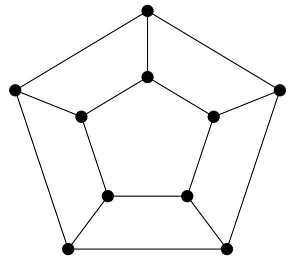
- $a_{2023} = a_1$,
- and for every $n \geq 3$ we have $a_n = \frac{a_{n-1}+a_{n-2}}{2} - 1$, so from the third number onwards, each number is one less than the average of the two preceding numbers.

Prove that $a_n \geq a_1$ holds for all $1 \leq n \leq 2023$.

(Proposed by Benedek Váli)

(Solution)

2.2.3 Category E



1. a) This is the map of the ten islands of the Düreric Ocean. There is a treasure hidden on one of the islands. Two islands are connected on the map if there is a direct ship connection between them. Leila, who has a friend on each of the islands, wants to find the treasure. Before she visits the archipelago, she wants to make sure she knows where to go, therefore she calls some of her friends on the phone. Any friend she calls can only tell her how many direct ship trips are needed to reach the treasure island from his/her own island. How many people does Leila need to call in order to be able to tell the location of the treasure with certainty, if first she calls Lily who lives on Scylla's Island?
b) This time Leila is visiting another archipelago made up of five islands and similarly to the previous part, one island holds a treasure. Leila managed to find out which islands are connected by direct ship connections. After a bit of thinking she discovered that she could definitely determine the location of the treasure by calling not more than two of her friends living on different islands. Based on this, what is the maximum number of direct ship connections between the islands?

The conditions are the same as in the first part: on each island, she has a friend whom she can call, and the friend will tell her how many direct ship trips are needed to reach the treasure island from his/her own island. Between two islands, there is at most one direct ship route, and ships travel in both directions. We also know that every island is reachable from every island via ship trips.

(Proposed by Áron Horváth and Bence Kovács)

(Solution)

2. Let $ABCD$ be a parallelogram, and let E be the midpoint of side CD. Denote the intersection of segments AE and BD by F. Suppose that the angle AEB is a right angle and $EB = ED$. Calculate the angle AFB.

(Proposed by József Osztényi)

(Solution)

3. There are 100 people seated around a round table: 50 knights who always tell the truth and 50 knaves who always lie. Mark enters the room, chooses someone sitting at the table, and starting from that person, moving clockwise, asks each person the question: "Among the answers given so far, was the number of 'yes' answers even?" Can the people be seated in such a way that no matter who Mark asks first, he always gets the same number of 'yes' answers?

(Proposed by Benjámin Móricz)

(Solution)

4. Let $a_1, a_2, \ldots, a_{2023}$ be real numbers such that

- $a_{2023} = a_1$,
- and for every $n \geq 3$ we have $a_n = \frac{a_{n-1}+a_{n-2}}{2} - 1$, so from the third number onwards, each number is one less than the average of the two preceding numbers.

Prove that $a_n \geq a_1$ holds for all $1 \leq n \leq 2023$.

(Proposed by Benedek Váli)

(Solution)

5. A round table is surrounded by $n \geq 2$ people, each assigned one of the integers $0, 1, \ldots, n-1$ such that no two people received the same number. In each round, everyone adds their number to their right neighbour's number, and their new number becomes the remainder of the sum when divided by n. We call an initial configuration of the integers *glorious* if everyone's number remains the same after some finite number of rounds, never changing again.
a) For which integers $n \geq 2$ is every initial configuration glorious?
b) For which integers $n \geq 2$ is there no glorious initial configuration at all?

(Proposed by Benjámin Móricz)

(Solution)

2.2.4 Category E$^+$

1. There are 100 merchants selling salmon for Dürer dollars around the circular shore of the island of Dürerland. Since the beginning of times good and bad years have been alternating on the island. (So after a good year, the next year is bad; and after a bad year, the next year is good.) In every good year all merchants set their price as the maximum value between their own selling price from the year before and the selling price of their left-hand neighbour from the year before. In turn, in every bad year they sell it for the minimum between their own price from the year before and their left-hand neighbour's price from the year before. Paul and Pauline are two merchants on the island. This year Paul is selling salmon for 17 Dürer dollars a kilogram. Prove that there will come a year when Pauline will sell salmon for 17 Dürer dollars a kilogram.
The merchants are immortal, they have been selling salmon on the island for thousands of years and will continue to do so until the end of time.

(Proposed by András Imolay)

(Solution)

2. One quadrant of the Cartesian coordinate system is tiled by dominoes of size 1 cm $\times$ 2 cm. The dominoes don't overlap with each other, they cover the entire quadrant and they all fit in the quadrant. Farringdon, the flea is sitting at the origin in the beginning and is allowed to jump from one corner of a domino to the opposite corner any number of times. Is it possible that the dominoes are arranged in a way that Farringdon is unable to move more than 2023 cm away from the origin?
A quadrant *is one quarter of the plane with its boundaries being two perpendicular rays from the origin. An example of a quadrant is* $\{(x, y) : x, y \geq 0\}$.

(Proposed by Kartal Nagy and Gábor Szűcs)

(Solution)

3. A round table is surrounded by $n \geq 2$ people, each assigned one of the integers $0, 1, \ldots, n-1$ such that no two people received the same number. In each round, everyone adds their number to their right neighbour's number, and their new number becomes the remainder of the sum when divided by n. We call an initial configuration of the integers *glorious* if everyone's number remains the same after some finite number of rounds, never changing again.
a) For which integers $n \geq 2$ is every initial configuration glorious?
b) For which integers $n \geq 2$ is there no glorious initial configuration at all?

(Proposed by Benjámin Móricz)

(Solution)

4. In the game of *Calculabyrinth* two players control an adventurer in an underwater dungeon. The adventurer starts with h hit points, where h is an integer greater than one. The dungeon consists of several chambers. There are some passageways in the dungeon, each leading from a chamber to a chamber. These passageways are one-way, and a passageway may return to its starting chamber. Every chamber can be exited through at least one passageway. There are 5 types of chambers:

- Entrance: the adventurer starts here, no passageway comes in here;
- Hollow: nothing happens;
- Spike: the adventurer loses a hit point;
- Trap: the adventurer gets shot by an arrow;
- Catacomb: the adventurer loses hit points equal to the total number of times they have been hit by an arrow.

The two players take turns controlling the character, always moving them through one passageway. A player loses if the adventurer's hit points fall below zero due to their action (at 0 hit points, the character stays alive). Show an example of a dungeon map, which consists of at most 20 chambers and contains exactly one Entrance, with the following condition: the first player has a winning strategy if h is a prime, and the second player has a winning strategy if h is composite.
If the game doesn't end after a finite number of moves, neither player wins.

(Proposed by Márton Németh)

(Solution)

5. For a given triangle $A_1A_2A_3$ and a point X inside of it we denote by X_i the intersection of line A_iX with the side opposite to A_i for all $1 \leq i \leq 3$.
Let P and Q be distinct points inside the triangle. We then say that the two points are *isotomic* (or we say they form an *isotomic pair*) if for all i the points P_i and Q_i are symmetric with respect to the midpoint of the side opposite to A_i.

Augustus wants to construct isotomic pairs with his favourite app, GeoZebra. In fact, he already constructed the vertices and sidelines of a non-isosceles acute triangle when suddenly his computer got infected with a virus. Most tools became unavailable, only a few are usable, some of which even require a fee:

Name of tool	**Description**	**Fee (per use)**
Point	select an arbitrary point (with respect to the position of the mouse) on the plane or on a figure (circle or line)	free
Intersection	intersection points of two figures (where each figure is a circle or a line)	free
Line	line through two points	5 Dürer dollars
Perpendicular	perpendicular from a point to an already constructed line	50 Dürer dollars
Circumcircle	circle through three points	10 Dürer dollars

a) Agatha selected a point P inside the triangle, which is not the centroid of the triangle. Show that Augustus can construct a point Q at a cost of at most 1000 Dürer dollars such that P and Q are isotomic.
b) Prove that for all positive integers n Augustus can construct n different isotomic pairs at a cost of at most $200 + 10n$ Dürer dollars.
*In both parts, partial points may be awarded for constructions exceeding Augustus's budget. The parts are unrelated, that is Augustus can't use his constructions from part **a)** in part **b)** .*

(Proposed by Áron Bán-Szabó)

(Solution)

2.3 Final round – day 1

2.3.1 Category C

1. The crew of the pirate ship drifting on the Düreric Sea consisted of three girls and three boys. They had been adrift on the sea for a long time when someone from the crew, seeking to add some excitement to their monotonous days, went down alone to the ship's hold and stole a bottle of rum. The crew wants to find out who the thief was with the help of their parrot. The parrot spoke four times, telling the truth twice and lying twice. Who was the thief if the parrot's statements were as follows?

- The boys were not on the ship on the day of the theft because of a raid.
- Jasmine and John always steal together. Jessica is two-wooden-legged.
- The thief was either one-legged or one-eyed. Joe was in the hold that day.
- Sometimes Jane or Jack goes down to the hold alone.

The female crew members are One-eyed Jane, Five-fingered Jasmine, and Two-wooden-legged Jessica, while the male crew members are One-legged Jack, Blind John, and Red-beard Joe. The pirates are characterized by their surnames, and the given attribute applies to no one else.

(Proposed by Lili Mohay)

(Solution)

2. Let ABC be an equilateral triangle. We draw a semicircle with the diameter BC on the outside of the triangle. Let D the trisection point of the semicircular arc closer to B, E the trisection closer to C. Let F be the intersection of segment AD with BC, and H be the intersection of segment AE with BC. Prove that segments BF, FH and HC are all of equal length.

(Proposed by Lili Mohay)

(Solution)

3. **a)** Fourteen people sat down around a table at the same time. After sitting down, in the first minute some people are cheerful, and everyone else is sad. As the cheerful people tell jokes, in every subsequent minute, precisely those people whose both neighbors were sad in the previous minute become sad, and everyone else becomes cheerful. Is it true that no matter who was sad initially, if there was at least one cheerful person, eventually everyone will always be cheerful?
b) What happens if 1001 people sit around the table?
If a cheerful person sits next to two sad people, they will become sad.

(Proposed by András Imolay)

(Solution)

4. Seven children (noted by A, B, C, D, E, F, and G) are playing a game called bomb-shield. At the beginning of each round, at the sound of a whistle, each child chooses two different children from the other six, one as their bomb and the other as their shield. From this point, they have 1 minute to position themselves on the plane, after which everyone must remain in place. At the end of a round, points are awarded to those children, who are aligned in such a way, that they are collinear with their bomb and shield, and the shield is in the middle.
a) In a game, the children chose their bomb and shield as shown in the table. Show an arrangement where all seven children earn points at the end of the round.
b) Is it possible for the children to choose their bomb and shield in such a way that there is no arrangement where everyone can earn points in the round?
c) Is it true that, if in a round, the children can position themselves so that everyone earns points, then they can also position themselves so that everyone earns points and they all stand on a single line?

child	A	B	C	D	E	F	G
shield	D	A	A	G	G	C	C
bomb	F	E	G	E	B	B	D

(Proposed by Kartal Nagy)

(Solution)

5. We call a subset H of $\{0, 1, \ldots, 9\}$ *sufficient* if any integer greater than 10 can be expressed as the sum of two non-negative integers, whose digits all come from H.
a) Prove that the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is sufficient.
b) Prove that there is no sufficent set that contains only 4 elements.
c) Provide a sufficient set with the smallest possible number of elements and prove its sufficiency.

(Solution)

6. Game: Researchers have discovered a female specimen of the endangered straight-moving catshark species. The shark moves deep in the water, so the researchers use three submarines to capture it. The researchers communicate with each other, they can see the shark and the shark sees them as well. The lake is square-shaped and divided into 4×4 square sectors. Every day at noon one of the submarines moves to another sector adjacent to it. (two sectors are adjacent if they share a side). The shark wants to lay its eggs peacefully in 11 days, so it tries to escape utnil then, moving up to two times to an adjacent sector every night. The resarchers move first from the following starting position on the first day.The researchers win if a submarine ends up in the same sector as the shark at any point until the 11th day. The shark wins if it remains free at the end of the 11th day.
Defeat the organizers twice in a row in this game. You can decide whether you want to play as the researchers or the shark.

(Proposed by Anita Páhán)

		K	K
			K
C			

(Solution)

2.3.2 Category D

1. The crew of the pirate ship drifting on the Düreric Sea consisted of three girls and three boys. They had been adrift on the sea for a long time when someone from the crew, seeking to add some excitement to their monotonous days, went down alone to the ship's hold and stole a bottle of rum. The crew wants to find out who the thief was with the help of their parrot. The parrot spoke four times, telling the truth twice and lying twice. Who was the thief if the parrot's statements were as follows?

- The boys were not on the ship on the day of the theft because of a raid.
- Jasmine and John always steal together. Joe did not go to the hold that day.
- The thief was either one-legged or one-eyed. Jessica does not like rum.
- Sometimes Jane or Jack steals alone. Jessica likes rum.

The female crew members are One-eyed Jane, Five-fingered Jasmine, and Two-wooden-legged Jessica, while the male crew members are One-legged Jack, Blind John, and Red-beard Joe. The pirates are characterized by their surnames, and the given attribute applies to no one else.

(Proposed by Lili Mohay)

(Solution)

2. **a)** Fourteen people sat down around a table at the same time. After sitting down, in the first minute some people are cheerful, and everyone else is sad. As the cheerful people tell jokes, in every subsequent minute, precisely those people whose both neighbors were sad in the previous minute become sad, and everyone else becomes cheerful. Is it true that no matter who was sad initially, if there was at least one cheerful person, eventually everyone will always be cheerful?
b) What happens if 1001 people sit around the table?
If a cheerful person sits next to two sad people, they will become sad.

(Proposed by András Imolay)

(Solution)

3. Seven children (noted by A, B, C, D, E, F, and G) are playing a game called bomb-shield. At the beginning of each round, at the sound of a whistle, each child chooses two different children from the other six, one as their bomb and the other as their shield. From this point, they have 1 minute to position themselves on the plane, after which everyone must remain in place. At the end of a round, points are awarded to those children, who are aligned in such a way, that they are collinear with their bomb and shield, and the shield is in the middle.
a) In a game, the children chose their bomb and shield as shown in the table. Show an arrangement where all seven children earn points at the end of the round.
b) Is it possible for the children to choose their bomb and shield in such a way that there is no arrangement where everyone can earn points in the round?
c) Is it true that, if in a round, the children can position themselves so that everyone earns points, then they can also position themselves so that everyone earns points and they all stand on a single line?

child	A	B	C	D	E	F	G
shield	D	A	A	G	G	C	C
bomb	F	E	G	E	B	B	D

(Proposed by Kartal Nagy)

(Solution)

4. Let I be the centre of the incircle of the triangle ABC, D the point of tangency with side AB and F the midpoint of AB. Prove that if $\frac{AD}{AF} = \frac{2}{3}$ and $\sphericalangle FIB = \sphericalangle ACI$ then the triangle is isosceles.

(Proposed by József Osztényi)

(Solution)

5. We call a subset H of $\{0, 1, \ldots, 9\}$ *sufficient* if any integer greater than 10 can be expressed as the sum of two non-negative integers, whose digits all come from H.
a) Prove that the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is sufficient.
b) Prove that there is no sufficent set that contains only 4 elements.
c) Provide a sufficient set with the smallest possible number of elements and prove its sufficiency.

(Solution)

6. Game: Researchers have discovered a female specimen of the endangered straight-moving catshark species. The shark moves deep in the water, so the researchers use four submarines to capture it. The researchers communicate with each other, they can see the shark and the shark sees them as well. The lake is square-shaped and divided into 5×5 square sectors. Every day at noon one of the submarines moves to another sector adjacent to it. (two sectors are adjacent if they share a side). The shark wants to lay its eggs peacefully in 15 days, so it tries to escape utnil then, moving up to two times to an adjacent sector every night. The resarchers move first from the following starting position on the first day.The researchers win if a submarine ends up in the same sector as the shark at any point until the 15th day. The shark wins if it remains free at the end of the 15th day.
Defeat the organizers twice in a row in this game. You can decide whether you want to play as the researchers or the shark.

(Proposed by Anita Páhán)

			K	K
			K	K
C				

(Solution)

2.3.3 Category E

1. Seven children (noted by A, B, C, D, E, F, and G) are playing a game called bomb-shield. At the beginning of each round, at the sound of a whistle, each child chooses two different children from the other six, one as their bomb and the other as their shield. From this point, they have 1 minute to position themselves on the plane, after which everyone must remain in place. At the end of a round, points are awarded to those children, who are aligned in such a way, that they are collinear with their bomb and shield, and the shield is in the middle.
a) In a game, the children chose their bomb and shield as shown in the table. Show an arrangement where all seven children earn points at the end of the round.
b) Is it possible for the children to choose their bomb and shield in such a way that there is no arrangement where everyone can earn points in the round?
c) Is it true that, if in a round, the children can position themselves so that everyone earns points, then they can also position themselves so that everyone earns points and they all stand on a single line?

child	A	B	C	D	E	F	G
shield	D	A	A	G	G	C	C
bomb	F	E	G	E	B	B	D

(Proposed by Kartal Nagy)

(Solution)

2. Let I be the centre of the incircle of the triangle ABC, D the point of tangency with side AB and F the midpoint of AB. Prove that if $\frac{AD}{AF} = \frac{2}{3}$ and $\sphericalangle FIB = \sphericalangle ACI$ then the triangle is isosceles.

(Proposed by József Osztényi)

(Solution)

3. We call a subset H of $\{0, 1, \ldots, 9\}$ *sufficient* if any integer greater than 10 can be expressed as the sum of two non-negative integers, whose digits all come from H. What is the smallest possible number of elements in a sufficient set?

(Solution)

4. On the island of Dürerland, the grand final of the ever-popular gameshow, *Merchant of the island* has just arrived! To determine a winner, first, the contesters, *Paul* and *Pauline* have to divide a salmon of size $2n$ equally amongst themselves (where n is a positive integer). They have a machine which upon receiving a piece of fish of size k, cuts it into two pieces with positive integer sizes, but the distribution cannot be predicted beforehand (k is an integer bigger than 1). What is the minimum number of cuts, after which Paul and Pauline can distribute the pieces, such that the sum of the sizes of the pieces they both receive is equal to n, no matter how the machine makes the cuts?
The machine might not cut pieces of equal size the same way every time. After each cut, the sizes of the resulting pieces are measured right away.

(Proposed by Csongor Beke)

(Solution)

5. Given any integer a and positive integer k, denote by $(a \mod k)$ the remainder when a is divided by k, so that $0 \leq (a \mod k) \leq k-1$ and $a-(a \mod k)$ is divisible by k. We call the infinite sequence of integers $a_1, a_2, \ldots$ *highly periodic* if for every positive integer k, the sequence $(a_1 \mod k), (a_2 \mod k), \ldots$ is periodic. Is there a highly periodic sequence that contains infinitely many zeros, but whose zero positions are not periodic?
A sequence $c_1, c_2, \ldots$ is called periodic if there exists a positive integer d such that $c_{i+d} = c_i$ for all positive integers i. We say that the zero positions of a sequence are periodic if, by replacing every nonzero number in the sequence with 1, we obtain a periodic sequence.

(Proposed by Gábor Szűcs)

(Solution)

6. Game: Two players take turns filling out a lottery ticket with six numbers. A filling is valid if none of the numbers on the ticket have been previously selected, and the greatest common divisor of the numbers on the ticket is greater than 1. The game ends when a player cannot fill out a new valid ticket, resulting in their loss.
Each lottery ticket must contain six different numbers chosen from the positive integers not greater than 45.
Defeat the organisers twice in a row in this game! You can decide whether you want to be the first or second player.

(Proposed by Kartal Nagy)

(Solution)

2.3.4 Category E$^+$

1. Describe all ordered sets of four real numbers (a, b, c, d) for which the values $a+b, b+c, c+d, d+a$ are all non-zero and

$$\frac{a+2b+3c}{c+d} = \frac{b+2c+3d}{d+a} = \frac{c+2d+3a}{a+b} = \frac{d+2a+3b}{b+c}.$$

(Proposed by András Imolay)

(Solution)

2. For every subset P of the plane let $S(P)$ denote the set of circles and lines that intersect P in at least three points. Find all sets P consisting of 2024 points such that for any two distinct elements of $S(P)$, their intersection points all belong to P.

(Proposed by Csongor Beke)

(Solution)

3. On the island of Dürerland, the grand final of the ever-popular gameshow, *Merchant of the island* has just arrived! To determine a winner, first, the contesters, *Paul* and *Pauline* have to divide a salmon of size $2n$ equally amongst themselves (where n is a positive integer). They have a machine which upon receiving a piece of fish of size k, cuts it into two pieces with positive integer sizes, but the distribution cannot be predicted beforehand (k is an integer bigger than 1). What is the minimum number of cuts, after which Paul and Pauline can distribute the pieces, such that the sum of the sizes of the pieces they both receive is equal to n, no matter how the machine makes the cuts?
The machine might not cut pieces of equal size the same way every time. After each cut, the sizes of the resulting pieces are measured right away.

(Proposed by Csongor Beke)

(Solution)

4. Let $\mathcal{H}$ be the set of all lines in the plane. Call a function $f : \mathbb{R}^2 \to \mathcal{H}$ from the points of the plane *polarising*, if for any points $P, Q \in \mathbb{R}^2$, $P \in f(Q)$ implies $Q \in f(P)$.
a) Show that there is no surjective polarising function.
b) Give an example of an injective polarising function.
c) Prove that for every injective polarising function there exists a point P on the plane for which $P \in f(P)$. *A function $f : A \to B$ is surjective, if for all $b \in B$, there is an $a \in A$ such that $f(a) = b$. f is injective, if for any two distinct $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$.*

(Proposed by Áron Bán-Szabó)

(Solution)

5. Let p be a fixed prime number.
a) How many 3-tuples (a_1, a_2, a_3) exist, for which all three numbers are non-negative integers less than p and $p \mid a_1^2 + a_2^2 + a_3^2$ holds?
b) Let now k be a fixed positive odd number. Determine the number of k-tuples where all k numbers are non-negative integers less than p and $p \mid a_1^2 + a_2^2 + \ldots + a_k^2$.

(Proposed by Csongor Beke)

(Solution)

6. Game: On a $1 \times n$ board there are $n - 1$ *separating edges* between neighbouring cells. Initially none of the edges contain matchsticks. During a move of size $0 < k < n$, a player chooses a $1 \times k$ sub-board which contains no matches inside, and places a matchstick on all of the separating edges bordering the sub-board that don't already have one. A move is considered legal if at least one match can be placed and if either $k = 1$ or k is divisible by 4. The two players take turns making moves, the player in turn must choose one of the available legal moves of the largest size $0 < k < n$ and play it. If someone does not have a legal move, the game ends and that player loses.
Beat the organisers twice in a row in this game! First the organisers determine the value of n, then you get to choose whether you want to play as the first or the second player.

(Proposed by Márton Németh)

(Solution)

2.4 Final round – day 2

2.4.1 Category C

1. Erik selected some edges of a cube in such a way that no two selected edges share a common endpoint. What is the maximum number of edges he could have selected?

(Proposed by Erik Füredi)

(Solution)

2. Zorka wanted to know how old Captain Panna is, so she asked her. Captain Panna replied that she is exactly 25 years older than his son, Tamás. What is Panna's age in years if Zorka knows that 12 years from now, Tamás will be exactly half as old as Panna is now?

(Proposed by Áron Horváth)

(Solution)

3. What is the only 3-digit number that is divisible by 4 and its first digit is one greater than its last one, moreover the product of its first and last digits is one greater than its middle digit?

(Proposed by Lili Mohay)

(Solution)

4. Albert, the captain, sailed his ship 15 km in a straight line on the Düreric Sea, then turned left at a right angle and sailed another 12 km in a straight line. After that, he turned left at a right angle again and sailed another 10 km straight, following the exact instructions of a treasure map, reaching the desired treasure island. What is the maximum distance Albert could have saved if he had known the location of the treasure island in advance and sailed there directly?

(Proposed by Lilla Tóthmérész)

(Solution)

5. Kartal fills the fields of a 5×5 table with integers from 1 to 25, using each number exactly once. After this, Benedek sums the numbers in each row and writes them beside their rows, and Dani sums the numbers in each column and writes them under their columns. Jóska subtracts the largest number written by Benedek from the smallest number written by Dani. What is the largest possible number that Jóska could have obtained?

(Solution)

6. A digital display can show an integer between 1 and 9999, and it counts up or down by one per second, either always increasing or always decreasing. Unfortunately, the top right segment of the digit at the units place is faulty (everything else works). Béla watched the display continuously for some time and could not determine whether the count was going up or down. What is the maximum number of different numbers that the display could have attempted to show during this time?

When the display reaches 9999 *in the increasing direction, or* 1 *in the decreasing direction, it stops. The display does not show leading zeros, and the four segments respectively show the digits at the thousands, hundreds, tens, and units places.*

The left side shows an example of how the display indicates the number 637*. The right side shows how the display would indicate each digit if all segments were functioning correctly.*

(Proposed by Áron Horváth)

(Solution)

7. A criminal octopus has eight tentacles. When caught by the police, they want to handcuff its tentacles in pairs using four handcuffs. Each tentacle is handcuffed to exactly one other tentacle. How many different ways can they do this?

Two ways of handcuffing are considered different if there is at least one tentacle that is handcuffed to a different tentacle.

(Proposed by Áron Horváth)

(Solution)

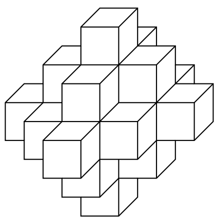
8. In the Kingdom of Letters, an election is held where the three candidates are A, B, and C. A total of 40 voters decide the winner. They are divided into 8 teams of 5 people each, and then each team casts 2 main votes. Only those candidates receive a main vote from a team who are supported by at least 2 members of that team. If there is only one such candidate, they receive 2 main votes from the team; if there are two, they receive 1 main vote each. Each of the 40 voters supports exactly one candidate: 16 support A, 12 support B, and 12 support C. Let x and y denote the minimum and maximum number of main votes A can receive out of the 16 main votes, respectively. What is $x \cdot y$?

(Proposed by Erik Füredi)

(Solution)

9. We glued together 25 standard dice in the manner shown in the figure so that the number of dots on any pair of faces that were glued together is the same. What is the maximum number of dots that could be on the surface?
Aside from the dots, the solid figure looks the same from the opposite direction. A die is called standard if the number of dots on opposite faces sum to 7.

(Proposed by Gyuri Győrffi)



(Solution)

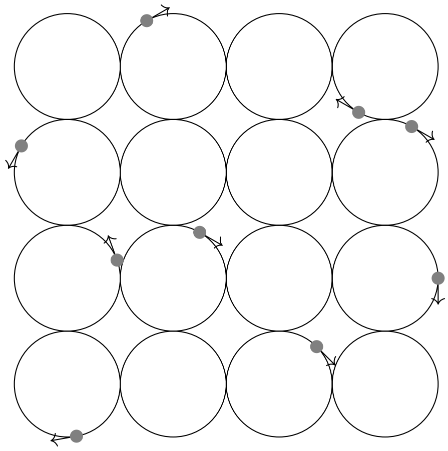
10. What is the smallest positive integer such that the half of N is a perfect square, the third of N is a perfect cube?

(Proposed by Gábor Szűcs)

(Solution)

11. On the circles of the diagram there are 9 ants walking around, all of them with the same constant speed and without turning around or stopping. Whenever an ant reaches a point of tangency of two circles, it can choose which of the two circles it continues on, but without turning around. An ant cannot turn around even when it switches to another circle. Each time two ants meet, one of them climbs over the other and if they have not met before, one of them shouts once. What is the maximum number of times that the ants shouted if their initial positions and directions are as shown on the diagram?
We know that it is not possible that two ants meet while travelling in the same direction.

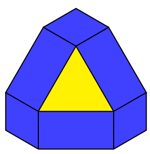
(Proposed by Alex Kempf)



(Solution)

12. On the beach of the island of Óxisz, a sandcastle with an equilateral triangle base was built, surrounded by a moat as shown in the diagram. The base of the moat consists of rectangles and rhombuses. The depth of the moat is 35 cm in the rectangular sections and $\sqrt{12}$ dm in the rhombus sections. The sides of the sandcastle are 2 meters, and the sides of the rhombuses are 1 meter long. How many liters of water were needed to fill the moat completely?

(Proposed by Leila Nagy)



(Solution)

13. A number is called a *duck number* if by reading it backwards, we obtain a greater number than the original one. How many 8-digit duck numbers can be written using only thedigits 1 and 2?
For example, 37145 *is a duck number because* $54173 > 37145$.

(Proposed by Benedek Kovács)

(Solution)

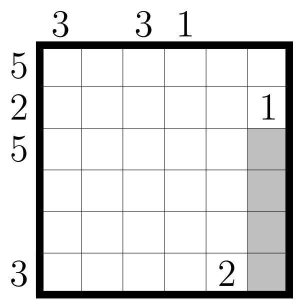
14. We glued together a $5 \times 7 \times 9$ cuboid from small cubes of the same size, where the middle cube is red and the rest of the cubes are yellow, each with a different shade.
Anita removed some yellow cubes (at least one) in such a way that the remaining object was a cuboid, with no visible red on its surface. How many different ways could the resulting cuboid have looked like?
Anita could have also removed cubes from the bottom of the cuboid, and all the faces of the remaining cuboid were visible.

(Proposed by Anita Kercsó-Molnár)

(Solution)

15. Fill in the grid with the digits $1, 2, 3, 4, 5, 6$ such that every row and column contains each digit exactly once. Furthermore, the numbers beside the grid indicate the highest absolute difference between any two consecutive digits in the row/column. **The answer is the four digit number obtained by reading the digits in the grey colored squares from top to bottom.**

(Proposed by Áron Horváth)



(Solution)

16. Ibolya, Hanga, Kamilla, and Rózsa are distributing flowers along the circular Bartolomeu Boulevard, which has 26-26 houses on each side. The houses are paired, meaning there is a house directly opposite to each one across the road. Ibolya and Hanga distribute flowers only on the inner side of the boulevard, while Kamilla and Rózsa do so only on the outer side. All four of them start from the same house pair, moving in the same direction and arriving simultaneously at each subsequent house pair. At the first house pair, all four distribute flowers, after which Ibolya and Kamilla distribute flowers to every second house on their respective sides, Hanga to every third house, and Rózsa to every fifth house. How many flowers do they distribute in total until each house has received at least one flower using this method? *Each of them distributes flowers individually to the visited houses. The boulevard forms a complete circle.*

(Proposed by Bence Kovács)

(Solution)

2.4.2 Category D

1. Zorka wanted to know how old Captain Panna is, so she asked her. Captain Panna replied that she is exactly 25 years older than his son, Tamás. What is Panna's age in years if Zorka knows that 12 years from now, Tamás will be exactly half as old as Panna is now?

(Proposed by Áron Horváth)

(Solution)

2. Leila, the one-eyed pirate, was in her ship at the center of a circular sea. At sunset, she set out in some direction and sailed straight for 24 kilometers. However, a huge storm arose, which turned the ship 90 degrees to the left. After the storm subsided, she sailed straight for another 7 kilometers in this new direction and finally reached the shore. What is the diameter of the sea? *During the storm, Leila anchored the ship, so it did not move, only turned.*

(Proposed by Leila Nagy)

(Solution)

3. Kartal fills the fields of a 5×5 table with integers from 1 to 25, using each number exactly once. After this, Benedek sums the numbers in each row and writes them beside their rows, and Dani sums the numbers in each column and writes them under their columns. Jóska subtracts the largest number written by Benedek from the smallest number written by Dani. What is the largest possible number that Jóska could have obtained?

(Solution)

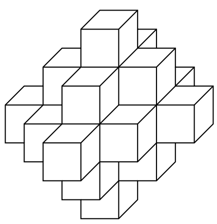
4. In the board game called Quadropoly, Csaba ended up in jail. A player can attempt to get out of jail by rolling two standard dice at once. The attempt is successful if the sum of the rolled numbers is divisible by 4. What is the probability that Csaba will get out of jail on his first attempt? **Provide the sum of the numerator and the denominator of the simplest form of the fraction as the answer!**

(Proposed by Leila Nagy)

(Solution)

5. We glued together 25 standard dice in the manner shown in the figure so that the number of dots on any pair of faces that were glued together is the same. What is the maximum number of dots that could be on the surface?
Aside from the dots, the solid figure looks the same from the opposite direction. A die is called standard if the number of dots on opposite faces sum to 7.

(Proposed by Gyuri Győrffi)



(Solution)

6. A swordfish has a special relationship with the number 8, so its favourite numbers are exactly those three-digit numbers for which the number itself, the sum of its digits and the product of its digits are all divisible by 8, but none the number's digits are divisible by 8. What is the sum of the swordfish's favourite numbers?

(Proposed by Kartal Nagy)

(Solution)

7. Given rectangles $ABCD$ and $AEFG$ such that E is an interior point of segment AB, and D is an interior point of segment AG. Let H be the intersection of segments EF and CD. We know that $AEHD$ is a square with an area of 36 units, and that the area of rectangle $ABCD$ is 10 times the area of rectangle $AEFG$. How long is segment AB if points B, H, and G are collinear?

(Proposed by András Imolay)

(Solution)

8. What is the smallest positive integer such that the half of N is a perfect square, the third of N is a perfect cube?

(Proposed by Gábor Szűcs)

(Solution)

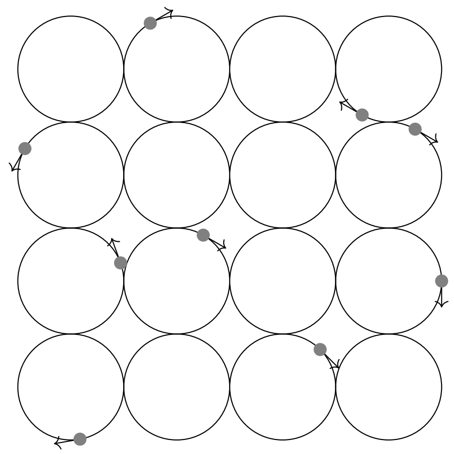
9. A number is called a *duck number* if by reading it backwards, we obtain a greater number than the original one. How many 8-digit duck numbers can be written using only thedigits 1 and 2?
For example, 37145 *is a duck number because* $54173 > 37145$.

(Proposed by Benedek Kovács)

(Solution)

10. On the circles of the diagram there are 9 ants walking around, all of them with the same constant speed and without turning around or stopping. Whenever an ant reaches a point of tangency of two circles, it can choose which of the two circles it continues on, but without turning around. An ant cannot turn around even when it switches to another circle. Each time two ants meet, one of them climbs over the other and if they have not met before, one of them shouts once. What is the maximum number of times that the ants shouted if their initial positions and directions are as shown on the diagram?
We know that it is not possible that two ants meet while travelling in the same direction.

(Proposed by Alex Kempf)



(Solution)

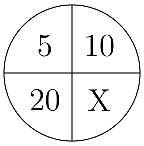
11. We glued together a $5 \times 7 \times 9$ cuboid from small cubes of the same size, where the middle cube is red and the rest of the cubes are yellow, each with a different shade.
Anita removed some yellow cubes (at least one) in such a way that the remaining object was a cuboid, with no visible red on its surface. How many different ways could the resulting cuboid have looked like?
Anita could have also removed cubes from the bottom of the cuboid, and all the faces of the remaining cuboid were visible.

(Proposed by Anita Kercsó-Molnár)

(Solution)

12. Captain Morgan threw 9 darts aiming at the board composed of four sectors, depicted on the diagram. If a dart hit the board, then the Captain received points equal to the number in the sector that the dart hit. If the dart did not hit the board, the Captain received no points for that dart. How many distinct positive integers could be in the place of the X, if the Captain scored a total of 175 points?
Captain Morgan didn't hit the border of any sector.

(Proposed by Bence Kovács)



(Solution)

13. At an individual maths competition there were 3 students participating from each of the two classes 12.a and 12.b. We know that the students got 6 different total scores, meaning that their overall positions were also different. Simply by looking at the 6 positions, their teacher was able to conclude that the sum of scores of the students from class 12.b is greater than the sum from 12.a. What is the number of the possible orders of the six students?
For any two students, the one having more points will have a higher position. If two students have the same score then their position is the same. The teacher does not even know the maximum possible score for the competition.

(Proposed by Erik Füredi)

(Solution)

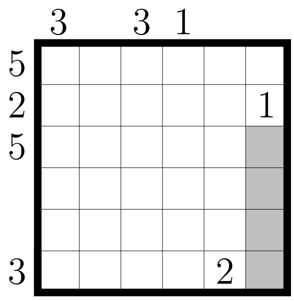
14. Dorka wrote some different positive integers on a piece of paper with the following property: for any integer $2 \leq K \leq 100$ that Lili thinks of, she can find some distinct numbers (at least one) among the ones written by Dorka whose product is exactly K. At least how many numbers did Dorka write on the paper?

(Proposed by Csongor Beke)

(Solution)

15. Fill in the grid with the digits $1, 2, 3, 4, 5, 6$ such that every row and column contains each digit exactly once. Furthermore, the numbers beside the grid indicate the highest absolute difference between any two consecutive digits in the row/column. **The answer is the four digit number obtained by reading the digits in the grey colored squares from top to bottom.**

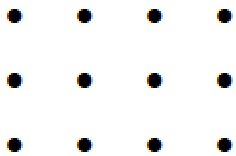
(Proposed by Áron Horváth)



(Solution)

16. The 3×4 grid shown on the diagram has 12 points. We connected these points in some order to form a polygon such that its perimeter passes through each grid point exactly once, and each of its vertex is one of these points. How many different polygons could we have obtained this way? *The polygon's perimeter cannot be self-intersecting or touch itself. Cases that can be transformed into each other by rotation or reflection are considered different.*

(Proposed by Benedek Nádor)



(Solution)

2.4.3 Category E

1. In the board game called Quadropoly, Csaba ended up in jail. A player can attempt to get out of jail by rolling two standard dice at once. The attempt is successful if the sum of the rolled numbers is divisible by 4. What is the probability that Csaba will get out of jail on his first attempt? **Provide the sum of the numerator and the denominator of the simplest form of the fraction as the answer!**

(Proposed by Leila Nagy)

(Solution)

2. Kartal fills the fields of a 5×5 table with integers from 1 to 25, using each number exactly once. After this, Benedek sums the numbers in each row and writes them beside their rows, and Dani sums the numbers in each column and writes them under their columns. Jóska subtracts the largest number written by Benedek from the smallest number written by Dani. What is the largest possible number that Jóska could have obtained?

(Solution)

3. A digital display can show an integer between 1 and 9999, and it counts up or down by one per second, either always increasing or always decreasing. Unfortunately, the top right segment of the digit at the units place is faulty (everything else works). Béla watched the display continuously for some time and could not determine whether the count was going up or down. What is the maximum number of different numbers that the display could have attempted to show during this time?
When the display reaches 9999 *in the increasing direction, or* 1 *in the decreasing direction, it stops. The display does not show leading zeros, and the four segments respectively show the digits at the thousands, hundreds, tens, and units places.*

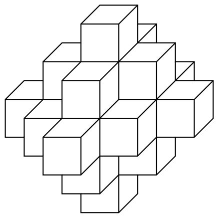
The left side shows an example of how the display indicates the number 637*. The right side shows how the display would indicate each digit if all segments were functioning correctly.*

(Proposed by Áron Horváth)

(Solution)

4. We glued together 25 standard dice in the manner shown in the figure so that the number of dots on any pair of faces that were glued together is the same. What is the maximum number of dots that could be on the surface?
Aside from the dots, the solid figure looks the same from the opposite direction. A die is called standard if the number of dots on opposite faces sum to 7*.*

(Proposed by Gyuri Győrffi)



(Solution)

5. Given rectangles $ABCD$ and $AEFG$ such that E is an interior point of segment AB, and D is an interior point of segment AG. Let H be the intersection of segments EF and CD. We know that $AEHD$ is a square with an area of 36 units, and that the area of rectangle $ABCD$ is 10 times the area of rectangle $AEFG$. How long is segment AB if points B, H, and G are collinear?

(Proposed by András Imolay)

(Solution)

6. A number is called a *duck number* if by reading it backwards, we obtain a greater number than the original one. How many 14-digit duck numbers can be written using only the digits 1 and 2?
For example, 37145 *is a duck number because* $54173 > 37145$*.*

(Proposed by Benedek Kovács)

(Solution)

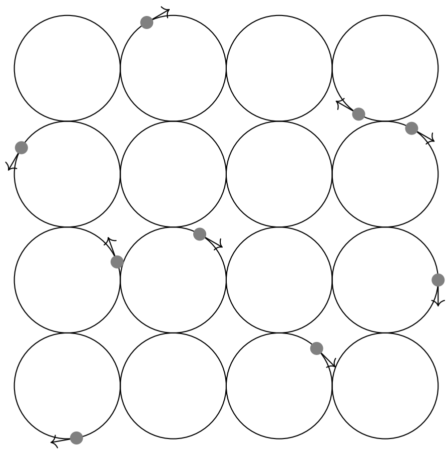
7. A swordfish has a special relationship with the number 8, so its favourite numbers are exactly those three-digit numbers for which the number itself, the sum of its digits and the product of its digits are all divisible by 8, but none the number's digits are divisible by 8. What is the sum of the swordfish's favourite numbers?

(Proposed by Kartal Nagy)

(Solution)

8. On the circles of the diagram there are 9 ants walking around, all of them with the same constant speed and without turning around or stopping. Whenever an ant reaches a point of tangency of two circles, it can choose which of the two circles it continues on, but without turning around. An ant cannot turn around even when it switches to another circle. Each time two ants meet, one of them climbs over the other and if they have not met before, one of them shouts once. What is the maximum number of times that the ants shouted if their initial positions and directions are as shown on the diagram?
We know that it is not possible that two ants meet while travelling in the same direction.

(Proposed by Alex Kempf)



(Solution)

9. Let N be the smallest positive integer such that the half of N is a perfect square, the third of N is a perfect cube and the fifth of N is a perfect fifth power. How many positive divisors does N have?

(Proposed by Gábor Szűcs)

(Solution)

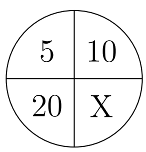
10. At an individual maths competition there were 3 students participating from each of the two classes 12.a and 12.b. We know that the students got 6 different total scores, meaning that their overall positions were also different. Simply by looking at the 6 positions, their teacher was able to conclude that the sum of scores of the students from class 12.b is greater than the sum from 12.a. What is the number of the possible orders of the six students?
For any two students, the one having more points will have a higher position. If two students have the same score then their position is the same. The teacher does not even know the maximum possible score for the competition.

(Proposed by Erik Füredi)

(Solution)

11. Captain Morgan threw 9 darts aiming at the board composed of four sectors, depicted on the diagram. If a dart hit the board, then the Captain received points equal to the number in the sector that the dart hit. If the dart did not hit the board, the Captain received no points for that dart. How many distinct positive integers could be in the place of the X, if the Captain scored a total of 175 points?
Captain Morgan didn't hit the border of any sector.

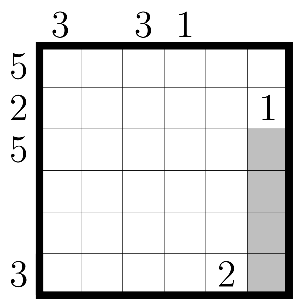
(Proposed by Bence Kovács)



(Solution)

12. Fill in the grid with the digits $1, 2, 3, 4, 5, 6$ such that every row and column contains each digit exactly once. Furthermore, the numbers beside the grid indicate the highest absolute difference between any two consecutive digits in the row/column. **The answer is the four digit number obtained by reading the digits in the grey colored squares from top to bottom.**

(Proposed by Áron Horváth)



(Solution)

13. We glued together a $3 \times 3 \times 5$ cuboid from small cubes of the same size, where the middle cube is red and the rest of the cubes are yellow, each with a different shade.
Anita removed some cubes (at least one) in such a way that the remaining object was a cuboid, with no visible red on its surface. How many different ways could the resulting cuboid have looked like?
Anita could have also removed cubes from the bottom of the cuboid, and all the faces of the remaining cuboid were visible.

(Proposed by Anita Kercsó-Molnár)

(Solution)

14. The circle k with centre A has a radius of 14 units, and B is a point on k. The circle ℓ is tangent to segment AB in its midpoint and is also tangent to k. Let C be the point for which the incircle of ABC is ℓ. How many units is the perimeter of triangle ABC?

(Proposed by Benedek Váli)

(Solution)

15. Six villages, Arka, Bőcs, Cák, Dég, Ete and Füzér are situated along a road in this order. There is a bus service between any pair of consecutive villages. For the five services, buses leave every 5, 7, 9, 11 and 12 minutes, but we don't know which pair of consecutive villages corresponds to which frequency (between any two given villages, the frequency is constant). Today, Alex, Aron and Benedek arrived at different times to the bus station in Arka: Alex arrived at 12:00, Aron at 12:01 and Ben at 13:00. They all aimed to get to Füzér; Alex arrived at the bus station in Füzér at 14:20, but Aron arrived only at 15:00. How many minutes elapsed between the arrivals of Aron and Benedek in Fuse?
The buses come on time, they leave and arrive in a whole minute, buses between the same pair of villages always take the same time, transfer doesn't take any time, and they all took the earliest possible bus at each station.

(Proposed by Kartal Nagy)

(Solution)

16. The altitudes of an acute triangle are 585, 600 and 936 units long. How many units is the perimeter of the triangle?

(Proposed by Erik Füredi)

(Solution)

2.4.4 Category E$^+$

1. A number is called a *duck number* if by reading it backwards, we obtain a greater number than the original one. How many 14-digit duck numbers can be written using only the digits 1 and 2?
For example, 37145 *is a duck number because* $54173 > 37145$.

(Proposed by Benedek Kovács)

(Solution)

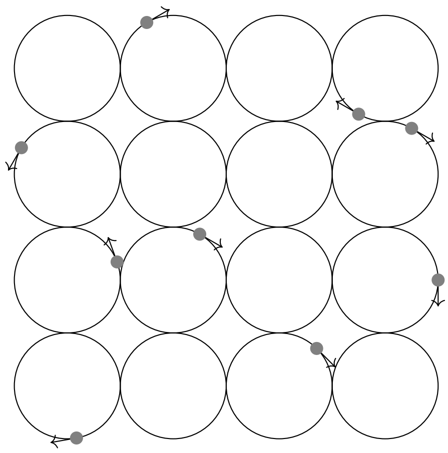
2. A swordfish has a special relationship with the number 8, so its favourite numbers are exactly those three-digit numbers for which the number itself, the sum of its digits and the product of its digits are all divisible by 8, but none the number's digits are divisible by 8. What is the sum of the swordfish's favourite numbers?

(Proposed by Kartal Nagy)

(Solution)

3. On the circles of the diagram there are 9 ants walking around, all of them with the same constant speed and without turning around or stopping. Whenever an ant reaches a point of tangency of two circles, it can choose which of the two circles it continues on, but without turning around. An ant cannot turn around even when it switches to another circle. Each time two ants meet, one of them climbs over the other and if they have not met before, one of them shouts once. What is the maximum number of times that the ants shouted if their initial positions and directions are as shown on the diagram?
We know that it is not possible that two ants meet while travelling in the same direction.

(Proposed by Alex Kempf)



(Solution)

4. Let N be the smallest positive integer such that the half of N is a perfect square, the third of N is a perfect cube and the fifth of N is a perfect fifth power. How many positive divisors does N have?

(Proposed by Gábor Szűcs)

(Solution)

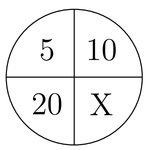
5. Dorka wrote some different positive integers on a piece of paper with the following property: for any integer $2 \leq K \leq 100$ that Lili thinks of, she can find some distinct numbers (at least one) among the ones written by Dorka whose product is exactly K. At least how many numbers did Dorka write on the paper?

(Proposed by Csongor Beke)

(Solution)

6. Captain Morgan threw 9 darts aiming at the board composed of four sectors, depicted on the diagram. If a dart hit the board, then the Captain received points equal to the number in the sector that the dart hit. If the dart did not hit the board, the Captain received no points for that dart. How many distinct positive integers could be in the place of the X, if the Captain scored a total of 175 points?
Captain Morgan didn't hit the border of any sector.

(Proposed by Bence Kovács)



(Solution)

7. At an individual maths competition there were 4 students participating from each of the two classes 12.a and 12.b. We know that the students got 8 different total scores, meaning that their overall positions were also different. Simply by looking at the 8 positions, their teacher was able to conclude that the sum of scores of the students from class 12.b is greater than the sum from 12.a. What is the number of the possible orders of the eight students?
For any two students, the one having more points will have a higher position. If two students have the same score then their position is the same. The teacher does not even know the maximum possible score for the competition.

(Proposed by Erik Füredi)

(Solution)

8. We glued together a $3 \times 5 \times 5$ cuboid from small cubes of the same size, where the middle cube is red and the rest of the cubes are yellow, each with a different shade.
Anita removed some cubes (at least one) in such a way that the remaining object was a cuboid, with no visible red on its surface. How many different ways could the resulting cuboid have looked like?
Anita could have also removed cubes from the bottom of the cuboid, and all the faces of the remaining cuboid were visible.

(Proposed by Anita Kercsó-Molnár)

(Solution)

9. Gabi asked Beni when his birthday was. Beni mysteriously replied that he is only telling the value of m^d, where m and d respectively represent the month and day of his date of birth. From this, Gabi has not yet been able to determine when Beni's birthday was. Based on this, how many possible days of the year are there on which Beni could have been born?

(Proposed by Áron Horváth)

(Solution)

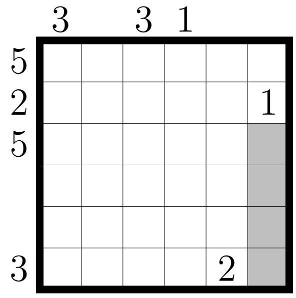
10. The circle k with centre A has a radius of 14 units, and B is a point on k. The circle ℓ is tangent to segment AB in its midpoint and is also tangent to k. Let C be the point for which the incircle of ABC is ℓ. How many units is the perimeter of triangle ABC?

(Proposed by Benedek Váli)

(Solution)

11. Fill in the grid with the digits $1, 2, 3, 4, 5, 6$ such that every row and column contains each digit exactly once. Furthermore, the numbers beside the grid indicate the highest absolute difference between any two consecutive digits in the row/column. **The answer is the four digit number obtained by reading the digits in the grey colored squares from top to bottom.**

(Proposed by Áron Horváth)



(Solution)

12. The altitudes of an acute triangle are 585, 600 and 936 units long. How many units is the perimeter of the triangle?

(Proposed by Erik Füredi)

(Solution)

13. Six villages, Arka, Bőcs, Cák, Dég, Ete and Füzér are situated along a road in this order. There is a bus service between any pair of consecutive villages. For the five services, buses leave every 5, 7, 9, 11 and 12 minutes, but we don't know which pair of consecutive villages corresponds to which frequency (between any two given villages, the frequency is constant). Today, Alex, Aron and Benedek arrived at different times to the bus station in Arka: Alex arrived at 12:00, Aron at 12:01 and Ben at 13:00. They all aimed to get to Füzér; Alex arrived at the bus station in Füzér at 14:20, but Aron arrived only at 15:00. How many minutes elapsed between the arrivals of Aron and Benedek in Fuse?
The buses come on time, they leave and arrive in a whole minute, buses between the same pair of villages always take the same time, transfer doesn't take any time, and they all took the earliest possible bus at each station.

(Proposed by Kartal Nagy)

(Solution)

14. What is the value of $\sum\limits_{k=1}^{17} \frac{1}{k(k+1)(k+2)(k+3)}$?
As an answer, give the numerator of the simplest form of the fraction.

(Proposed by Erik Füredi)

(Solution)

15. Benjamin thought of a real number x and told Timi the value of $\lfloor x^3 \rfloor$. We know from Timi that this is a positive integer not larger than 100 and that by knowing the value of $\lfloor x^3 \rfloor$, the value of $\lfloor x^2 \rfloor$ cannot be determined uniquely. Let A denote the number of possible values that Timi could have heard. After this Benjamin thought of a real number y and told Timi the value of $\lfloor y^4 \rfloor$. Timi also says that this second value is a positive integer not larger than 100 and that by knowing the value of $\lfloor y^4 \rfloor$, the value of $\lfloor y^3 \rfloor$ cannot be determined uniquely. Let B denote the number of possible values that Timi could have heard as the second value. What is the value of $A + B$?
The notation $\lfloor r \rfloor$ denotes the integer part of the real number r, which is the largest integer not greater than r.

(Proposed by Erik Füredi)

(Solution)

16. A positive integer n is called *infernal* if $\frac{1! \cdot 2! \cdot \ldots \cdot 100!}{n!}$ is an integer and also a square of an integer. What is the sum of the infernal numbers?

(Proposed by Áron Horváth)

(Solution)

3 Solutions – grammar school categories

3.1 Regional round

3.1.1 Tables - Category A

#	ANS	Problem	P
A-1	1	Three ships want to reach point X	3p
A-2	3300	Szása is a centipede who lives with their mother,	3p
A-3	8	Hanga's morning walk to school takes 7 minutes,	3p
A-4	115	As a young child, Anita went to the beach	3p
A-5	9	Anchorless Jack, the pirate buried his treasure	4p
A-6	8	Five children want to play a sailor-themed game,	4p
A-7	55	After their daily fishing, the fishermen would like to	4p
A-8	35	Liza's birthday cake has as many candles as the number of her age.	4p
A-9	11	Blackbeard is currently 160 centimeters tall	5p
A-10	8	Two pieces were placed on a square of a large 8×8 board.	5p
A-11	16	Five sailors — Ali, Bod, Csed, Doma, and Ede —	5p
A-12	26	Marvin wrote down all pairs of numbers composing of two integers	5p
A-13	32	A siren spends every night on one of the 7 cliffs	6p
A-14	245	The sail area of the ship shown in the diagram is 700 m^2.	6p
A-15	8	Zsuzsi wants to make a necklace for Ludmilla using 3 orange	6p

(Back to problems)

3.1.2 Tables - Category B

#	ANS	Problem	P
B-1	3300	Szása is a centipede who lives with their mother,	3p
B-2	4	Five ships want to reach point X, where the treasure is located,	3p
B-3	115	As a young child, Anita went to the beach	3p
B-4	16	Hanga's morning walk to school takes 7 minutes,	3p
B-5	8	Five children want to play a sailor-themed game,	4p
B-6	11	Blackbeard is currently 160 centimeters tall	4p
B-7	8	Two pieces were placed on a square of a large 8×8 board.	4p
B-8	16	Five sailors — Ali, Bod, Csed, Doma, and Ede —	4p
B-9	26	Marvin wrote down all pairs of numbers composing of two integers	5p
B-10	36	How long are the edges of cube A shown in the diagram?	5p
B-11	32	A siren spends every night on one of the 7 cliffs	5p
B-12	245	The sail area of the ship shown in the diagram is 700 m^2.	5p
B-13	1450	Jack the pirate stumbled upon a treasure chest	6p
B-14	820	As the three-masted sailing ship named Victoria	6p
B-15	20	Zsuzsi wants to make a necklace for Ludmilla using 4 orange	6p

(Back to problems)

3.2 Final round – day 1

3.2.1 Category A

1. The ship's dock number is divisible by 5, so the last digit can only be 0 or 5. However, we know that 5 does not appear among the digits, so the last digit must be 0.

Therefore, there is no single-digit solution. Furthermore, it is evident that it is not a two-digit number, and that 1000 cannot be appropriate. So we only need to search among the three-digit numbers.

Such a number's first two digits' sum is divisible by 5 and can be at most $9 + 9 = 18$. Therefore, the sum of these digits is 5, 10, or 15.

These numbers can be expressed as sums of digits between 0 and 9, excluding 5, in the following ways:

$5 = 4 + 1 = 3 + 2 = 2 + 3 = 1 + 4$

$10 = 9 + 1 = 8 + 2 = 7 + 3 = 6 + 4 = 4 + 6 = 3 + 7 = 2 + 8 = 1 + 9$

$15 = 9 + 6 = 8 + 7 = 7 + 8 = 6 + 9$

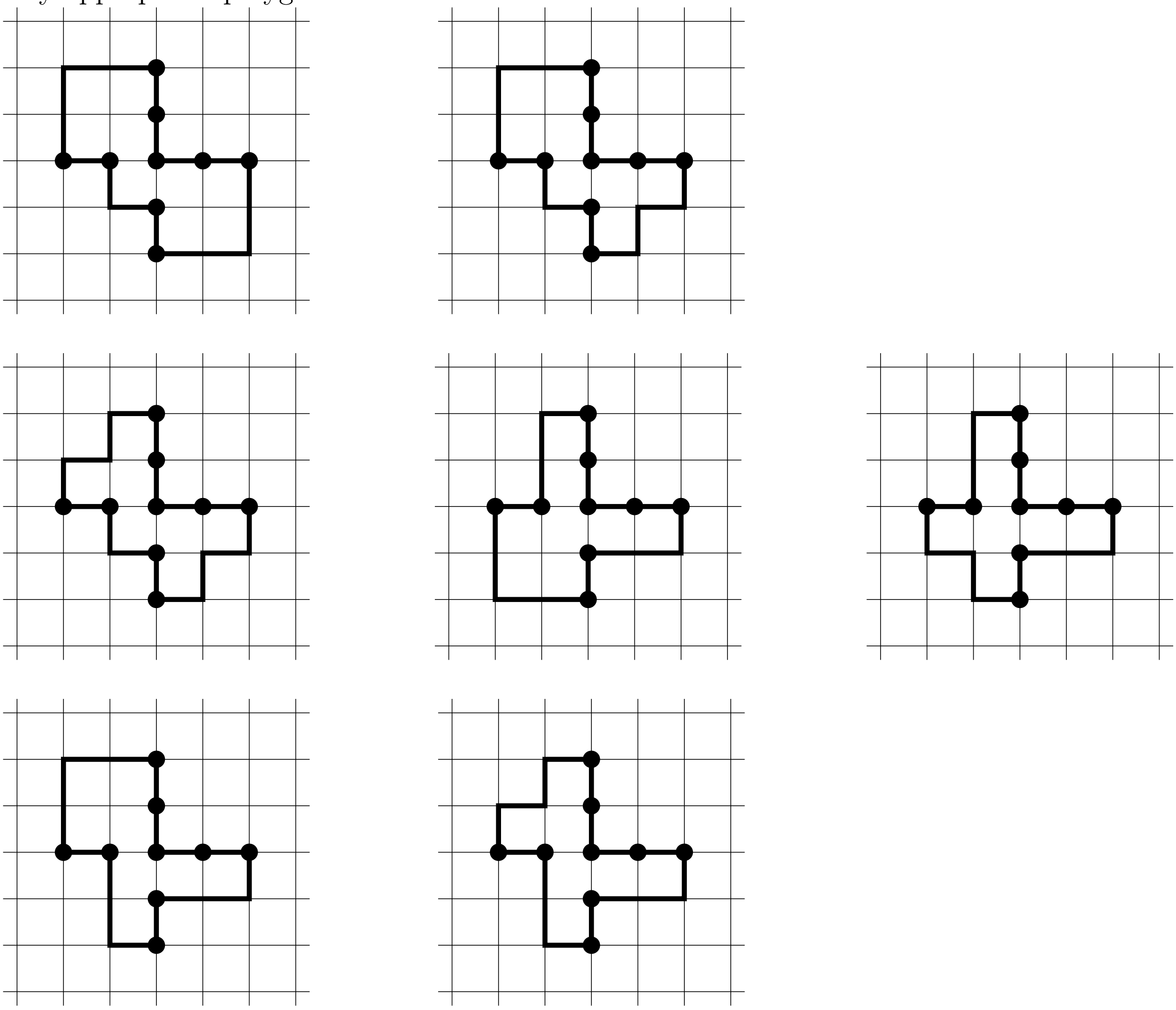
So the possible dock numbers are $410, 320, 230, 140, 910, 820, 730, 640, 460, 370, 280, 190, 960,$ 870, 780 and 690.

(Back to problems)

2. The are only seven options, which are shown below. The reason why there cannot be any more is very difficult to explain in a complete and precise way, so we will not do so, only summarise the main ideas of the proof.

Any grid angle covering the 9 marked grid points appears to be at least four wide when viewed from all four directions (top, bottom, right, left). Thus, the perimeter can only be 16 if the polygon is exactly 4 wide in all directions, and there can be no side of the polygon which is not visible in any direction because it is obscured by another side. From this it is not hard to see that the only way to cover the central vertex is to have an angle of 270° there, and the two sides that meet here are 2 long, so they cover two more marked points. Since it is enough to determine the solution up to rotation and reflection, we can assume that these two sides start from the middle point to the right and upwards, as shown in all the plotted figures.

Finally, one can see by checking all possibilities that in order to have a perimeter of 16 these are the only appropriate polygons.



(Back to problems)

3. Dani said he preceded all people taller than him. Anita is taller than Dani, so according to Dani's statement he outran Anita as well. However, Anita said she preceded Dani. The two statements contradict each other, so one of them was lying.

Only one of the children lied, so we know that every other competitor told the truth. Either Dani's or Anita's statement is true so one of them preceded the other, which means they couldn't have been partners.

According to Csenge's statement Ferenc preceded her by two places, so we know that Ferenc either took first or second place. According to Ferenc he didn't finish second, so Ferenc took first place and Csenge took third place, which is two places worse.

Benedek said he was Csenge's partner, so Benedek also took third place.

Anita, Benedek, Csenge and Dani are taller than Eszter. Therefore, Eszter claims exactly one of them was in the team immediately succeeding hers. Benedek and Csenge's team finished third, so Eszter can't have been in the team finishing second or fourth, as she preceded some team according to her statement. Thus, Eszter took first place and her partner was Ferenc.

Therefore, 4 people remain for the teams taking second and fourth place: Anita, Dani, Gábor and Hanga.

According to Hanga's statement, Anita and Gábor weren't partners, and we have already shown that Anita couldn't have been Dani's partner either. Therefore, Anita could have only been Hanga's partner and Gábor Dani's.

Gábor claims he surpassed Hanga, so only Gábor and Dani's team could have been second, while Hanga and Anita's team must have been fourth.

So, the only possible sequence is the following:

helyezés	egyik tag	másik tag
1.	Eszter	Ferenc
2.	Dani	Gábor
3.	Csenge	Benedek
4.	Hanga	Anita

This is really a solution, since in this case only Anita is not telling the truth.

(Back to problems)

4. Let's denote the four candidates as A, B, C, and D. Since everyone voted for two people, each ballot contains two letters. Therefore, there are a total of 1 vote for A, 26 votes for B, 3 votes for C, and we don't know how many votes for D were on the ballots.

A total of $1+26+3 = 30$ votes were cast not for D, of which two were cast by D themselves, so there can be at most 28 other people, as everyone voted for at least one candidate other than D. Thus, D could have received at most 28 votes, since everyone voted for D at most once.

The letter B appears on 26 ballots, and each of them has another letter alongside it. There are a total of 4 ballots with A and C combined. Moreover, B's own ballot does not include B, so at least one A or C must appear, meaning A or C can be alongside B on at most 3 ballots. Therefore, D appears alongside B on at least 23 ballots.

An additional observation is that since each ballot contains two names, the total number of names on all ballots is even. Since the total number of A, B, and C letters is even, the number of D letters must also be even.

Considering all these points, we can conclude that D could theoretically have received only 24, 26, or 28 votes.

All of these are indeed possible in the following ways:

D receives exactly 24 votes:
A's vote: B, C

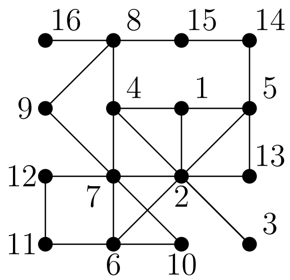
B's vote: A, C
C's vote: B, D
D's vote: B, C
Everyone else's vote (23 people): B, D

D receives exactly 26 votes:
A's vote: B, D
B's vote: A, D
C's vote: B, D
D's vote: B, C
Everyone else's vote (24 people): one C, B, one C, D, everybody else B, D

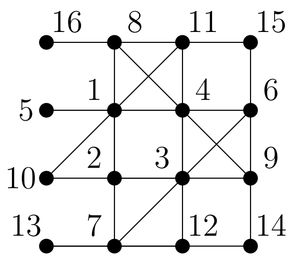
D receives exactly 28 votes:
A's vote: C, D
B's vote: C, D
C's vote: B, D
D's vote: A, C
Everyone else's vote (25 people): B, D

(Back to problems)

5. a) The figure can arise from the following order:

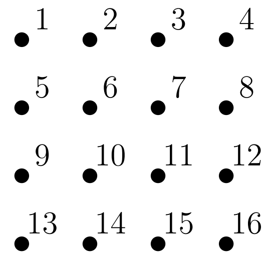


b) The following path visits cities 35 times:



More visits are not possible. In the following proof, we show that 37 is not possible.

Clearly we will visit all 16 cities. The question is, how many times it will occur that we visit a city multiple times. Before visiting a city for the first time, let us count how many cities do we revisit before we visit that city again. So that we can refer to the cities, let us number them.



We can only visit cities 6, 7, 10 and 11 by passing through at least one other city which has already been visited before. As for the other 12 cities, we can pass through at most 2 other cities beforehand.

We can only reach city 5 by passing through two cities by coming from city 8, and the same is true for reaching city 8 by passing through two cities only from city 5. As we cannot fulfill both of these, we pass through at most one city when reaching one of these cities. The same thought is valid for the city pairs 2-14, 3-15 and 9-12 too.

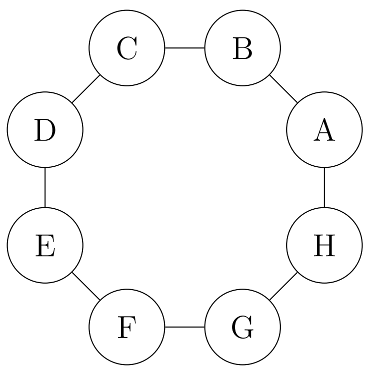
Also, by considering the four corners (1, 4, 13, 16), we can see that these cities can also be only reached from another corner so that we pass through two previously visited cities. So the corner which Dani visits first, he can visit at most one other city when going there.

So there can be at most 7 other cities, for which Dani visits 2 other towns when going there (4 cities on the sides and 3 of the corner cities), and for the other 9 cities, he can visit at most 1. But there must be 3 cities before which he cannot visit another city: the first city, the second city and the first city which is not on the line determined by the first two cities. So there are at most $7 \cdot 2 + 9 - 3 = 20$ times when he revisits a city. So Dani can have at most $16 + 20 = 36$ jugs of raspberry soda.

Our construction was for 35 jugs, and we only proved that 37 is impossible. We do not detail the case of 36 jugs as it is long, but we point out that at the end of the proof, we can show by a small case distinction that by adding a city, the maximum proved above will decrease, so 36 cities is also impossible.

(Back to problems)

6. Let's name the vertices starting from A as A, B, C, D, E, F, G, H according to the diagram.



The architect has a winning strategy. Let his path on the first day be A $\to$ B $\to$ C $\to$ D $\to$ E. Afterwards the bandits can destroy one of these five towers.

If the bandits demolish the towers A or E, then the architect can build all towers by the second day by taking the path E → F → G → H → A. Afterwards no matter which tower the bandits destroy, he can always rebuild it the following day, meaning that he wins by sunset on the fourth day.

If the bandits demolish tower C, then the architect should take the path E → F → G → H → G, and if they demolish the tower at vertex D, then he should go E → F → G → H. (In other words in both cases he stops across the tower that has been demolished.) Then, no matter what the bandits demolish the following day, the architect will build every tower by the end of the third day. The same way he can achieve that all towers are built at the end of the fourth day, and therefore he wins.

Finally, if the bandits demolish tower B after the first day, then the architect should take the following route: E → F → G → H → A. If afterwards the bandits are demolishing one of A, C, D or E, then the architect can build everything by the third evening by taking the route A → B → C → D → E, and we have seen that in this case he wins. If instead the bandits demolish F, G or H, then the architect should first rebuild the tower at B and afterwards should stop across the tower that was demolished during the second night. Then no matter what the bandits choose to demolish during the third night, the architect will always be able to rebuild all towers by the end of the fourth day.

(Back to problems)

3.2.2 Category B

1. For the solution, see Category A Problem 2.

(Back to problems)

2. For the solution, see Category A Problem 3.

(Back to problems)

3. For the solution, see Category A Problem 4.

(Back to problems)

4. Artúr needs at least 6 pieces of chocolate, because even if Picur has only a size and shape preference there are already 6 pairs of chocolates that should belong to different pairs (small and round, medium and round, large and round, small and square, medium and square, large and square).

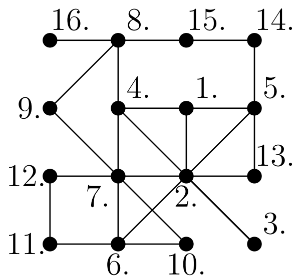
It can be checked that 6 chocolates are enough:

Size	Shape	Filling	Type
small	round	orange	milk
small	square	marzipan	dark
medium	round	marzipan	milk
medium	square	orange	dark
large	square	orange	dark
large	square	marzipan	milk

Therefore Gombóc Artúr needs at least six chocolates.

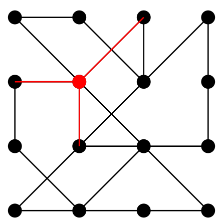
(Back to problems)

5. a) I. We get this diagram from the following numbering:



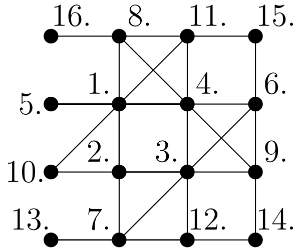
II. We present two reasons why there is no solution. First, from the city numbered 16, only one road can depart, and there is no such city on the diagram, so there really is no solution. The other solution is a bit more complex, but we describe it because it is also instructive.

Let's examine the city marked in red on the diagram.



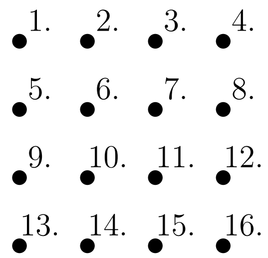
Along the three lines marked in red, one can only travel through if going from one end of the line to the other, as these lines do not continue to other cities. Therefore, on each of these three red lines, one can only traverse paths that either start or end in the city marked in red. However, Dani can only stop once in this city, so there can only be one path that starts or ends in the city marked in red, meaning he cannot traverse along any of these three red lines.

b) The following route visits cities 35 times:



More than this is not achievable. Now in the following proof, we will show that 37 cannot be reached.

It is certain that we will visit all 16 cities; the question is how many times we will revisit a city. Let's examine how many cities we pass through where we have previously been on a direct route to each city before its first visit. To refer to the cities, let's number them.



Cities 6, 7, 10, and 11 shown on the diagram can only be reached by passing through at most one other previously visited city. For each of the remaining 12 cities, we can pass through at most two other cities before reaching them.

Notice that to reach city 5 through two other cities, we can only do so if coming from city 8, and similarly, we can only go through two other cities to reach city 8 from city 5. Since both cannot be fulfilled simultaneously, when traveling to these cities, we can pass through at most one other city. The same reasoning applies to city pairs 2-14, 3-15, and 9-12.

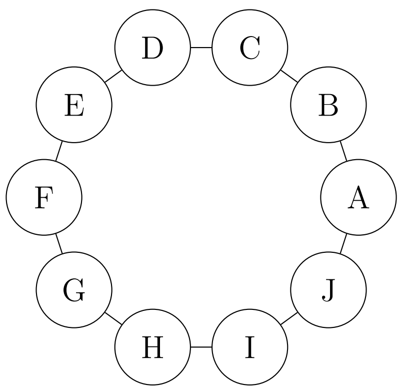
In addition, if we look at the four corners (1, 4, 13, 16), we can see that we can only reach them from one of the other corners while passing through two previously visited cities. This means that when Dani first goes to a corner, he can touch at most one other city along the way.

Therefore, there can be at most 7 cities where Dani visits two other cities when he goes to that city (4 cities among the sides, and 3 cities among the corner cities). For the remaining 9 cities, he can visit at most one other city. However, it can be seen that there will be 3 cities where he cannot visit another city beforehand: the first city, the second city, and the first city that is not on the line of the first and second cities. Thus, there can be at most $7 \cdot 2 + 9 - 3 = 20$ instances where he steps into a city where he has previously been. Therefore, Dani can drink at most $16 + 20 = 36$ glasses of raspberry syrup.

We showed a construction for 35, and only proved that 37 is not possible. We do not describe the case for 36 properly because it is lengthy, but we note that at the end of the proof, with a little case analysis, it can be seen that even with one more town, the previously proven possible maximum will decrease. Thus, it is clear that Dani won't be able to drink 36 glasses of raspberry syrup either.

(Back to problems)

6. Let's name the vertices starting from A as A, B, C, D, E, F, G, H, I, J according to the diagram.



The architect has a winning strategy. Let his path on the first day be A → B → C → D → E → F. Afterwards the bandits can destroy one of these six towers.

If the bandits destroy the towers A or F, then the architect can build all towers by the second day by taking the path F → G → H → I → J → A. Afterwards no matter which tower the bandits destroy, he can always rebuild it the following day, meaning that he wins by sunset on the fourth day.

If the bandits demolish tower D, then the architect should take the path F → G → H → I → J → I, and if they demolish the tower at vertex E, then he should go F → G → H → I → J. (In other words in both cases he stops across the tower that has been demolished.) Then, no matter what the bandits demolish the following day, the architect will build every tower by the end of the third day. The same way he can achieve that all towers are built at the end of the fourth day, and therefore he wins.

Finally, if the bandits demolish towers B or C after the first day, then the architect should take the following route: F → G → H → I → J → A. If afterwards the bandits are demolishing one of A, B, C, D, E or F, then the architect can build everything by the third evening by taking the route A → B → C → D → E → F, and we have seen that in this case he wins. If instead the bandits demolish G, H, I or J, then the architect should first rebuild the demolished one among B and C, and afterwards should stop across the tower that was demolished during the second night. Then no matter what the bandits choose to demolish during the third night, the architect will always be able to rebuild all towers by the end of the fourth day.

(Back to problems)

3.3 Final round – day 2

3.3.1 Tables

#	ANS	Problem	P
A-1	62	In today's date (2024-01-13) every digit	3p
A-2	6	Bogi, Dorka, Andris, and Kristóf are going fishing.	3p
A-3	5	On the map below a pirate hid his treasure in city X	3p
A-4	48	Jack, the one-legged pirate is playing on the hopscotch	3p
A-5	4	The star-shaped iceberg shown in the picture floated	4p
A-6	10	Anett is spending her 17-day winter break at Lake Balaton.	4p
A-7	5096	A prisoner sent his mate the code for a safe,	4p
A-8	12	Andris, Balázs, and Kristóf went fishing on three consecutive days.	4p
A-9	175	A few sailors bring a 24 × 54 meter rectangular sail	5p
A-10	72	Two large and five small octopuses are heading home	5p
A-11	7	In Zsombor's house, each room has exactly three doors.	5p
A-12	526	Csenge drew a rectangle in her grid notebook	5p
A-13	32	Lili would like to sail with her ship overnight	6p
A-14	15	The captain of the Wind Queen hid 13 gold blocks	6p
A-15	20	The boat wants to travel from A to B following the lines	6p
A-16	163	There are 180 colored balls in a bag.	6p

#	MO	Problem	P
B-1	6	Bogi, Dorka, Andris, and Kristóf are going fishing.	3p
B-2	48	Jack, the one-legged pirate is playing on the hopscotch	3p
B-3	4	The star-shaped iceberg shown in the picture floated	3p
B-4	10	Anett is spending her 17-day winter break at Lake Balaton.	3p
B-5	5096	A prisoner sent his mate the code for a safe,	4p
B-6	175	A few sailors bring a 24×54 meter rectangular sail	4p
B-7	72	Two large and five small octopuses are heading home	4p
B-8	7	In Zsombor's house, each room has exactly three doors.	4p
B-9	526	Csenge drew a rectangle in her grid notebook	5p
B-10	32	Lili would like to sail with her ship overnight	5p
B-11	15	The captain of the Wind Queen hid 13 gold blocks	5p
B-12	20	The boat wants to travel from A to B following the lines	5p
B-13	20	A flea wants to travel from the bottom-left corner	6p
B-14	2697	Which occurrence in the current millennium	6p
B-15	108	In rectangle $ABCD$ the midpoint of side AB	6p
B-16	163	There are 180 colored balls in a bag.	6p

3.3.2 Category A

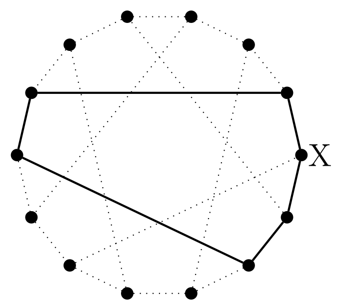
1. If we find such a day this year, as we will soon see is possible, then since the year (2024) contains the digits 0, 2, and 4, the description of the month and day need to include the digits 1, 3, and 5. Since no month has 35 or more days, one of the digits 3 or 5 must be in the description of the month. Therefore, before March cannot be an appropriate time, and in March there is one good day, the fifteenth, so 2024-03-15 is the next suitable day, which will be 62 days from now.

(Back to problems)

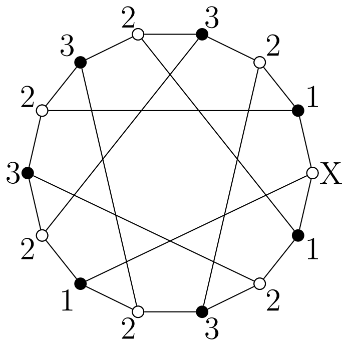
2. Since everyone has to bring exactly one item, Dorka will definitely bring the fishing rod. Therefore, the net must be brought by Andris, leaving Bogi to bring the bucket and Kristóf to bring the worm box. So, the answer to the question, on behalf of Andris, is the number of letters in his name, which is 6.

(Back to problems)

3. It can return by touching five cities, an example of this:



With fewer than five cities touched, one cannot return to the city marked with X. Note that if we alternately color the cities black and white, we can only travel from a white city to a black city and vice versa. Therefore, to return to the city marked with X, we need to touch an odd number of other cities. Touching just one city is insufficient because it would require turning back. Now let's examine why touching three cities is not enough. Write down the distance of each city from the city marked with X:



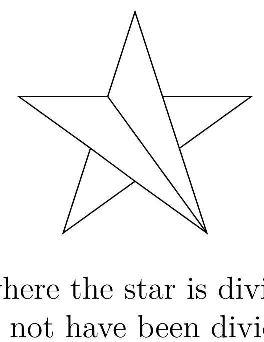
If it were possible to achieve this with three cities touched, it would look like this: X - a city at distance one - a city at distance two - a city at distance one - X. However, such a scenario does not exist because from the cities at distance two, there is only one route leading to a city at distance one. Therefore, to create such a route, one would also need to turn back.

(Back to problems)

4. It cannot be that Jack did not jump on field 9, because 9 is not a divisor of 120, so it couldn't come out as a product. Consequently, Jack did not jump on field number 10. $120 : 10 = 12$, so the product of the other two missed fields must be 12. Since 12 is not divisible by 5, Jack certainly jumped on field 5, hence he did not jump on field 6. Since $12 : 6 = 2$, the third skipped number is 2. Therefore, Jack jumped on fields $1, 3, 4, 5, 7, 8, 9$ and 11 and their sum is 48.

(Back to problems)

5.



The diagram shows an example where the star is divided into 4 triangles.

Let's prove that the ice floe could not have been divided into 3 triangles.

The star has five sharp corners, which we call "nice corners"; each of these must also be a corner of some triangle. It's clear that among these nice corners, no two adjacent ones can belong to the same triangle. Additionally, there cannot be a triangle that covers three nice corners. Furthermore, since a triangle cannot cover two adjacent nice corners, there cannot be two triangles that cover four different nice corners in total, because then these triangles would "intersect" each other. From this, it follows that three triangles are not enough.

(Back to problems)

6. If the two days before the break were cold, then on the first day of the break, she can definitely go ice skating. Following this, on the first day of the break, let the temperature be above $-5°$C, and then below on the 2nd and 3rd days. This guarantees that on the 2nd, 3rd, and 4th days they will be able to go ice skating again. Repeating this pattern, let the temperatures be above $-5°$C on the 4th, 7th, 10th, 13th, and 16th days, and below on the other days. In this case, she can go ice skating every day because from any set of three consecutive days, only one day will be warm. Let's note that on the last day, it doesn't matter what the weather is like, so it could still be warm. Thus, there are a total of 7 warm days, which means that on 10 days the temperature is below $-5°$C. There cannot be fewer cold days than this, because if the first 3 days do not have at least 2 cold days, then on the 4th day they could not go ice skating. Similarly, among the 4-6, 7-9, 10-12, and 13-15 days, there must also be two cold days in each period. Therefore, at least 10 days have temperatures below $-5°$C.

(Back to problems)

7. Let us go through the possible values of the integer used for the encoding based on the last digits of the numbers in the sum. We can observe that 7 works and it can be checked that nothing else works. Therefore the code is 5096.

(Back to problems)

8. The three boys caught a total of $37 + 30 + 26 = 93$ fish over the three days. From this, Kristóf caught $93-(40+24) = 29$ fish. On the first two days, the three boys caught $37+30 = 67$ fish together, of which Andris and Balázs caught 50 fish, leaving Kristóf with $67 - 50 = 17$ fish from the first two days. Therefore, on the third day, Kristóf caught $29 - 17 = 12$ fish.

(Back to problems)

9. The area of the sail is $24\text{m} \cdot 54\text{m} = 1296\text{m}^2$. It doesn't matter which way they fold the sail in half, since its area gets halved either way.

Similarly, if they fold one thirds, then the area reduces to two-thirds. Therefore if they had proceeded according to the original instructions, the area would have halved four times, meaning at the end it would have been $1296 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 81$ square metres. Instead the area ended up being $1296 \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = 256$ square metres.

This means that the difference in area is $256 - 81 = 175$ square metres.

(Back to problems)

10. The ideal formation is when the octopuses are in one line with the two big ones on the ends, and each of the octopuses hold one tentacle with each neighbour.

Then the two big octopuses have 7 tentacles free while the five small ones have 6 tentacles free each. Therefore they can take $2 \cdot 3 \cdot 7 + 5 \cdot 6 = 72$ pearls in total.

Now we will show that more is not possible. Initially, when no one is connected with anyone, then the seven octopuses form seven "groups". When two of them join tentacles, then they become one group, and with each join the number of groups reduces by at most one. Since we need one group at the end, there there have to be at least 6 joins. This means that of the $7 \cdot 8 = 56$ tentacles total, at least 12 have to be used to form joins, therefore at most 44 tentacles can be used to carry pearls.

Each octopus needs to use at least one tentacle to join to the others, therefore the big octopuses can use at most 14 of their tentacles to carry pearls. Since they can carry more on each tentacle than the small ones, they can carry more pearls altogether if they use as many tentacles of the big ones as possible. Therefore 72 is the maximum number of pearls they can carry.

(Back to problems)

11. Let's figure out how many rooms there are on the sides of the doors. There are 10 doors between two rooms, and one door separates a room and Zsombor's courtyard. So, if we add up the total number of doors in the rooms, we get $2 \cdot 10 + 1 \cdot 1 = 21$. Since each room has three doors, the total number of rooms is $21 : 3 = 7$.

(Back to problems)

12. The small rectangle can be oriented in two ways. Either its 10-unit side is parallel to the 30-unit side of the large rectangle, or its 8-unit side is parallel to the 30-unit side.

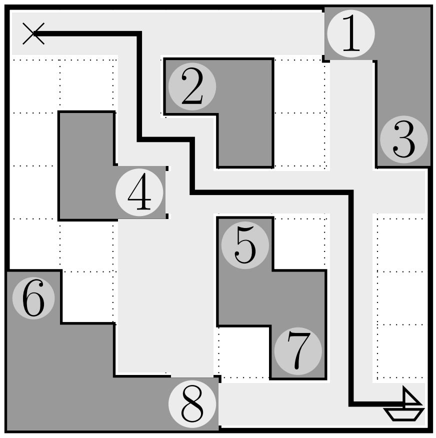
If the 8-unit long side is parallel to the 30-unit long side, then in this orientation, the rectangle can be positioned in $30 - 8 + 1 = 23$ different ways. With the 20-unit long side parallel to the 10-unit long side, allowing $20 - 10 + 1 = 11$ possibilities in this direction. Therefore, in the first case, the smaller rectangle can be positioned in a total of $23 \cdot 11 = 253$ ways.

If the 30-unit long side is parallel to the 10-unit long side, then in this orientation, there are $30 - 10 + 1 = 21$ possibilities. With the 20-unit long side parallel to the 8-unit long side, providing $20-8+1 = 13$ options in this direction. Therefore, in this case, there are $21 \cdot 13 = 273$ possibilities.

Therefore, there are a total of $253 + 273 = 526$ possible positions for the small rectangle.

(Back to problems)

13. By turning on lighthouses 1, 4 and 8, Lili can get to the cell marked with $\times$ the following way:



Nothing can illuminate the cell to the bottom left from lighthouse 1, therefore we cannot go to the right of the island of lighthouse 2. If we want to go to the left of the island of lighthouse 4, then both lighthouses 5 and 6 need to be turned on, and by turning on only one more lighthouse, Lili couldn't make her trip.

Therefore the path must go between the islands of lighthouses 2 and 4. This means that lighthouse 4 must be turned on, and in order to reach the end, one of 1 and 6 has to be turned on, and for the start at least one of 3 and 8. For getting across the strait between the islands of lighthouses 2 and 4, lighthouse 8 must be turned on, and therefore lighthouse 1 is also needed.

Therefore the only possibility is when lighthouses 1, 4 and 8 are turned on, meaning that the answer is 32.

(Back to problems)

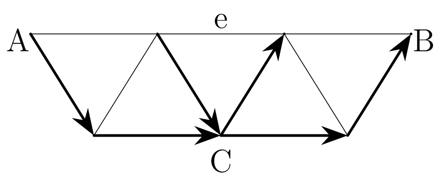
14. The pirate must break into at least 8 chests, because if at least 4 chests remained sealed throughout, and if the captain hid 2 gold blocks in each of those chests every day, the pirate could steal at most $13-4\cdot 2=5$ gold blocks. Therefore, the pirate must spend at least 8 nights breaking into new chests. It is always possible that a chest could be empty, so on those nights, it's uncertain if they can acquire any gold blocks. Hence, the pirate also needs at least another 7 nights. Thus, the pirate needs a total of at least 15 nights.

That's sufficient because if the pirate breaks into a new chest each of the first 8 nights and then inspects all the chests for the next 7 nights, they will always find a gold block during these nights. This is because in the 3 chests that remained sealed, there can be at most 6 gold blocks each night.

(Back to problems)

15. If the ship ever reaches the top row, it cannot reach B without passing through previously visited points: If it enters the top row at any point, then it can only proceed from there by moving left and up relative to the previous path. Therefore, in order to return to the other side of this arc, it must intersect at some point during subsequent steps, which will create a repetition.

Therefore, we can disregard the top row and the paths leading to it; it is sufficient to count the paths in the diagram below:



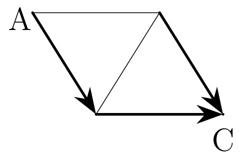
If at any point during the path it crosses the line labeled e or the point labeled C, then it cannot cross the other one. Therefore, there are two cases where the vertical bisector crosses either the line e or the point C.

If they pass through the line marked e:

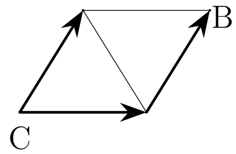
At the left end of the line, there are only 2 ways to reach it, and from the right end of the line to B, there are also 2 possible routes. These can be combined in any way, so there are a total of $2 \cdot 2 = 4$ possible ways to navigate from $A \to B$.

If they pass through the point marked C:

To get from A to C is shown in the figure below:



It can be seen here that there are 4 possible routes to reach there. Similarly, there are 4 ways to get from C to B:



Here, $A \to C$ and $C \to B$ paths can also be combined in any way, so there are $4 \cdot 4 = 16$ such paths.

Therefore, the ship can reach from A to B in a total of $4 + 16 = 20$ ways.

(Back to problems)

16. Firstly, let's observe that the number of colors is definitely a divisor of 180, because there are an equal number of balls in each color.

Let's consider when two different collections of 180 balls each can be mixed after drawing some balls. If one collection has a colors and the other has b colors, where $a < b$, then in one collection there are $180/a$ balls of each color, and in the other collection there are $180/b$ balls of each color. Therefore, if we see at most a different colored balls, with at most $180/b$ balls of each color among them, we cannot determine whether there are a colors or b colors in the collection. Hence, it is possible to draw $a \cdot 180/b$ balls without being able to determine the number of colors in the bag. However, if we draw more than this number, either we will have more than a different colors among the drawn balls, or there will be a color from which we draw more than $180/b$ balls, allowing us to exclude either a colors or b colors.

Based on this, one can consider that the task depends on which divisors a and b of 180 are chosen, where $a < b$, and maximizing the ratio a/b.

It's easy to see that these two divisors are 9 and 10. Based on the findings so far (with $a = 9$ and $b = 10$), if we draw $9 \cdot 180/10 = 162$ balls, we still cannot be certain whether there are 9 or 10 colors. However, if we draw at least 163 balls, we will definitely know the number of colors. This is because if we were to draw 163 balls and still couldn't determine whether there are a or b colors in the deck, where a and b are divisors of 180 and $a < b$, then according to the above observation, we could have drawn at most $a \cdot 180/b = 180 \cdot a/b$ balls. Hence, $a \cdot 180/b = 180 \cdot a/b \geq 163$, implying $a/b > 9/10$. However, we know that 9/10 was the highest ratio, so this is not possible.

Therefore, the solution is 163.

(Back to problems)

3.3.3 Category B

1. For the solution, see Category A Problem 2.

(Back to problems)

2. For the solution, see Category A Problem 4.

(Back to problems)

3. For the solution, see Category A Problem 5.

(Back to problems)

4. For the solution, see Category A Problem 6.

(Back to problems)

5. For the solution, see Category A Problem 7.

(Back to problems)

6. For the solution, see Category A Problem 9.

(Back to problems)

7. For the solution, see Category A Problem 10.

(Back to problems)

8. For the solution, see Category A Problem 11.

(Back to problems)

9. For the solution, see Category A Problem 12.

(Back to problems)

10. For the solution, see Category A Problem 13.

(Back to problems)

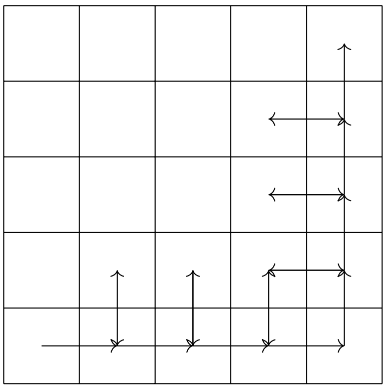
11. For the solution, see Category A Problem 14.

(Back to problems)

12. For the solution, see Category A Problem 15.

(Back to problems)

13. The following sequence works well in 20 steps: right, up, down, right, up, down, right, up, down, right, up, left, right, up, left, right, up, left, right, up. See the diagram.



We will now show that fewer than 20 steps are not sufficient. If the flea jumps to the right or upward (let's call these good steps), it gets closer to the goal, but if it jumps to the left or downward (let's call these bad steps), it gets farther away. Therefore, to reach the goal, the flea needs to take 8 more good steps than bad ones. According to the problem's condition, any three consecutive jumps must differ pairwise, ensuring there is at least one bad step among them. Based on this, the difference between good and bad jumps can increase by at most 1 every three jumps, and within three jumps, it can increase by at most 2. Hence, after 18 steps, the flea can take at most 6 more good steps than bad ones, and fewer than this number of steps are not enough. Therefore, after 18 steps, two more good steps are needed, totaling 20 steps in all.

(Back to problems)

14. In the current millennium, the years so far that include the digit 3 are: $2003, 2013, 2023$. In these years, there were a total of $3 \cdot 365 = 1095$ days, each of which satisfies the condition of the problem.

Outside of these years, from 2001 to 2023, there are 20 years. March is the third month, and none of the other month descriptions include the digit 3. March has 31 days, so in these twenty years, there were a total of $20 \cdot 31 = 620$ days in March, each of which meets the conditions of the problem.

The days of each month that contain the digit 3 (excluding March, as we have already counted the days in March):

January, May, July, August, October, and December have 31 days, so the suitable days in these months are: $03, 13, 23, 30, 31$. This is totals 5 days per month. In February, only the days $03, 13, 23$ are suitable. April, June, September, and November have 30 days each, so the suitable days in these months are: $03, 13, 23, 30$. Therefore, the total number of suitable days in a year excluding March is: $6 \cdot 5 + 1 \cdot 3 + 4 \cdot 4 = 49$.

Thus, from 2001 to the end of 2023, the number of suitable days is: $1095+620+20\cdot49 = 2695$.

In 2024, before today (January 13th), there was only one day (January 3rd) that contained the digit 3.

So today is the $2695 + 2 = 2697$.th such day in the millennium.

Second solution:

Including the leap days on February 29th, there would be a total of $23 \cdot 366 = 8418$ days from 2001 to 2023.

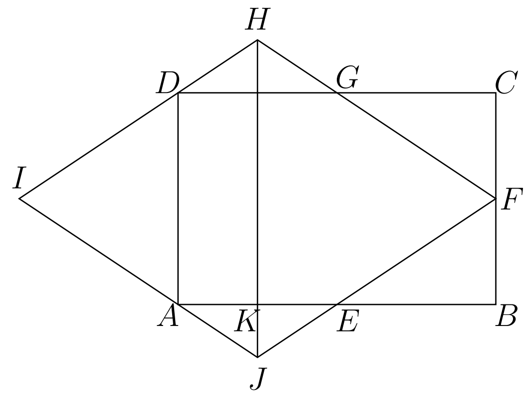
Let's count the days without the digit 3. From 2001 to 2023, there are 20 such years, each with 11 months, and each month has 26 days without the digit 3. Therefore, there are $20 \cdot 11 \cdot 26 = 5720$ days without the digit 3.

Based on this, there are $8418 - 5720 = 2698$ days that contain the digit 3. However, this count includes February 29th in the years 2003, 2013, and 2023, which are not actual days. Therefore, there were only 2695 days that contained the digit 3.

In 2024, before today (January 13th), there was only one day (January 3rd) that contained the digit 3. Therefore, today is the $2695 + 2 = 2697$th such day in the millennium.

(Back to problems)

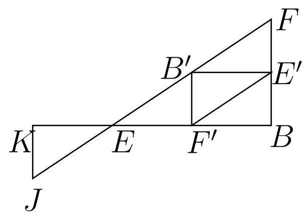
15.



Since $FHIJ$ is a rhombus, segment IH is of the same length as segment FJ, therefore it is enough to calculate the length of the latter.

Triangles GCF and EBF are congruent as their sides are pairwise the same: $CG = BE = \frac{AB}{2}, CF = BF = \frac{BC}{2}$. Therefore their hypotenuses are also of the same length and since triangle HFJ is isosceles, segment HJ is parallel to segments GE and BC.

Let K be the point of intersection of segments HJ and AB. Segment JK bisects andgle AJE, since it is a diagonal of a rhombus. We know that JK is perpendicular to AB, since it is parallel to BC. From this we get that triangles AJK and EJK are congruent, meaning that the length of segment KE is the half of segment AE, which is the quarter of segment AB. Now let's draw the section of the diagram consisting of triangles KEJ and FEB. The the midpoints of sides in triangle FEB be points E', F', B'.



By drawing the triangle $F'E'B'$ it cuts triangle FEB into four congruent triangles, furthermore JEK is also congruent with these triangles. This means that $JE = EB' = B'F$. Therefore $JF = JE + EB' + B'F = 3EB'$ and $EF = EB' + B'F = 2EB'$. From this we get that $JF = \frac{3}{2}EF = 108$ millimeters.

(Back to problems)

16. For the solution, see Category A Problem 16.

(Back to problems)

4 Solutions – high school categories

4.1 Online round

4.1.1 Tables

#	ANS	Problem	P
C-1	252	There is a lighthouse on each of the two each sides of a bay.	3p
C-2	8	A pirate crew wants to travel from Pineapple Island	3p
C-3	71	The pirates left a chest filled with treasure	4p
C-4	8	The crew of a ship consists of pirates and monkeys.	4p
C-5	78	The sail factory orders pieces of cloth, sized $5\,\text{m} \times 6\,\text{m}$.	4p
C-6	32	In a square-shaped room, there is a statue	5p
C-7	18	On the map seen below three letters A, B and C	5p
C-8	14	Timi received 27 identical white cubes	6p
C-9	44	How many such triangles are there that their vertices	6p

#	ANS	Problem	P
D-1	8	A pirate crew wants to travel from Pineapple Island	3p
D-2	71	The pirates left a chest filled with treasure	3p
D-3	8	The crew of a ship consists of pirates and monkeys.	4p
D-4	3552	Four sailors are stranded on a desert island.	4p
D-5	4	Unlucky Ubul and Unfortunate Ulrik sail out	4p
D-6	18	On the map seen below three letters A, B and C	5p
D-7	14	Timi received 27 identical white cubes	5p
D-8	21	Two pirates, Zorka and Kristof want to make a deal.	6p
D-9	26	Let P be a point indside the square $ABCD$	6p

#	ANS	Problem	P
E-1	8	The crew of a ship consists of pirates and monkeys.	3p
E-2	3552	Four sailors are stranded on a desert island.	3p
E-3	4	Unlucky Ubul and Unfortunate Ulrik sail out	4p
E-4	720	Captain Alex the pirate is preparing for	4p
E-5	18	On the map seen below three letters A, B and C	4p
E-6	14	Timi received 27 identical white cubes	5p
E-7	21	Two pirates, Zorka and Kristof want to make a deal.	5p
E-8	26	Let P be a point indside the square $ABCD$	6p
E-9	2727	The pirates left a chest filled with treasure	6p

4.1.2 Category C

1. We need to calculate the lowest common multiple of 18 and 28. Since $18 = 2 \cdot 3^2$ and $28 = 2^2 \cdot 7$ the lowest common multiple eguals $2^2 \cdot 3^2 \cdot 7 = 252$ Therefore the next time they send signals at the same time will be in 252 seconds.

(Back to problems)

2. There are three cases depending on whether we enter only the upper group of islands, only the lower group, or both. In the first case, there are 2 routes; in the second case, there are also 2 routes; and in the third case, we can choose between two routes both in the upper and the lower groups, giving us a total of 4 possibilities. Adding up all three cases, the total number of possible routes is $2 + 2 + 4 = 8$.

(Back to problems)

3. Let the number opening the lock be noted by n. Then, $n+1$ is divisible by both 3 and 8, and leaves a remainder of 2 when divided by 5. If a number is divisible by both 3 and 8, it is also divisible by 24. The first few multiples of 24 are: 24, 48, 72; their remainders when divided by 5 are 4, 3, and 2 respectively. Thus, the possible smallest value for $n+1$ is 72, leading to $n = 71$.

(Back to problems)

4. If we note the size of the original crew by n, then they had $1.75n = \frac{7}{4}n$ legs in total. After the arrival of the new monkey, the $n+1$ members had $\frac{7}{4}n+4$ legs in total and the new average number of legs was $\frac{\frac{7}{4}n+4}{n+1} = 2$. So, $2(n+1) = \frac{7}{4}n+4$, from which $2n+2 = \frac{7}{4}n+4$, meaning that $\frac{1}{4}n = 2$, so $n = 8$.

(Back to problems)

5. Since we cut a cloth with area of 30 m^2 into congruent rectangles with integer number of metres side-lengths, the area of each rectangle must be an integer, that is a divisor of 30. This gives us 8 possible areas: 1,2,3,5,6,10,15 and 30. The rectangle with area of 1 can only be a 1×1 rectangle. Since 2,3 and 5 are prime numbers, the rectangles with these areas can only be 1×2, 1×3 and 1×5 respectively, and the cloth can indeed be cut into such pieces. If the area is 6 the rectangle can be either 1×6 or 2×3, and the cloth can be covered with both kinds of rectangles. Rectangles with areas of 10,15 or 30 can only be 2×5 , 3×5 and 5×6 respectively, since if we divide 10, 15 or 30 with other numbers that are at most 6, the result is always greater than 6, meaning the cloth cannot be cut into such pieces. The total area of all the possible sails: $1+2+3+5+2\cdot 6+10+15+30 = 78$.

(Back to problems)

6. Since we cannot move the bottom left statue, it can only face either up or right. In both cases, the other three statues have the same number of suitable position combinations, so it is sufficient to count how many such cases exist when the bottom left statue is facing up (and then multiply by 2).

Note that it does not matter in which order the two buttons are pressed, only how many times each button is pressed. Thus, we can assume that we first press the button that moves the two statues at the top as many times as we think is needed, and only then press the other button. If we press a button 4 times, the situation does not change anything in total, so we can assume that each button is pressed at most 3 times. In this case, if we count the good initial positions based on whether each button is pressed 0, 1, 2, or 3 times separately, there are a total of $4\cdot 4$ cases since each option yields a different initial position. Therefore, there are $2\cdot 4\cdot 4 = 32$ suitable initial positions, in total.

(Back to problems)

7. In order to solve the problem we will write an arrow in each cell that points exactly in the direction of the wind. Since the ship departing from X has reached cell B, there exists a path from the arrows that leads from X to B. This path, including the endpoints, consists of at least 7 squares (3 arrows to the right and 3 arrows upward, plus cell B). If we join a path at any point, from then on, we cannot deviate from it and must follow its line. Therefore, if we want to reach cell A from any other cell, our path must not have any common squares (i.e., they cannot intersect) with either the $X\to B$ or the $Y\to C$ paths.

This means that from the squares below the $Y\to C$ path, it is definitely impossible to reach A. Thus, we get the maximum number of cells from which we end up in A if the $Y\to C$ path runs along the bottom edge of the map, as shown in the picture.

					B
					A
Y					
$\downarrow$		X			C
$\downarrow$					$\uparrow$
$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\uparrow$

Then, there are $36 - 7 - 11 = 18$ cells left from which it is still possible to reach A. This is actually possible from all cells, if the arrows are arranged in the following way:

$\rightarrow$	$\downarrow$	$\leftarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
$\rightarrow$	$\downarrow$	$\rightarrow$	$\uparrow$	$\rightarrow$	$\rightarrow$
$\downarrow$	$\downarrow$	$\uparrow$	$\rightarrow$	$\uparrow$	$\uparrow$
$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\rightarrow$
$\downarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\uparrow$	$\uparrow$
$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\uparrow$

(Back to problems)

8. After both of the paintings exactly one cube will have 3 faces painted, 6 cubes will have 2 faces painted and 12 cubes will have 1 face painted. This means that every cube that has at least 3 red faces after the second painting must have 2 faces that were painted during the same painting session. Therefore there are at most 14 cubes, that can have at least 3 red faces. This can be achieved if the 1-1 cubes that have 3 faces painted during one painting do not have any more faces painted during the other session and the 6-6 faces cubes that had 2 faces painted in one session have one other face painted in the other session.

(Back to problems)

9. Let the vertices of the square be A, B, C, D and the midpoints of the sides be E, F, G, H in clockwise direction such that E is on the side AB. We consider the following two possibilities: The vertex of the triangle with the largest angle is either a vertex of the square or the midpoint of a side of the square.

Case 1: The vertex of the square: Let's assume that the vertex is A, since the case is symmetric for other vertices. Then, one of the remaining two vertices of the triangle is either D or H, the other is either B or E. Any combination of these does form a triangle giving us 4 possibilities. Since the same argument works for all 4 vertices of the square, we have $4 \cdot 4 = 16$ possibilities.

Case 2: The vertex with the largest angle is the midpoint of one of the sides of the square: We can assume this vertex is E, since the case is symmetric for the other midpoints. Assuming that E is the midpoint of the top horizontal side, the remaining two vertices cannot be on the same vertical line, because then the largest angle would not be at E. Therefore, one vertex has to be to the left of the other vertex. The vertex to the left can be any of A, H, D or G. The number of possible vertices on the right is 3,2,1,1 respectively, which adds up to 7. Since this is true for all midpoints, the number of possible triangles is $4 \cdot 7 = 28$.

In total that is $16 + 28 = 44$ possible triangles.

(Back to problems)

10. (Game) We can win by playing as the second player. In this case, the opponent must start by taking one chip, leaving 16 chips in the pile. From then on, whenever it is our turn, we should take one chip from the pile, changing the currently even-sized pile to an odd number. After that, the opponent won't be able to halve the odd-sized pile, so they will have to take one chip again, making the pile even once more. We should continue this strategy until there are 4 chips left. At that point, we halve the pile, leaving 2 chips. The opponent can either halve these 2 chips or take 1 chip, but these options are equivalent, so 1 chip will remain, which we will take to win.

(Back to problems)

4.1.3 Category D

1. For the solution, see Category C Problem 2.

(Back to problems)

2. For the solution, see Category C Problem 3.

(Back to problems)

3. For the solution, see Category C Problem 4.

(Back to problems)

4. At first, we do not distinguish the sailors from one another and just count the number of possible time intervals during wich the stations are occupied. At the end of the solution we multiply this by $4 \cdot 3 \cdot 2 \cdot 1$ since that is the number of possible ways to assign the sailors to the designated places and time intervals. We examine the two stations according to the number of sailors signaling at each:

(i) If there are 3 sailors sending signals from the northern station, their scheduling can be done in 4 ways, since out of the 7 hours between 8 am and 3 pm 6 hours are occupied and the o free hour can be before the first sailor, between the first and second sailor, between the second and third or after the third sailor.Meanwhile, the one sailor at the southern station can begin signaling at the start of any hour between 8 am and 1 pm: that gives us 6 possibilities.Since the events happening at the two stations are independent so we multiply the possible cases. That is $4 \cdot 6 = 24$.

(ii) If there are 3 sailors signaling at the southern stations, the same argument works, giving us 24 cases.

(iii) If there are 2 sailors signaling at both stations the sailors at the northern station can start signaling at 8 am and 10 am, 8 am and 11 am, 8 am and 12 pm, 8 am and 1 pm, 9 am and 11 am, 9 am and 12 pm, 9 am and 1 pm, 10 am and 12 pm, 10 am and 1 pm, 11 am and 1 pm, giving us 10 possibilities. The same can be stated about the two sailors at the southern station.Therefore the number of possibilities is $10 \cdot 10 = 100$

The total number of possibilities is $(24 + 24 + 100) \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3552$

(Back to problems)

5. Our aim is to place as few treasures as possible into the 16 sectors such that each sector or one of the sectors that share a side with it contains a treasure. Note that no treasure can be adjacent with more than one corner of the sea.Therefore we need at least 4 treasures since covering the 4 corners already requires 4 treasures. We show that 4 is enough to cover the whole sea, as shown in the figure below, where the letter T denotes the treasures.

		T	
T			
			T
	T		

(Back to problems)

6. For the solution, see Category C Problem 7.

(Back to problems)

7. For the solution, see Category C Problem 8.

(Back to problems)

8. Two pirates can trade only one three-dollar bill, by exchanging a three-dollar and a one-dollar-bill and the one who initially had the three-dollar bill, also gives back the one-dollar bill. So, in this exchange, only a three-dollar bill changed owner. This means that if someone has a one-dollar bill and the other a three-dollar one, the first person can get the three-dollar bill without other notes changing owner.

Similarly, two pirates can trade only one five-dollar bill if they have at least one three-dollar and one one-dollar bill, too. First, the receiver gets a three-dollar bill with the previously described method, if they don't already have one. Then, they exchange the five-dollar and three-dollar bills. Finally, they return the three-dollar bill to its original owner by the previously described method.

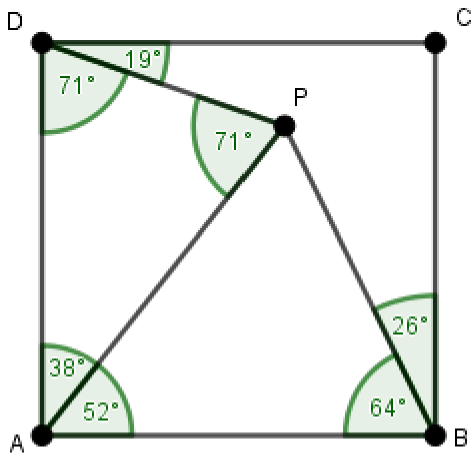
Since Zorka and Kristof have at least one of all three kind of notes, they can actually exchange one-, three-, and five-dollar bills freely, by the previously described methods, so that they follow the Ancient Pirate Law. So, Zorka can have any sum that can be made by choosing some notes from one one-dollar bill, two three-dollar ones, and three five-dollar ones. So, the highest possible value is $1 + 2 \cdot 3 + 3 \cdot 5 = 22$. Between 0 and 22, every value is possible, except for 2 and 20, because 2 would require two one-dollar bills then Kristof would have to have 2. Other values are possible with the following constructions:

$0 = 0$
$1 = 1$
$3 = 3$
$4 = 3 + 1$

$5 = 5$
$6 = 5 + 1$
$7 = 3 + 3 + 1$
$8 = 5 + 3$
$9 = 5 + 3 + 1$
$10 = 5 + 5$
$11 = 5 + 5 + 1$
$12 = 5 + 3 + 3 + 1$
$13 = 5 + 5 + 3$
$14 = 5 + 5 + 3 + 1$
$15 = 5 + 5 + 5$
$16 = 5 + 5 + 5 + 1$
$17 = 5 + 5 + 3 + 3 + 1$
$18 = 5 + 5 + 5 + 3$
$19 = 5 + 5 + 5 + 3 + 1$
$21 = 5 + 5 + 5 + 3 + 3$
$22 = 5 + 5 + 5 + 3 + 3 + 1$
So, in total, 21 different values are possible to be on Zorka's paper.

(Back to problems)

9. We know that $\sphericalangle PAD = 90° - \sphericalangle PAB = 38°$ and $\sphericalangle ADP = 90° - \sphericalangle CDP = 71°$. So, due to the sum of the angles of triangle ADP, $\sphericalangle DPA = 71°$ which means that triangle ADP is isosceles and $AP = AD$. Furthermore, $AD = DB$ since they are both the sides of the square, so ABP is also an isosceles triangle, which means that $\sphericalangle ABP = \sphericalangle APB$. Then, due to the sum of the angles of triangle ABP, $\sphericalangle PBA = \frac{180° - \sphericalangle PAB}{2} = 64°$, so $\sphericalangle PBC = 90° - \sphericalangle PBA = 26°$.



(Back to problems)

10. Game: Let us call a number n a *winning number* if a player starting with a pile of n chips can win. (Otherwise, we will call n a *losing number*.)

Clearly, 1 is a winning number.

2 is a losing number because regardless of whether we take one chip or halve the pile, there will always be one chip left, allowing the other player to win.

3 is a winning number because by taking one chip, we reduce the pile to 2, from which the other player loses, meaning we win.

Next, we will use complete induction to prove that any n greater than 3 is a winning number if and only if it is even.

From the position $n = 4$, we can move to the losing position 2 by halving the pile, thus 4 is a winning number. Now, let $n \geq 5$. We consider the following two cases based on the parity of n:

- If n is even, by taking one chip we can move to $n - 1$, which is odd. According to the induction hypothesis, $n - 1$ is a losing number, thus n is a winning number.
- If n is odd, we can only take one chip (since we cannot halve the pile), moving to $n - 1$, which is even. According to the induction hypothesis, $n - 1$ is a winning number, thus n is a losing number.

With this description, we have determined whether starting with a given number is advantageous or not and provided the winning strategy.

(Back to problems)

4.1.4 Category E

1. For the solution, see Category C Problem 4.

(Back to problems)

2. For the solution, see Category D Problem 4.

(Back to problems)

3. For the solution, see Category D Problem 5.

(Back to problems)

4. Place the chests from largest to smallest, and keep track of how many free slots are available for placing the next chest.

The largest chest can be placed in 1 way, and there are 2 available slots (the golden and the silver slots) for the next chest. Therefore, the next chest can be placed in 2 ways, and when placing it, the number of free slots increases by 1 (since the new chest occupies one slot but has two slots, opening up 2 additional slots for further chests). Similarly, with each subsequent chest placement, the number of free slots increases by 1, so there is one more option for placing each chest than for the previous one. Therefore, the total number of possibilities is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6! = 720$.

(Back to problems)

5. For the solution, see Category C Problem 7.

(Back to problems)

6. For the solution, see Category C Problem 8.

(Back to problems)

7. For the solution, see Category D Problem 8.

(Back to problems)

8. For the solution, see Category D Problem 9.

(Back to problems)

9. We know that the code contains either 4 different digits, or 3 different digits with one digit repeated twice, or 2 different digits with each appearing twice or one appearing once and the other three times, or all four digits are the same. In each case, upon listing the possible numbers, it is observed that the sum will be divisible by 1111 as it. Therefore, the code itself is divisible by 101, meaning it is of the form $\overline{abab}$.

So, the sum of the numbers formed by rearranging the digits will be $\overline{abab}+\overline{abba}+\overline{aabb}+\overline{baba}+\overline{bbab}+\overline{baab}=3333(a+b)$, implying that $303(a+b)=\overline{abab}$. Since the left side is divisible by 3, the right side must also be divisible by 3, indicating that $2a+2b$ is divisible by 3, due to the divisibility rule for 3, so $a+b$ is divisible by 3, too.

Now, since the left side is divisible by 9, the right side must also be divisible by 9, too, meaning that $a+b$ is divisible by 9, due to the divisibility rule for 9 and for the same reason we had for 3, previously. Therefore, the left side is divisible by $101 \cdot 27$, and hence so is the right side, yielding that the possible codes are $2727, 5454$, and 8181. Checking these, only 2727 satisfies all conditions.

Therefore, the code is 2727.

(Back to problems)

10. For the solution, see Category D Problem 10.

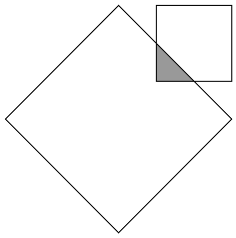
(Back to problems)

4.2 Regional round

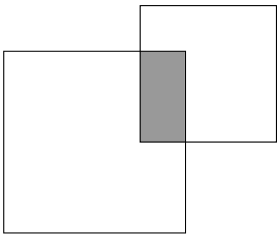
4.2.1 Category C

1. We examine the possible intersection according to the number of sides and right angles:

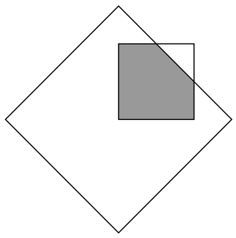
The intersection has 3 sides and 1 right angle:



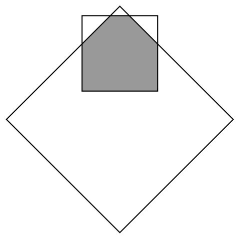
The intersection has 4 sides and 4 right angles:



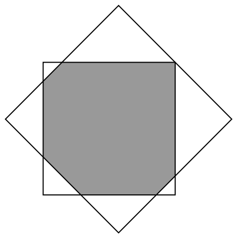
The intersection has 5 sides and 3 right angles:



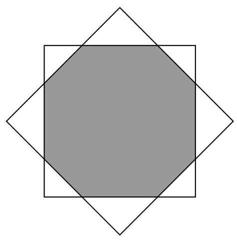
The intersection has 6 sides and 2 right angles:



The intersection has 7 sides:



The intersection has 8 sides and no right angles:



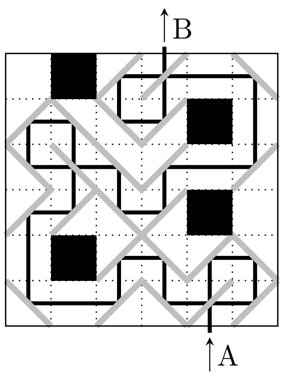
a) The figures above show that it is possible to draw examples with $3, 4, 5, 6, 7$ and 8 sides. Clearly, the intersection cannot have less than 3 sides, since every polygon has at least 3 sides. Furthermore, since squares are convex, a line cannot intersect its perimeter at more than 2 points. Therefore, if we fix one square and place the other square in any position, each of its sides intersect the fixed square at two points, at most. This means that the perimeters of the two squares intersect each other at 8 points, at most, which implies that the intersection, as a polygon, cannot have more than 8 vertices.

b) It can be seen in the figures above that $0, 1, 2, 3, 4$ right angles are possible. We will show that more right angles are never possible. The sum of the exterior angles of any convex polygon is $360°$. The exterior angle corresponding to a right angle is $90°$. The intersection of the two squares is a convex polygon, so it cannot have more than 4 right angles, because then the sum of the exterior angles corresponding to the right angles alone would be greater than $360°$.

(Back to problems)

2. Mesi's diet can be successful for 4 days in a row at most, starting from Monday, if she eats $3, 2, 1, 0$ Turo Rudis. Therefore, her diet will be unsuccessful by Friday at the latest. Similarly, her diet will be unsuccessful at Thursday, the earliest, because the longest successful sequence up to Sunday can be similarly, $3, 2, 1, 0$, which is 4 days. Therefore, only Thursday or Friday can be the days when her diet is not successful. Hence, in the four-day successful sequence, she eats $3, 2, 1, 0$, and in the other sequence, she eats $3, 2, 1, 0$ with one of these numbers left out. So the possible sequences are:
$2, 1, 0, 3, 2, 1, 0$
$3, 1, 0, 3, 2, 1, 0$
$3, 2, 0, 3, 2, 1, 0$
$3, 2, 1, 3, 2, 1, 0$
$3, 2, 1, 0, 2, 1, 0$
$3, 2, 1, 0, 3, 1, 0$
$3, 2, 1, 0, 3, 2, 0$
$3, 2, 1, 0, 3, 2, 1.$

(Back to problems)



3. In the cells where there are no mirrors in this construction (there are 7 such cells), no mirror can be placed that would break the path of light, anyway. This is because, in that case, the light would either collide into a pillar or exit the room elsewhere. Therefore, the maximum number of mirrors that can be placed in this way, is 25.

(Back to problems)

4. Let the 3 numbers be $1, 2$ and 4. Using these we can generate all ten numbers as follows:
$1 = 4 - 2 - 1$
$2 = 4/2 \cdot 1$
$3 = 4 + 1 - 2$
$4 = 4/(2 - 1)$
$5 = 4 + 2 - 1$
$6 = 4 + 2/1$
$7 = 4 + 2 + 1$
$8 = 4 \cdot 2 \cdot 1$
$9 = 4 \cdot 2 + 1$
$10 = 2 \cdot (4 + 1)$

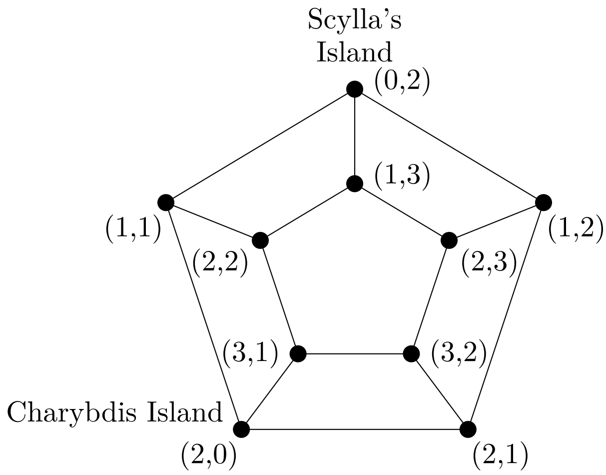
Remark : A curious reader may rightly ask, "Okay, okay, it's great that this is the best solution because we can generate all ten numbers with it, but how can one come up with this?" We don't have a completely precise answer to this. After a number of tries one can notice that smaller numbers allow for more combinations. The set $1, 2, 3$ also seems like a promising candidate, but the problem with it is that 10 is too large and cannot be achieved. However, it's not bad either, as the other nine numbers can all be created. Actually, there is no other set of three numbers besides $1, 2, 4$ that allows all numbers to be generated.

For this task, we did not expect the teams to find this best solution. We assumed that even among the best teams, very few would come up with it. However, mathematics is like this; often, even mathematicians have no chance of finding the best answer to an unsolved problem. In such cases, they try to find the best possible solution. The grading of this task was also like that: the more numbers a team could generate, the more points they earned.

(Back to problems)

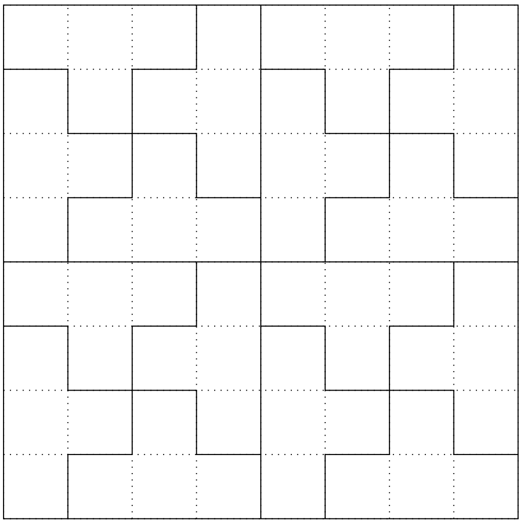
5. It is not enough for Leila to only call Lily: if, for instance, the treasure is just one ship trip away from Scylla's Island, she would not be able to determine which neighbouring island it is on.

Let's call the bottom left island on the map Charybdis Island. We will see that it is sufficient for Leila to only call Scylla and Charybdis Islands, and based on the obtained information, she can determine the treasure island uniquely. The treasure island can be determined uniquely if and only if for any two islands, it holds that they are not the same distance away from either Scylla or Charybdis. On the map, we have written two numbers for each island, the first being the minimum number of ship trips needed from Scylla to get there, and the second being the number of trips needed from Charybdis. It can be seen that there are indeed no two islands for which we wrote the same pair of numbers. Therefore, by calling only Scylla and Charybdis Islands, Leila can indeed determine where the treasure is, whatever her friends' replies are, hence the answer to the problem is two.

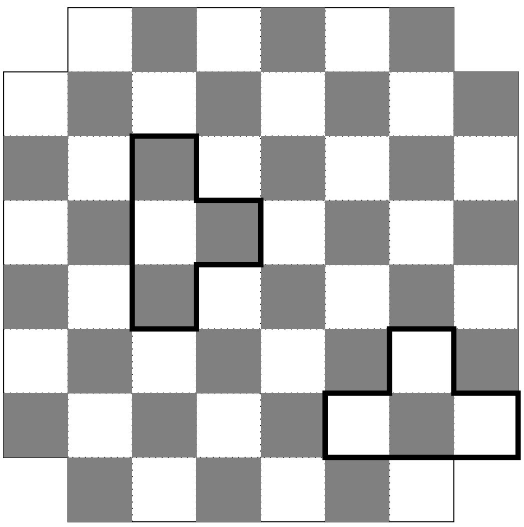


(Back to problems)

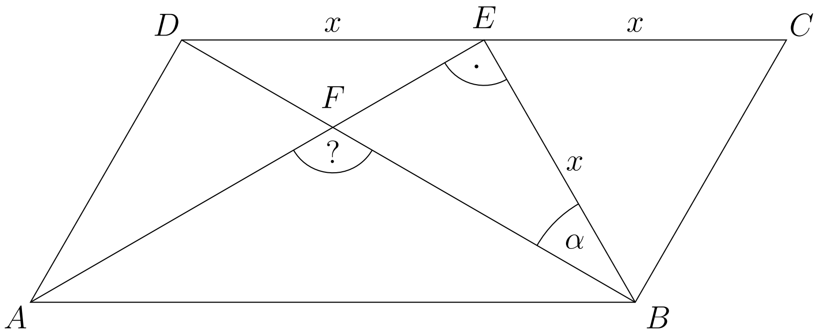
6. **a)** Yes, it is possible, for example, as shown below:



b) Let us colour the board with black and white, like a chessboard. This way, the board consists of 30 black and 30 white squares. No matter how we place a T-tetromino, it always covers either 3 black and 1 white squares or 1 black and 3 white squares. Since the number of black squares is the same as the number of the white squares, we must use as many tetrominos with 3 black squares and 1 white square as tetrominos with 3 white squares and 1 black square. However, the number of tetrominos is 15, which is an odd number, so this is not possible. Therefore it is not possible to cover the board with 15 T-tetrominos.



(Back to problems)



7. Let $\angle DBE = \alpha$. As $EB = ED = x$, triangle DEB is isosceles, therefore $\angle BDE = \angle DBE = \alpha$. As $\angle BDE$ and $\angle ABD$ are alternate angles, we have $\angle ABD = \alpha$.

Since E is the midpoint of side CD, we have $ED = EC = x$, from which $CD = 2x$. As $ABCD$ is a parallelogram, we have $AB = CD = 2x$, so in the right-angled triangle AEB, the leg EB is half as long as the hypotenuse AB, therefore AEB is half of an equilateral triangle. Because of this, $ABE\sphericalangle = 2\alpha = 60°$, so $\alpha = 30°$, meaning that $EAB\sphericalangle = 30°$.

Now considering the angles of triangle AFB and rearranging the equality we get that

$$AFB\sphericalangle = 180° - 2 \cdot 30° = 120°.$$

(Back to problems)

4.2.2 Category D

1. Since every digit must appear at least once, the total sum is at least $1 + 2 + \ldots + 9 = 45$. As the sum of the numbers must be the same in each column the total sum of all the digits on the board has to be divisible by 4, so at least 48 (as 48 is the smallest multiple of 4 that is not less than 45). In that case, the sum of the digits in each row and column is 12. This can be achieved as shown below:

9	3	0	0
2	8	1	1
0	0	7	5
1	1	4	6

(Back to problems)

2. For the solution, see Category C Problem 3.

(Back to problems)

3. For the solution, see Category C Problem 4.

(Back to problems)

4. If the greatest common divisor of two integers is 1 then we call them coprime integers.

The smallest possible example consists of 12 numbers. An example of 12 numbers originally found on a table: 6, 7, 7, 7, 7, 7, 8, 8, 8, 9, 9, 11. It can be seen that for every number here the number of numbers it is coprime with equals the number itself and there are two numbers that are not coprimes (for example 6 and 9).

Let's prove by contradiction that a construction like this cannot be created using less than 12 numbers. Suppose that there is an example consisting of less than 12 numbers, that satisfies the conditions. 1 can not be on the board since it is coprime with any positive integer, so we can only write 1 under it if there is only 1 number on the board, which is not allowed.

Every number on the board is at most 10, since 1 is the only positive integer that is coprime with itself and there is no 1 on the board. So, if a number would be more than 10, all numbers on the board should be coprime with it, including itself. Therefore the possible numbers that can appear on the board, range from 2 to 10.

Now, we prove that there cannot be two different numbers on the board which have the same prime factors. Suppose there are two numbers a and b such that $a \neq b$ and for every prime p, $p|a \Leftrightarrow p|b$. In this case, it is true for every positive integer x that $(x, a) = 1 \Leftrightarrow (x, b)$ Therefore a and b are coprimes with the exactly the same numbers, so we write the same number under them, but this a contradiction, as under every number, we write itself, and $a \neq b$. Therefore, we cannot have more than one number from the sets $\{2, 4, 8\}$ and $\{3, 9\}$, each.

If neither 6 nor 10 would be on the board, then all the numbers would be from the set $\{2, 3, 4, 5, 7, 8, 9\}$. However, this set only contains powers of primes and we cannot use 2 different powers of the same prime. Therefore, any two numbers would be coprimes which contradicts the conditions, so either 6 or 10 must be on the board. 10 can only be on the board if there are 11 numbers and 10 is coprime with every other number. This is only possible if every other number is an element of the set $\{3, 7, 9\}$ but this makes it impossible to have two different numbers that are not coprimes, since we already proved that 3 and 9 cannot both appear on the board and all the other pairs are coprimes. Therefore, 10 cannot be on the board.

Now, we know that the number 6 must be on the board. 2 cannot be on the board, since for every positive integer n, $(n, 6) = 1 \Rightarrow (n, 2) = 1$. Therefore, the number written under 2 would be at least as large as the one written under 6, which contradicts the conditions. The same argument implies that 3 or 4 cannot be on the board either.

This means that numbers on the board come from the set $\{5, 6, 7, 8, 9\}$ Let a_i be the number of times i appeared on the table initially for all i. Then, from the number of numbers that are coprimes with 6 we get $6 = a_5 + a_7$. Let's assume that there is no 8 or 9 on the board. Then, every number would be an element of the set $\{5, 6, 7\}$ so there would be no pair of different numbers that are not coprimes, which contradicts our conditions. Therefore, 8 or 9 appears on the board.

If there is an 8 then $8 = a_5 + a_7 + a_9$ so $8 = 6 + a_9$ meaning $a_9 = 2$. If there is a 9 we have: $9 = a_5 + a_7 + a_8 = 6 + a_8$ meaning $a_8 = 3$ Therefore both numbers are on the board, with 3 eights and 2 nines.

If there is a 5 then $5 = a_6 + a_7 + a_8 + a_9$. However, since we have at least 1 six, we have $a_6 \geq 1$ and we know that $a_8 + a_9 = 3 + 2 = 5$, so $a_6 + a_7 + a_8 + a_9 \geq 6$ which gives us a contradiction. Therefore we have no 5.

If there is a 7, $7 = a_5 + a_6 + a_8 + a_9 = 0 + a_6 + 3 + 2$, from wich we have $a_6 = 2$. We have 9 on the board so $9 = a_5 + a_7 + a_8 = 0 + a7 + 3$ from which $a_7 = 6$ so the total number of numbers on the board is $a_6 + a_7 + a_8 + a_9 = 2 + 6 + 3 + 2 = 13$, which is too many.

There is no 7 on the board. For the 9 we have: $9 = a_5 + a_7 + a_8 = 0 + 0 + 3$ which is a contradiction. So, there were at least 12 numbers on the board initially.

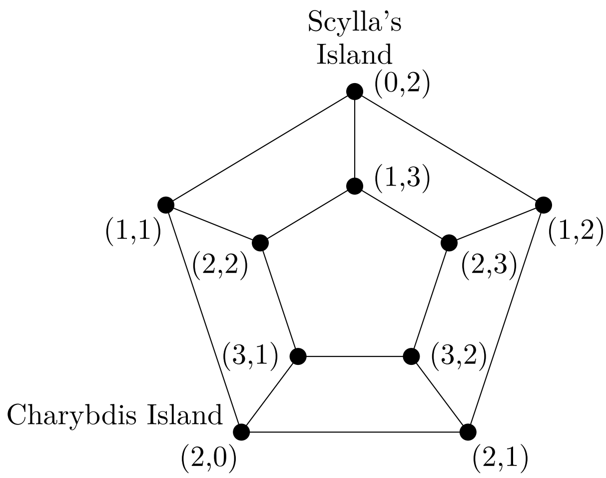
(Back to problems)

5. For the solution, see Category C Problem 7.

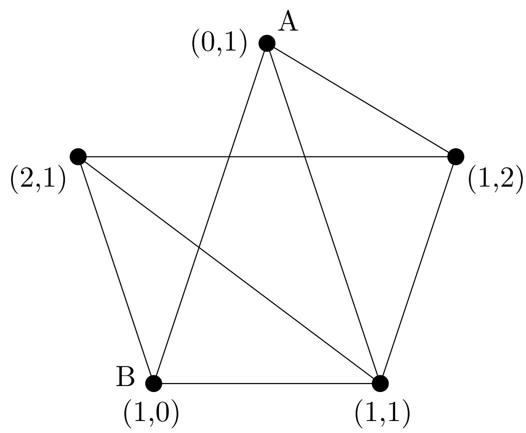
(Back to problems)

6. a) It is not enough for Leila to only call Lily: if, for instance, the treasure is just one ship trip away from Scylla's Island, she would not be able to determine which neighbouring island it is on.

Let's call the bottom left island on the map Charybdis Island. We will see that it is sufficient for Leila to only call Scylla and Charybdis Islands, and based on the obtained information, she can determine the treasure island uniquely. The treasure island can be determined uniquely if and only if for any two islands, it holds that they are not the same distance away from either Scylla or Charybdis. On the map, we have written two numbers for each island, the first being the minimum number of ship trips needed from Scylla to get there, and the second being the number of trips needed from Charybdis. It can be seen that there are indeed no two islands for which we wrote the same pair of numbers. Therefore, by calling only Scylla and Charybdis Islands, Leila can indeed determine the treasure island uniquely, hence the answer to the problem is two.



b) Consider the following configuration:



If we label the top island as A and the bottom left one as B, then similarly to the previous part of the problem, by assigning pairs of coordinates based on these, each island will have different coordinates (see diagram). Thus, it is sufficient to call these two islands to determine the location of the treasure. In this setup, there are 8 ship connections, and it only remains to show that there cannot be more than this.

Having 10 ship connections is clearly impossible because no matter which two islands we choose to call, the other three islands are at a distance of 1 from both, hence they cannot be distinguished.

Similarly, 9 ship connections cannot work because calling any two islands would result in another pair of islands, each at a distance of 1 from both of the called islands. (There is one pair of islands not connected, say A and B, and one has to check three cases depending on whether we call both of A and B, or just one of them and one other island, or two other islands.) Therefore, the location of the treasure cannot be determined uniquely in this case either.

Therefore the answer to the problem is 8.

(Back to problems)

7. We will prove the claim by contradiction: let us suppose there exists an a_n such that $a_n < a_1$, and we will show that it leads to a contradiction.

First we show by induction that if there is a value k for which $a_k < a_1$ and $a_{k+1} < a_1$ both hold, then $a_{2023} < a_1$, which is a contradiction. After this we will show that if there is an a_n such that $a_n < a_1$, then $a_{n+1} < a_1$ holds as well, which is sufficient to complete the proof.

From the first assumption let us show that for all $l > k+1$, $a_l < a_1$ holds. The assumption was $a_k < a_1$ and $a_{k+1} < a_1$, now we get that $a_{k+2} = \frac{a_k + a_{k+1}}{2} - 1 < \frac{2a_1}{2} - 1 < a_1$. So we have that if two consecutive elements are smaller then a_1, then all later elements are smaller than a_1. So this would mean $a_{2023} < a_1$, which is a contradiction.

Now we only have to show that if n is the smallest for which $a_n < a_1$, then $a_{n+1} < a_1$ holds as well. If $n = 2$, then

$$a_3 = \frac{a_1 + a_2}{2} - 1 < \frac{2a_1}{2} - 1 < a_1.$$

If $n > 2$, then we know that $a_n < a_1 \leq a_{n-2}$. And $a_{n+1} = \frac{a_{n-1} + a_n}{2} - 1 < \frac{a_{n-1} + a_{n-2}}{2} - 1 = a_n < a_1$.

So we showed the following, which is sufficient to prove that $a_n \geq a_1$ for all $1 \leq n \leq 2023$. If there is an n for which $a_n < a_1$, then there is also an n for which $a_n < a_1$ and $a_{n+1} < a_1$.

And if there is such an n, then for all $k > n$ we have $a_k < a_1$, namely $a_{2023} < a_1$, which is a contradiction.

(Back to problems)

4.2.3 Category E

1. For the solution, see Category D Problem 6.

(Back to problems)

2. For the solution, see Category C Problem 7.

(Back to problems)

3. Let us examine how the parity of the total number of 'yes' answers depends the last person.

If the last person to answer is a knight and the number of 'yes' answers before his response is even, then he answers 'yes'. This results in an odd number of 'yes' answers overall. On the other hand, if the number of 'yes' answers before the last knight's response is odd, then the answer is 'no'. This also leads to an odd number of 'yes' answers overall.

However, if the last person to answer is a knave, and the number of 'yes' answers before his response is even, then he answers 'no'. This results in an even number of 'yes' answers overall. Conversely, if the number of 'yes' answers before the last knave's response is odd, then the last answer is 'yes'. This also leads to an even number of 'yes' answers overall.

Since there are both knights and knaves among the people, there will be arrangements where the last responder is a knight and others where it is a knave. However, in one case, the number of 'yes' answers is odd, while in the other case, it is even. Therefore, there is no arrangement where, regardless of the starting person, Mark receives the same number of 'yes' answers.

In conclusion, there is no seating for which Mark gets the same number of 'yes' answers regardless of whom he asks first.

(Back to problems)

4. For the solution, see Category D Problem 7.

(Back to problems)

5. We will show that if n is odd then there is no glorious initial configuration; if n is a power of 2 then all initial configurations are glorious, and if n is even but not a power of 2 then there exist both glorious and non-glorious initial configurations.

A configuration is glorious if and only if it reaches the all-0 state, since if one person's number does not change, then their right neighbour had to have 0, therefore if no one's number changes, then it means that everyone had 0.

Firstly we show that if n is odd, then there is no way of reaching all zeros. Suppose the opposite. Then before the all zero state let the number of the ith person be k_i. We know that $k_i + k_{i+1} \equiv 0 \ (mod\ n)$ for all $i \ (mod\ n)$, meaning that $k_i \equiv -k_{i+1} \ (mod\ n)$, therefore $k_i \equiv k_{i+2} \ (mod\ n)$. Continuing this we get that $k_i \equiv k_{i+2} \equiv \ldots \equiv k_{i+2n-2} \ (mod\ n)$, this includes everyone since 2 is coprime to n. Therefore $k_i \equiv k_{i+1} \ (mod\ n)$ holds as well, but since $k_i \equiv -k_{i+1} \ (mod\ n)$, this means that $k_{i+1} \equiv 0 \ (mod\ n)$, therefore all k_i are zero. Therefore before the all zeros state there had to be all zeros as well, therefore this position cannot be reached from any starting position if n is odd.

Now we show that if n is even but not a power of 2, then there are both glorious and non-glorious configurations. Let the numbers of the $n = 2l$ people be $0, 1, -2, 3, -4, \ldots, -(2l-2), 2l-1$ in this order. (These are $2l$ numbers and are all different modulo n.) Then after the first step the numbers are $1, -1, 1, -1, \ldots, 1, -1$, meaning that they reach all zeros after the second step. However if the initial configuration is $0, 1, \ldots, 2l-2, 2l-1$, then it is not glorious: let $n = 2^a b$ where a is a positive integer and $b > 1$ is odd. After the first step the numbers are $1, 3, 5, \ldots, 4l-5, 4l-3, 2l-1$, meaning that the difference between neighbours is always 2. This means that in the next step the difference will be 4 everywhere and after the a steps the difference will be 2^a. Since 2^a is coprime to b, this means that after a steps the first b numbers will be different modulo b and the $b+1$th one will be the same as the first one modulo b. Moreover modulo b the numbers form 2^a cycles, each cycle containing all remainders. Therefore from here on each cycle will behave as if there were only b people, and since b is odd, the all zero state can never be reached. Therefore if n is even but not a power of 2, then both types of configurations exist.

Finally we show that if n is a power of 2, then all configurations are glorious. Let $n = 2^a$ and the initial numbers in order be $k_1, k_2, \ldots, k_{2^a}$. Since in every step everyone adds the right hand neighbour's number to theirs, after m steps the ith person will have

$$\binom{m}{0}k_i + \binom{m}{1}k_{i+1} + \binom{m}{2}k_{i+2} + \cdots + \binom{m}{m}k_{i+m}$$

where the index of k is considered modulo n. Let us observe the case where $m = 2^{a+1} - 1$. We know that $\binom{2^{a+1}-1}{j}$ is odd for all $0 \le j \le 2^{a+1} - 1$, therefore after $2^{a+1} - 1$ steps the number of the ith person is

$$\binom{2^{a+1}-1}{0}k_i + \binom{2^{a+1}-1}{1}k_{i+1} + \cdots + \binom{2^{a+1}-1}{2^{a+1}-1}k_{i+2^{a+1}-1},$$

where there are 2^{a+1} terms, meaning that every k_j appears in exactly two of the terms. Since every coefficient of k_j is of the form $\binom{2^{a+1}-1}{j}$, which is odd, therefore after summing these we get that the coefficient of k_j in the whole sum is even. This means that after $2^{a+1} - 1$ steps everyone will have an even number. After performing $2^{a+1} - 1$ more steps, everyone's number will have another factor of 2, so will be divisible by 4. Therefore we can reach a state where every number is divisible by 2^a, meaning that modulo 2^a we reached all zeros, therefore we have proven that in this case all initial configurations are glorious.

(Back to problems)

4.2.4 Category E^+

1. During the proof, indices are always used modulo 100. Number the merchants from 1 to 100, starting from some merchant and proceeding to the left. Let a_1, a_2, ..., a_{100} denote the salmon prices in one specific year, called year A, and let b_1, b_2, ..., b_{100} denote the salmon prices in the year immediately following year A, called year B. Finally let $c_1, c_2, ..., c_{100}$ denote the prices in the year C immediately following year B. The conditions of the problem can be reworded to say that for every $1 \leq i \leq 100$, $b_i = \max(a_i, a_{i+1})$ if year B is good, and $b_i = \min(a_i, a_{i+1})$ if year B is bad; and the transition between years B and C can be described similarly.

Suppose that year B is good. Observe that in this case there cannot be an index i such that $b_{i-1} < b_i$ and $b_{i+1} < b_i$, since

$$\max(b_{i-1}, b_{i+1}) = \max(a_{i-1}, a_i, a_{i+1}, a_{i+2}) \geq \max(a_i, a_{i+1}) = b_i.$$

So in a good year, no merchant can have a price strictly greater than both neighbours' prices, and similarly in a bad year, no merchant can have a price strictly lower than both neighbours' prices.

If we use this observation for bad year A, this means that for every i, at least one of $b_i = \max(a_i, a_{i+1})$ and $b_{i+1} = \max(a_{i+1}, a_{i+2})$ is equal to a_{i+1}. Furthermore it is clear that both quantities are at least a_{i+1}, so

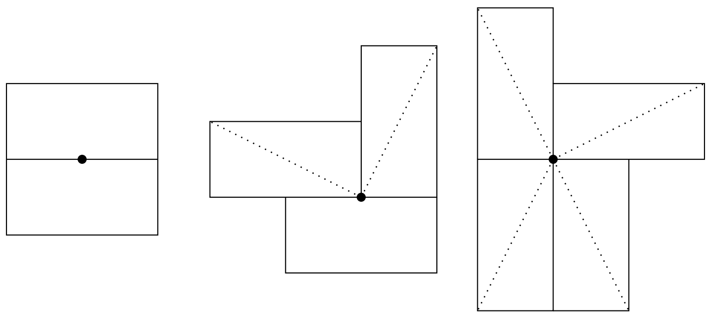
$$c_i = \min(b_i, b_{i+1}) = a_{i+1}.$$

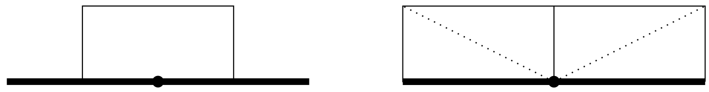
So in year C, all merchants will use exactly the same prices as what their left-hand neighbour used two years ago. Clearly this justification is valid for any year: in particular, a similar reasoning can be used if A and C are good years and B is bad.

So if Pauline is k spaces to the right of Paul along the shore of the island, then after $2k$ years Pauline will sell salmon for 17 Dürer dollars.

(Back to problems)

2. Let us consider the graph where the vertices are the corners of the dominoes, and two vertices are joined by an edge if and only if they are opposite corners of the same domino. This is equivalent to saying that Farringdon can directly jump from one vertex to the other. We can see that for each grid point of the quadrant, there are an even number of edges meeting there, except at the origin where there is only one (the figures show each distinct case). So if the set of vertices Farringdon can reach only consisted of finitely many vertices, then in this subgraph the sum of the degrees of the vertices would be odd, which is impossible. So the subgraph reachable by Farringdon cannot be finite, so Farringdon can reach points arbitrarily far from the origin.





(Back to problems)

3. For the solution, see Category E Problem 5.

(Back to problems)

4. Let us denote the first player with A and the second player with B. The idea is the following: we are constructing a series of chambers where firstly B can decide how many times the the adventurer gets hit by an arrow (but at least twice), then again B decides how many catacombs the player goes through (again at least twice). During this the adventurer loses hit points, which can be any composite number of B's choice. Finally we take A into a spike, who loses if the adventurer had 0 lives, otherwise B loses by getting into an infinite series of spikes.

Now we will detail the more rigorous proof. Consider the map below where E denotes the entrance, T is a trap, C is a catacomb, S is a spike, and the arrows denote passageways. The shape of the chambers and the rooms **X**, **Y**, **Z** are only for demonstration purposes.

Firstly observe that A can only visit the square-shaped rooms and B can only visit the circular ones. Moreover, players only have a choice in rooms **X**, **Y** and **Z**, everywhere else the next step is determined. Now we are going to consider two cases.

<u>Case 1:</u> h is a prime.

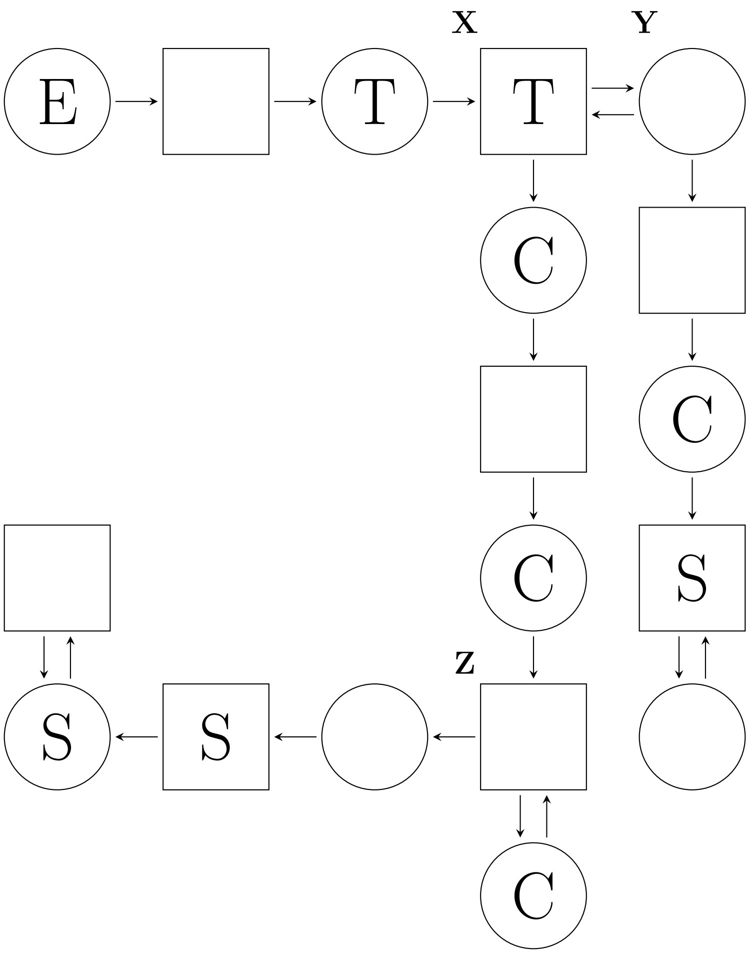
We show that in this case A has a winning strategy. Let the strategy be the following. If we are in room **Y** and the adventurer has been hit by more than h arrows, then move down, otherwise move left. This is a strategy for A, since they do not have a choice in the other rooms. Now let us investigate what B can do. If B moves right the first h times at **X**, then the first $h-1$ times A moves back to **X** from **Y**, and when the player has been hit by $h+1$ arrows, A moves down and B loses in the catacomb. If after some $0 \leq c < h$ moves to **Y** B moves downwards, then the player has been hit by $c+1$ arrows. Then if the adventurer is still alive, B has a choice in room **Z**. Observe that until B only moves down, A cannot lose. Assume that B moved down from **Z** $k \geq 0$ times, therefore the adventurer lost $(c+2)(k+2)$ hit points. Since we assumed that the player is still alive and h prime (therefore $h-(c+2)(k+2) > 0$), the adventurer has a positive number of hit points. This means that after moving left, A will not lose, but then in the infinite series of spikes B will. Therefore we can conclude that this is a winning strategy for A.

<u>Case 2:</u> h is composite.

Let $h = (c+2)(k+2)$ where $c, k \geq 0$. The strategy of B will be as follows. In room **X** move right the first c times, then down. In room **Z** move down the first k times and then to the left. Player A only has a choice in room **Y**. According to B's strategy the player visits room **Y** at most c times. Therefore if A decides to move downward from **Y**, the play will be hit by at most $c+1 < h$ arrows, therefore B does not die in the catacomb, but A will in the infinite series of spikes. If A goes back to **X** every time, then after the $(c+1)$st time in **X**, B will move down and A will not have any more choice. After moving down from **Z** k times, B moves to the left.

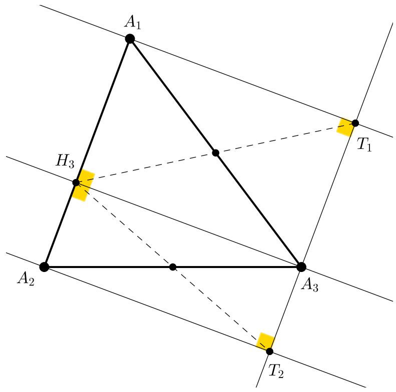
At this point the player has been hit by $(c+2)$ arrows and visited $(k+2)$ catacombs, therefore has exactly $h-(c+2)(k+2)=0$ hit points remaining. This means that when A enters the spike, they die, proving that this is indeed a winning strategy for B, and this concludes the proof.

Note: We believe that any proof will be based on a similar idea, but there are many different such maps, including ones consisting of fewer chambers. In this proof the aim was to provide a solution that is easy to follow, not necessarily to include a minimal example. Luckily this is still under the 20 chamber limit.



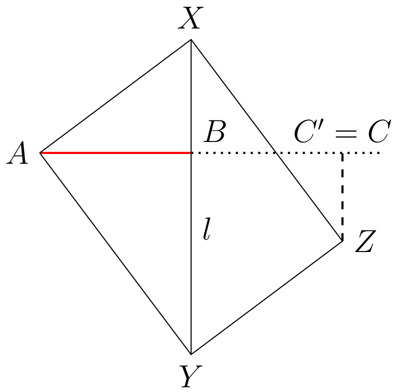
(Back to problems)

5. In both parts of the problem, D$ represents Dürer dollars. Let the vertices of our triangle be A_1, A_2, A_3. We will frequently use the notation X_i as mentioned in the problem.
a) We will construct Q straight from the definition. For this, we need the midpoints of two sides of the triangle. We demonstrate a relatively simple but less cost-effective construction (a more economical solution will be presented in part **b)**). Draw a perpendicular from A_3 to line A_1A_2 (50 D$), denote the point of intersection as H_3. Additionally, draw a perpendicular to line A_3H_3 at A_3 (50 D$), then draw perpendiculars from points A_1 and A_2 to this new line ($2 \cdot 50$ D$ $=$ 100 D$); let T_1 and T_2 denote the resulting intersections. Notice that due to the right angles, quadrilaterals $A_1T_1A_3H_3$ and $A_2H_3A_3T_2$ are rectangles, so their diagonals bisect each other. Therefore, if we draw lines T_1H_3 and T_2H_3 ($2 \cdot 5$ D$ $=$ 10 D$), they intersect sides A_1A_3 and A_2A_3 at their midpoints.



It remains to reflect points P_1 and P_2 in the midpoints. For this, we will use the following construction lemma: we can reflect a point in another point.

Construction: Suppose that we wish to reflect point A in point B. Construct the line through B perpendicular to AB, and call it l. Let $X \neq B$ be an arbitrary point of line l. Draw a perpendicular to line AX at A, and let it intersect l at point Y. Finally, let the perpendicular to AX at X and the perpendicular to AY at Y meet at Z. Drop a perpendicular from Z to AB, with its foot being C'. Then in triangle $AC'Z$, line l is parallel to the base $C'Z$, and bisects segment AZ, since the diagonals of a rectangle bisect each other. So l is a midsegment of $AC'Z$, therefore B is the midpoint of AC'. So the requested point is $C' = C$.



We can check that since we already had line AB constructed (as a sideline of ABC), the construction above costs us $5 \cdot 50$ D\$ + 5 D\$ = 255 D\$. To summarize: we take point P, construct points P_1 and P_2 ($2 \cdot 5$ D\$ = 10 D\$), then we reflect them in the midpoints of the corresponding segments ($2 \cdot 255$ D\$ = 510 D\$), hence getting points Q_1 and Q_2. The intersection of lines A_1Q_1 and A_2Q_2 is Q ($2 \cdot 5$ D\$ = 10 D\$). We prove that Q and P form an isotomic pair indeed. Clearly, by the method of construction of Q, it suffices to show that points P_3 and Q_3 are symmetrical with respect to the midpoint of side A_1A_2. Write down Ceva's theorem for points P and Q:

$$\frac{A_1P_3}{P_3A_2} \cdot \frac{A_2P_1}{P_1A_3} \cdot \frac{A_3P_2}{P_2A_1} = 1 = \frac{A_1Q_3}{Q_3A_2} \cdot \frac{A_2Q_1}{Q_1A_3} \cdot \frac{A_3Q_2}{Q_2A_1}.$$

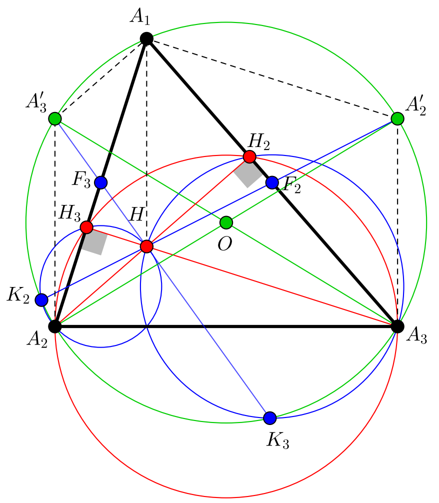
Since $A_2P_1 = Q_1A_3$, $P_1A_3 = A_2Q_1$, $A_3P_2 = Q_2A_1$, $P_2A_1 = A_3Q_2$, we see that $\frac{A_1P_3}{P_3A_2} = \frac{Q_3A_2}{A_1Q_3}$. However, as a point X moves along side A_1A_2, the ratio $\frac{A_1X}{XA_2}$ takes every positive real number precisely once, so the previous equality can only hold if $A_1P_3 = Q_3A_2$ and $P_3A_2 = A_1Q_3$, that is, if points P_3, Q_3 are indeed symmetrical in the midpoint.

Altogether we spent $50 + 50 + 100 + 10 + 10 + 510 + 10 = 740$ D\$, so we have nicely remained within our budget. It is easy to see that $P \neq Q$, otherwise P would be the centroid.

b) The task consists of two parts: first we show that we can construct one isotomic pair for 210 D\$, and then we show that from here we can always construct new pairs for 10 D\$ each.

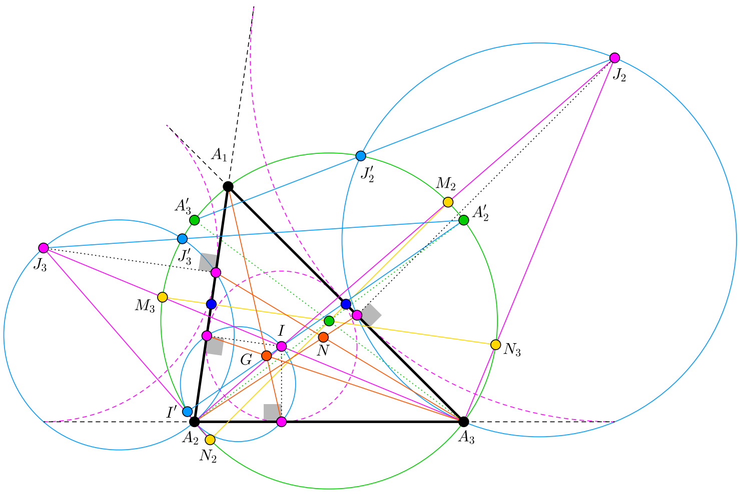
As the centroid cannot be used because of the definition, we need to find another isotomic pair. We can recall that the touchpoints of the incircle and excircles lie symmetrically on each side. From Ceva's theorem, it is also clear that if we connect each vertex with the opposite touchpoint of the incircle, we get three concurrent lines (since if we write down the subdivision ratios on each side, each of three tangent lengths from A, B, C to the incircle will appear once as a numerator and once as a denominator). We can make the same observation about the touchpoints of the excircles on each side. The two common intersection points are called the Gergonne and Nagel points of the triangle. These are what we will construct. (Since the triangle is not equilateral, the two points differ.)

Drop a perpendicular from A_2 to line A_1A_3 (50 D\$), let its foot be H_2. Then construct circle $(A_2A_3H_2)$ (10 D\$). Since this will be the circle with diameter A_2A_3, the intersection of the circle and A_1A_2 will be the foot of the altitude belonging to A_3. If we also draw the line A_3H_3 (5 D\$), we will get the orthocentre H too. Now draw circles $(A_1A_2A_3)$, (A_3HH_2), and (A_2HH_3) too ($3 \cdot 10$ D\$ $= 30$ D\$). Denote the new intersection points by $(A_1A_2A_3) \cap (A_3HH_2) = K_3$ and $(A_1A_2A_3) \cap (A_2HH_3) = K_2$. Draw line K_2H (5 D\$), let it intersect side A_1A_3 at F_2 and let its second intersection with the circumcircle be A_2'. Then $A_2'K_2A_2\angle = 90°$, so A_2A_2' is a diameter of the circumcircle. Furthermore, $A_1HA_3A_2'$ is a parallelogram, since $A_1H, A_2'A_3 \perp A_2A_3$ és $A_1A_2', HA_3 \perp A_1A_2$. Since the diagonals of a paralellogram bisect each other, F_2 will be the midpoint of side A_1A_3. Similarly draw line K_3H (5 D\$), intersecting line A_1A_2 at its midpoint F_3, and having a second intersection A_3' (opposite to A_3) with the circumcircle. So by constructing lines A_2A_2' and A_3A_3', we also get the circumcentre O ($2 \cdot 5$ D\$ $= 10$ D\$).



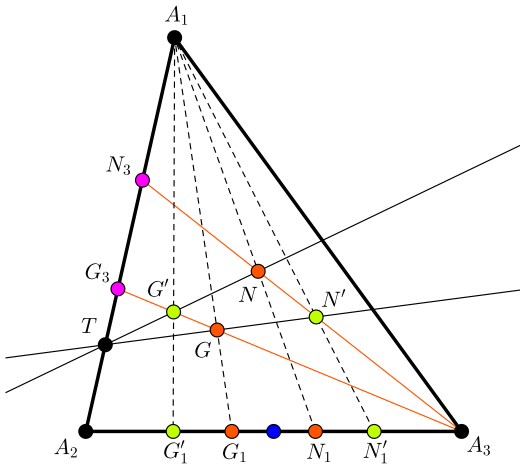
Construct the perpendicular bisectors OF_2 and OF_3 ($2 \cdot 5$ D\$ $= 10$ D\$). These intersect the circumcircle at the midpoints of arcs belonging to the sides A_1A_3 and A_1A_2. Let these be denoted by M_2, N_2, M_3, N_3. So if we draw the lines $A_2M_2, A_2N_2, A_3M_3, A_3N_3$, that is, the internal and external angle bisectors ($4 \cdot 5$ D\$ $= 20$ D\$) we will have constructed the incentre and the excentres. Denote these by I, J_1, J_2, J_3. Let line $A_2'I$ intersect the circumcircle for the second time at I' (5 D\$). Then circle (A_2II') will be the circle with diameter A_2I (since $90° = A_2I'A_2'\angle = A_2I'I\angle$), so it intersects sides A_2A_1, A_2A_3 at the touchpoints of the incircle (10 D\$). If we connect these with the opposing vertices, we get the Gergonne point ($2 \cdot 5$ D\$ $= 10$ D\$). We proceed similarly for the excircles. Let line $A_3'J_2$ intersect the excircle at

J_2' (5 D\$), then draw circle $(A_3J_2J_2')$ (10 D\$), which will actually be the circle of diameter A_3J_2, so it intersects side A_1A_3 at the projection of point J_2, that is, the touchpoint of the excircle for the second time. We can do the same for J_3 (5 D\$ + 10 D\$ = 15 D\$), and if we connect the two touchpoints with the opposing vertices, we get the Nagel point ($2 \cdot 5$ D\$ = 10 D\$).



Our total expense was $50+5+30+5+5+10+10+20+5+10+10+5+10+15+10 = 200$ D\$, so 10 D\$ remains. From half of the remaining 10 D\$, we buy an antivirus software, and the other half will be used later.

Now we will elaborate on how we can find a further isotomic pair. Take an arbitrary point G' on the part of line A_3G which is inside the triangle. Let line NG' meet the extended side A_1A_2 at point T, and let line TG meet line A_3N at N' ($2 \cdot 5$ D\$ = 10 D\$). We will prove that G' and N' are isotomic too.



For symmetry reasons, it suffices to prove that G_1' and N_1' are symmetric with respect to the midpoint of side A_2A_3. Observe that $(A_2, A_3; G_1, G_1') \overset{A_1}{=} (G_3, A_3; G, G') \overset{T}{=} (N_3, A_3; N'; N) \overset{A_1}{=} (A_2, A_3; N_1', N_1) = (A_3, A_2; N_1, N_1')$. If the reflection of G_1' to the midpoint of side A_2A_3 is denoted by G_1^*, then as reflection in a point preserves cross-ratios, $(A_2, A_3; G_1, G_1') = (A_3, A_2; N_1, G_1^*)$. So $(A_3, A_2; N_1, N_1') = (A_3, A_2; N_1, G_1^*)$, therefore $N_1' = G_1^*$, which is what we needed to prove.

However we have to be careful as T needs to exist. We can ensure this by choosing T to lie inside the segment A_1A_2. The only problem can happen if we choose the intersection point $GN \cap A_1A_2$ exactly, but this can be avoided by constructing line GN (5 D$).

(Back to problems)

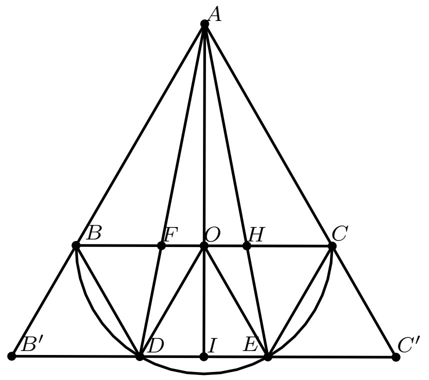
4.3 Final round – day 1

4.3.1 Category C

1. The second time the parrot said 'Jessica has two wooden legs'. This statement is true, and so is the other half of the utterance, that 'Jasmine and John always steal together'. Since the thief was alone, Jasmine and John cannot be the pirate who stole the rum. If the 3rd statement is true, then One-legged Jack or One-eyed Jane was the thief, but then the fourth statement is also true because than one of them went down alone into the hold during the heist. This way 3 of the statements would be true, so the 3rd statement can't be true. So neither Jack nor Jane was the thief. In addition, the second statement is also false, so Joe couldn't be the thief either, cause he hadn't gone to the hold that day.
Conclusion: the thief couldn't have been Jasmine nor John nor Jane nor Jack nor Joe. Therefore the rum was stolen by Two-wooden-legged Jessica.

(Back to problems)

2. Let O be the midpoint of segment BC. Then BO, DO, EO and CO segments are equally long, and their length is exactly the same as the radius of the half circle and as the half of the sides of triangle ABC. Then, because of the thirding of the arc: there are $60°$ angles around the point O, therefore BDO, DOE and EOC triangles are equilateral. (*) Let's observe that ABF and DFO triangles are similar because their sides are parallel, as OD and AB has angle of $60°$ with BO. We also know that the ratio of similarity is 2 since the length of AB is twice of the length of OD. From that we get that the length of FB is the twice of the length of OF, so F is the thirding-point of BC, when O is the half. The reasoning works symmetrically for H.
2nd solution after (*): Let B' and C' be points according to the figure, for which $BB'D$ and $CC'E$ triangles are equilateral. From point A if we enlarge B', C' points with the ratio of $\frac{2}{3}$, we'll get the points B and C since the length of segments CC' and BB' lengths are half of the lengths of the segments AB and AC. So a similarity takes points B', D, E, C' to points B, F, H, C. Therefore as $B'D = DE = EC'$ so after the similarity we get that $BF = FH = HC$, so we're done.



(Back to problems)

3. a) It's not true, for instance let's see a placement where 7 boys and 7 girls sitting by the table alternating. So every second person is a girl, and at first, every girl is happy and every boy is sad. In the next minute, every girl will be sad since both of their boy neighbours were sad, and all boys will be happy in the next minute, because all of their girl neighbours are happy, and so on. Therefore in every minute exactly 7 sad person will sit around the table.
b) In case of 1001 person the statement is true: if there was at least one happy person at the beginning, everyone will be happy eventually. We will show with each minute the number of happy people grows (therefore after at most 1000 minutes everyone will be happy.)

If there are k of happy people around the table at a given minute, they each give their neighbours (on a piece of napkin) a joke, so a total of $2k$ jokes are given out. The task says that in the next minute, exactly the people who didn't get a joke (i.e., both of their neighbors were sad) will be sad. Each person has two neighbors, so each person got up to 2 of jokes. Thus, out of $2k$ of jokes, at least k of people received a joke. Furthermore, if there is a person who only got one joke, then at least $k + 1$ person got a joke, so the number of happy people grows by at least one, due to pigeonhole principle.
If there is no such person who got exactly one joke, then everyone who got one, got exactly 2. This means that a happy person must have been sitting two to the left and two to the right of a happy person, since their neighbour got 2 jokes. So if Aladar was happy, and we can jump from him to Béla by two along the circle, then Béla must have been happy too. In case of odd number of people, this only can happen if everyone's already happy, because Aladár's distance to Béla is odd in both directions.
This way we are done since unless everyone is happy, the number of happy people grows in every minute, so after at most 1000 minutes everyone will be happy.
Comment: The solution shows that the statement of the problem is false in case on any even number of people, and true in case of any odd number of people.

(Back to problems)

4. a) If the children line up in a straight line in the order $FDCBAGE$, they all earn points. This is easily verifiable. To figure this out, we need to examine more and more columns step by step. Due to the fourth column, the order of DGE is given. Then, because of the seventh column, the order DCGE is also fixed. Due to the third column, the order DCAGE is fixed.

From the first column, the order FDCAGE is derived, and finally, B's position relative to the others is clear due to the two remaining columns.

b) Among the seven children, let's examine only the choices of A and C. If A's shield is B and bomb is C, then these three children must definitely be positioned on a straight line with B in the middle. However, if in addition to this, C's shield is A and bomb is B, then the three children must stand on a straight line with A in the middle among them. Thus, the children cannot line up in a way that both A and C receive a point.

c) Let's consider an arrangement of the children on the coordinate plane where everyone scores a point. We can assume that any line determined by two children is a line that is not parallel to the y-axis: the children (since there are a finite number of them) can rotate in such a way that this condition is indeed met, and the arrangement remains good. In this case, each child corresponds to a point in the coordinate system, and no two children's points have the same x-coordinate.

Now have every child walk over to the x-axis as follows: if they were standing at point (x, y) before, they should move to point $(x, 0)$. This will still be an arrangement where everyone gets a point, because if three children were standing in a straight line in the original arrangement, their left-to-right order on that line matched the ascending order of their x-coordinates, and this is preserved in the new arrangement, thus we are done.

Note: This geometric transformation is the perpendicular projection onto the line. We are utilizing the fact that there exists a line in the plane for the finite number of children that is not parallel to any of their pairwise lines. This exists because only a finite number of lines are excluded. Instead of projection, other geometric transformations are also suitable. For example, we can determine an order for the children by rotating a line around a point, thereby 'sweeping' across the plane and observing in which order the children cross the line. Here, we need to ensure that the center of rotation is not on any line determined by two children. If they line up on a straight line according to the order obtained this way, we also get a suitable solution.

(Back to problems)

5. a) Let's take an arbitrary integer n which is greater than 10. The solution will build on pencil-and-paper addition. We build two numbers, a and b from the right, the first position in a way that neither will contain the digit 0 and that $a + b = n$. First, we choose the digits of a and b in the first position in a way that the addition is correct in the first position and neither of the two chosen digits is 0. It's clear, that we can do this. From now, in each step, after we know the last k digits of a and b, the next digits in the $k + 1$. position can be determined as we know the value of the carry and we can choose adequate digits for a and b in the $k + 1$. position, too. It is important to note that the carry cannot be more than 1 in this case. When we get to the last position, we note its value by m and if we have a carry, let the remaining digit of a be $m - 1$. If this would be 0, we are not putting anything in that position. If there is no carry, then this digit of a should be m. With this process, we create the correct a and b numbers.

In order to make this method clear, we also show it through an example. Let $n = 2564$. Let the digits of a and b in the first position 6 and 8 as $6 + 8 = 14$ that ends with 4. The carry is 1, so the digits of a and b in the second position need to add up to a number that gives a remainder of $6-1=5$ when divided by 10. Let these be 2 and 3, so a ends with 26, while b ends with 38. Then, there is no carry, so at the third position position, the sum of the two chosen digits need to give 5 as a remainder when divided by 10, so $6+9$ is a correct choice, which means that a ends with 626, while b ends with 938. Finally, the carry is 1, so the first digit of a is 1. Now, we have $2564 = 1626 + 938$.

So, the set $\{1; 2; 3; 4; 5; 6; 7; 8; 9\}$ is indeed sufficient.

b) The last digit of the sum of two numbers is equal to the last digit of the sum of their last digits. So, in order for a set H to be sufficient, all ten digits should be produced as the last digit of the sum of two elements in H as the last digit of any number is produced in this way.

For now, we assume, that there is a sufficient 4-element set and we note it by $\{a, b, c, d\}$. The sums that can be created from these elements are $a+a$, $a+b$, $a+c$, $a+d$, $b+b$, $b+c$, $b+d$, $c+c$, $c+d$, and $d+d$. From these, $a+a, b+b, c+c$, and $d+d$ must be even. So, in order for all digits to be produced from these sums, from the $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ numbers, 5 has to be odd. If there are three elements in H with the same parity, their 3 distinct sums will be even, so there could only be 3 odd numbers among the 6. If this is not the case, that means that there are two odd and two even numbers in H. Then, the sum of the two odd, and of the two even numbers will be even, so there could only be $6-2=4$ odd numbers among the 6. So, we cannot create all 5 odd digits as the last digits of the sums of the elements of H, which means that there is no sufficient set with only 4 elements.

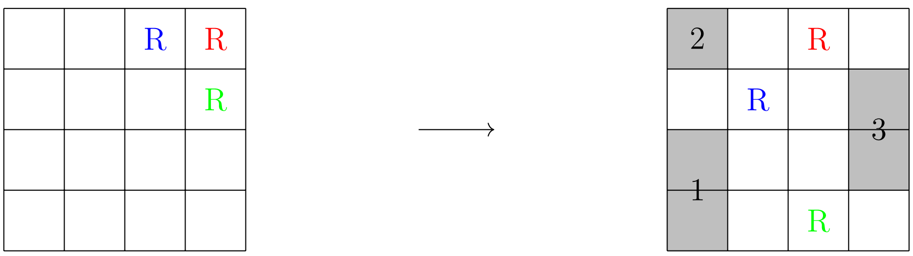
c) In part b) we proved that there are no sufficient sets with only 4 elements. However, there are sufficient sets with 5 elements, for example, $\{0; 1; 3; 4; 5\}$. First, we show that all digits can be produced as the sum of two elements from the set:

$$\begin{aligned} 0 &= 0 + 0 \\ 1 &= 1 + 0 \\ 2 &= 1 + 1 \\ 3 &= 3 + 0 \\ 4 &= 4 + 0 \\ 5 &= 5 + 0 \\ 6 &= 5 + 1 \\ 7 &= 4 + 3 \\ 8 &= 5 + 3 \\ 9 &= 5 + 4 \end{aligned}$$

Then, we can create any integer greater than 10 as the sum of exactly two non-negative integers which have all their digits from H, the following way:
We assign the two digits to each digit of the number that add up to that from H, as we listed it above. In each case, these two will be the digits of the terms in the corresponding positions. For example, if there is a 6 in the third position of the sum, then, in the first term, on the third position, there will be a 5 while in the second term, there will be a 1. If any of the terms would begin with 0, then we delete those digits. For example, in the case of 106579, the sum would be $106579 = 105545 + 001034$, so the final solution will be $106594 = 105545 + 1034$. As the sum of any two digits will never be more than 9, there is no carry in any case, so the method really works.

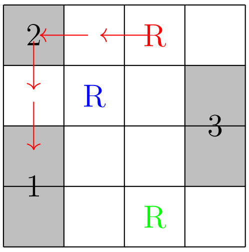
(Back to problems)

6. The researchers have a winning strategy, which we describe below. We will denote the researchers by R. In the first six steps the researchers move to the following position regardless of the shark's movement:

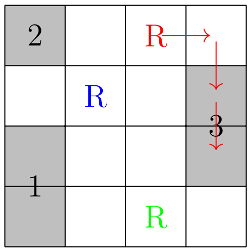


If they do not catch the shark until day 6, and in the morning of day 7 the shark is not in the grey area of the figure above then it is either in the same sector as one of the submarines or it is next to one.Therefore they can catch him that day. From now on we examine the different cases based on the position of the shark.

If the shark is either in sector 1 or sector 2 then the blue and green submarines only move if the shark moves next to them, otherwise only the red submarine moves. The red submarine moves two to the left, then if the shark did not move to the same sector with a submarine or next to the blue or green submarine it can only be in sector 1 in the morning of day 9. Then the red submarine moves downwards, this way in the morning of day 11 the shark will either be in the same sector as a submarine or next to one.



If the shark is in sector 3, similar to the previous case, unless the shark moves next to the green or blue submarine only the red submarine moves the following way.



After the three steps marked with red the shark must be in the same sector as one of the submarines or next to one. Therefore they catch it on day 10 at the latest.

Alternative solution:

An alternative strategy for the researchers:

Let the columns of the table from left to right be A,B,C,D and the rows from the bottom to the top 1,2,3,4.
During the first four days the submarines in sectors C4 and D3 move to B3 and C2.After that the shark cannot be above the main diagonal A4-B3-C2-D1 since the researchers can reach each of those sectors so they would catch the shark the next day.

If the shark is in the sector A4 then the submarine in B3 moves to A3. The shark can only move to B4 without getting caught.Then the submarine in D4 moves to C4 so in the next step the shark can only move to either the same sector as a submarine or next to one.Therefore they catch him on day 7. If the shark is in D1 on day 5 the researchers can catch it with a similar strategy.
The only remaining cases are where the shark is in one of the sectors A1, A2 and B1 on day 5. Then the submarine in D4 moves to C3 in two days: if the shark is in A2 on day 5 then using the steps C4-C3 , if the shark is on B1 then with steps D3-C3. This way they can catch the shark in both cases if it moves to either A4 or D1. Now we only need to examine the cases where the shark is in one of the sectors A2,A1 and B1.

(S)			R
	R		
		R	

Day 4

	S	R	
R			
		R	

Figure 2

	R	R	
		R	

Day 6

If the shark is in B1 then the submarine in B3 moves to B2. After that the shark can only move to A1 or D1. If it moves to A1 then the researcher in C3 moves to B3 creating a position similar to the case where the shark moves to D1 so we only examine this case. (Figure 4) In this case the submarine in B2 moves to A2 on day 9 so the shark can only escape to B1 after the submarine in C2 moves to C1 covering all the possible positions of the shark. Therefore we catch the shark on day 11. If the shark is in A2 we can move similarly.

The only case left is where the shark is in A1 after day 6. In this case the submarine in C2 moves to B2. If the shark stays put the submarine in C3 moves to C2 reaching the position of Figure 4. therefore we catch it with the strategy mentioned above. We can assume that the shark moves to another sector which can only be C1. Then the submarine in C3 moves to C2 and the shark has to escape again. If it moves back to A1 we get the position of Figure 4 once again on day 8, so using the strategy discussed above the researchers catch the shark. Now we can assume that the shark moves to D1 (the only possible position apart from A1) and we get the position of Figure 6 on day 8. After that the submarine in C2 moves to D2 so the shark can only move to C1. Then the submarine in B2 moves to B1 giving us a position on day 10 where regardless of the next move of the shark the researchers can catch it.
We have proved that the resarchers can always catch the shark in 11 days if they play the game perfectly, therefore it is in the players interest to choose to play as the researchers.

	R		
	R	R	
S			

Figure 4

	R		
R			
	S	R	

Figure 5

	R		
	R	R	
			S

Figure 6

	R		
			R
	R	S	

Figure 7

(Back to problems)

4.3.2 Category D

1. The parrot said as his 3rd sentence that 'Jessica doesn't like rum, and as his fourth sentence: 'Jessica likes rum'. Exactly one of these is true. If it's the 3rd, then Jack or Jane is the thief, this way the first part of the fourth statement is would be true as well since the thief was alone. So if the 3rd sentence is true, then the fourth should be true and false at once, which yields a contradiction. So the 3rd statement isn't true, and the fourth is true. Since we already know one true and one false statement, therefore among the first two statements exactly one is true. If the second one is false, then Joe went to the hold that day, this way he couldn't be at the outroad all day, so the first statement is false also, it's a contradiction. So the second statement is true.
From the second statement it follows that neither Jasmine nor John were the thief (since the thief was alone), furthermore, Joe couldn't be it either, cause he wasn't in the hold. From the

falsehood of the 3rd statement it follows that the thief is neither Jack nor Jane. Therefore Two-jury-legged Jessica stole the rum.
Note: Furthermore, she even had motivation, as she likes rum.

(Back to problems)

2. For the solution, see Category C Problem 3.

(Back to problems)

3. For the solution, see Category C Problem 4.

(Back to problems)

4. Let the length of section DF be 1 unit, from which $AD = 2$ and $FB = 3$.

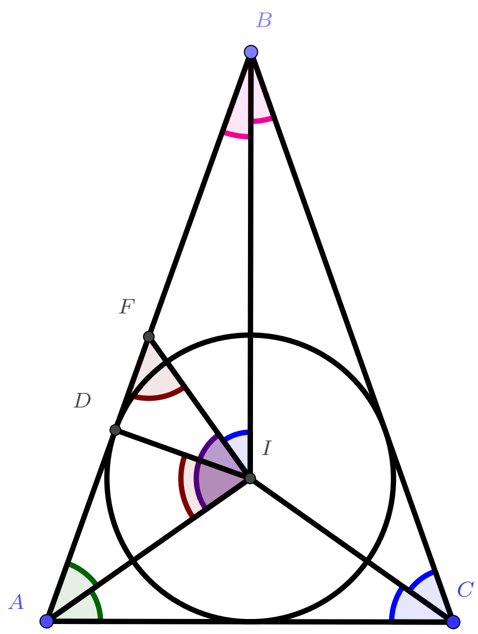
Let the angles of the triangle be α, β and γ, in the traditional way, from which we know that $\sphericalangle BAI = \sphericalangle IAC = \frac{\alpha}{2}$, $\sphericalangle ABI \sphericalangle = IBC = \frac{\beta}{2}$, and $\sphericalangle BCI = \sphericalangle ICA = \frac{\gamma}{2}$.

Let's calculate $\sphericalangle AIB$ based on the fact that we already know the other angles of triangle AIB, so $\sphericalangle AIB = 180° - \sphericalangle IAB - \sphericalangle ABI = 180° - \frac{\alpha+\beta}{2} = 90° + \frac{\gamma}{2}$. Then, since $\sphericalangle FIB = \sphericalangle ACI$, $\sphericalangle AIF = \sphericalangle AIB - \sphericalangle FIB = \sphericalangle AIB - \sphericalangle ACI = 90°$, so $\sphericalangle AIF$ is a right angle.

Then, one of the heights of right triangle AIF is IF, so triangles AID, IFD, and AFI are similar. Due to the similarity, $\frac{AD}{ID} = \frac{ID}{FD}$, from which we get the length of section ID, as $ID = \sqrt{AD \cdot FD} = \sqrt{2}$.

Then, as we know the length of section ID, we can express the lengths of sections IF and IB from the Pythagorean theorem, as $FI = \sqrt{FD^2 + DI^2} = \sqrt{3}$ and $BI = \sqrt{BD^2 + DI^2} = 3\sqrt{2}$.

In order to complete the solution, we only need one similarity. Triangles FBI and IBC are similar, because $\sphericalangle ICB = \sphericalangle FIB$ and $\sphericalangle CBI = \sphericalangle IBF$. Due to the similarity, $\frac{BI}{BF} = \frac{BC}{BI}$, from which we get $BC = \frac{BI^2}{BF} = 6$. Based on this, $AB = BC$, so triangle ABC is indeed isosceles.



(Back to problems)

5. For the solution, see Category C Problem 5.

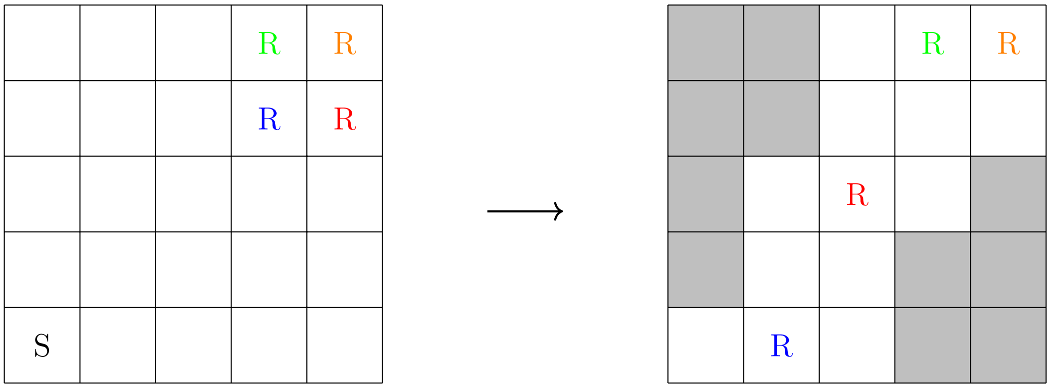
(Back to problems)

6. The researchers can always catch the shark with the right strategy.

The key observation is that if two submarines are in the position shown on the figure below then the shark cannot move between the ships without getting caught the next day.

	R	
S		
		R

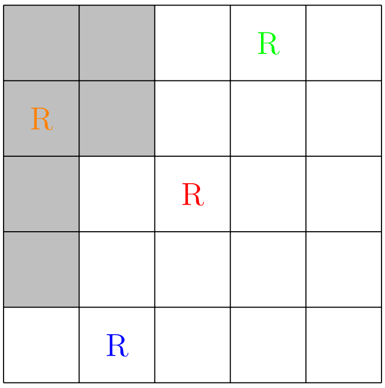
For this reason the submarines move into the position below in 8 steps regardless of how the shark moves.



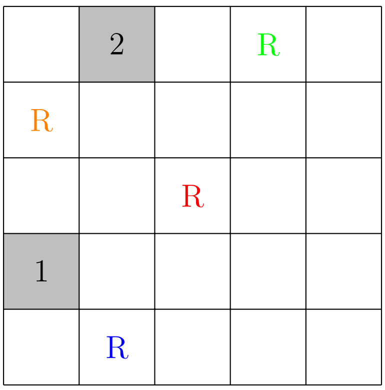
After the 8th night if not caught yet the shark stops in one of the sectors. If it is in one of the grey sectors then the submarines catch it the next day. Therefore we can assume that the shark is in one of the grey sectors.

From now on we divide the strategy into two parts based on the position of the shark.

If the shark is in the top left sector:
In this case the orange submarine moves to the position below in 5 days. If during these 5 days the shark leaves the grey area then the next day the green, red or blue can catch it.

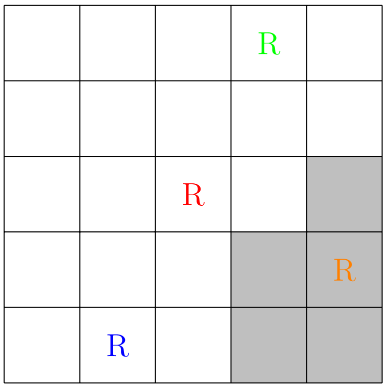


This way if they did not catch the shark until the 13th day and cannot catch it after its movement during the night of day 13 the it can be in the following two sectors:

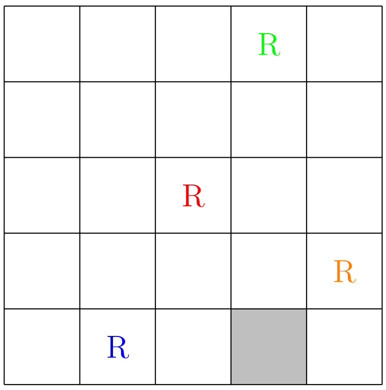


If the shark is in sector 1 then on the 14th day the orange submarine moves to the sector below it. Then regardless of how the shark moves it will end up either in the same sector as one of the submarines or in a sector next to a submarine on day 15 so one of the submarines will catch it.
If the shark is in sector 2 then similary to the previous case the green submarine moves to the left. This way they catch the shark in 15 days.

If the shark is in the bottom right sector:
In this case the orange submarine moves to the position shown below in 3 days. If during these days the shark leaves the grey area then the next day the green, red or blue submarine can catch it.



If they do not catch the shark until noon on day 11 and they cannot catch it after its movement during the night of day 11 either then it can only be in the following sector:



Now the blue submarine moves to the left on day 12 so the shark can only move from the grey sector to the same sector as one of the submarines or next to one. Therefore on day 13 one of the submarines can catch it.

(Back to problems)

4.3.3 Category E

1. For the solution, see Category C Problem 4.

(Back to problems)

2. For the solution, see Category D Problem 4.

(Back to problems)

3. We get the same last digit if we sum the two integers or sum the last digits of the two integer. Thus a criteria for a sufficient set is that we should be able to get all 10 digits as the last digit of a sum of two digits from H.

Tegyük fel, hogy van egy négyelemű elégséges halmaz. Legyen ez $\{a; b; c; d\}$. Az ezekből képzelhető összegek: $a + a,\ a + b,\ a + c,\ a + d,\ b + b,\ b + c,\ b + d,\ c + c,\ c + d$ és $d + d$. Ezek közül $a + a, b + b, c + c$ és $d + d$ biztosan páros. Tehát, hogy minden jegy elő tudjon állni két H-beli elem összegének utolsó jegyeként, ahhoz $a + b,\ a + c,\ a + d,\ b + c,\ b + d,\ c + d$ között 5 páratlan számnak kell lennie. Ha ebben a halmazban van valamilyen paritású elemből legalább 3, akkor ennek a 3 elemnek a páronkénti összegei mind párosak (mivel két azonos paritású szám összege páros), tehát legfeljebb 3 páratlan szám szerepelhetne az összegek között. Ha a halmazban semelyik paritású elemből nincs 3, akkor az azt jelenti, hogy 2 páros és 2 páratlan elemből áll, ekkor a két páros elem összege páros és a két páratlané is, azaz legfeljebb $6 - 2 = 4$ páratlan szám lehetne az $a+b,\ a+c,\ a+d,\ b+c,\ b+d,\ c+d$ összegek között. Tehát sehogyan nem állhat elő mind az 5 páratlan számjegy, mint két, elégéges halmazbeli elem összegének utolsó jegye, azaz nincs négyelemű elégséges halmaz.

Ötelemű elégséges halmaz viszont már van. Például a $\{0; 1; 3; 4; 5\}$. Először is vegyük észre, hogy minden számjegy előáll két halmazbeli elem összegeként:

$$\begin{gathered}0 = 0 + 0\\ 1 = 1 + 0\\ 2 = 1 + 1\\ 3 = 3 + 0\\ 4 = 4 + 0\\ 5 = 5 + 0\\ 6 = 5 + 1\\ 7 = 4 + 3\\ 8 = 5 + 3\\ 9 = 5 + 4\end{gathered}$$

Ekkor egy tetszőleges 10-nél nagyobb egész számot fel tudunk írni pontosan két olyan nemnegatív egész szám összegeként, melyeknek a számjegyei csak H-ból kerülnek ki az alábbi módon:

A szám minden számjegyét felírjuk két, H-beli elem összegeként a fent mutatott módon, ez a két jegy lesz a megfelelő helyiértékű jegye a két összeadandónak. Például ha az összegben a százas helyiértéken 6-os áll, akkor az első összeadandóban legyen 5-ös a százas helyiértéken, a másodikban pedig 1-es. Ha a két szám valamelyike valahány 0-val kezdődne, akkor azokat

töröljük. Például a 106579 esetén ez a felírás így néz ki: $106579 = 105545 + 001034$, a számok elejéről a nullákat törölve: $106594 = 105545 + 1034$. Mivel soha nem lesz két jegy összege 10-nél nagyobb (azaz nem keletkezik maradék az összeadás során), tudunk jegyenként összeadni, tehát ez a módszer tényleg működik.
Összefoglalva: Egy elégséges halmaznak legalább **5** eleme van.

(Back to problems)

4. We claim that they have to use the machine at least n times in order to halve the salmon. First, we show a case where n steps (of using the machine) are needed. If the machine always cuts off a piece of size 1, then we have no choice, as we always have pieces of size 1 and a big piece (and we cannot put the small ones into the machine). Therefore n steps are needed, since that is when the size of the big piece reaches n.

Now let us show that n steps are always sufficient. Let us proceed the following way: if there is a piece larger than n, then put one into the machine that is not the largest and not size 1. If there is no such, then put the biggest piece. We stop the process when we cut the largest and it gets cut to two pieces both at most size n.

This way we have a piece of size k, another one of size l and $2n-(k+l)$ pieces of size 1, where $k \leq n$, $l \leq n$ and $k+l > n$. Since in every step the number of pieces increases by exactly one, therefore we made at most $(2n-(k+l)+2-1) \leq n$ steps. And indeed, they can divide the salmon equally as one of them takes the piece of size k, the other one the one of size l, and each take pieces of size 1 to reach n in total.

Second solution: We show another way of proving that n cuts are sufficient. Imagine the salmon as a circular cake, where there are $2n$ radii from the centre to the edge, and these are the lines through which we always cut. The first time, the cut is made along two radii, and afterwards on one of the radii. Since after using the machine n times there will be $n+1$ cuts, we can choose two of them that make up a diameter and the salmon can be halved along it, since the $2n$ radii create n diameters in distinct pairs, and if we randomly choose $n+1$ of them, by the pigeonhole principle, there will be at least one diameter of which both radii have been chosen.

(Back to problems)

5. We construct a sequence, for which $(a_1 \text{mod } n), \ldots$ sequence is periodic according to 10^{n+1}, for every n. The n-th element of the sequence, a_n is consturcted from n's shape in base 10. If n is a power of 10, let $a_n = 0$, otherwise, let $k_1! + k_2! + \ldots + k_l!$, where $k_1, k_2, \ldots. k_l$ are the ordinal number of local values where n contains a non-0 digits. For istance in case of $n = 1065$: $a_n = 4! + 2! + 1! = 27$, while in case of $n = 25000$: $a_n = 5! + 4! = 144$.
This sequence's 0 values are the 10 powers, which form a non-periodic series, but for any k the series $(a_i \bmod k)$ is periodic according to 10^{k+1}, because:
It is sufficient to show, that if $i - j = 10^k$, then $a_i \equiv a_j \bmod k$. This is true, because, $a_i - a_j$ is the sum of factorials, so that $a_i - a_j$ is the sum of the factorials of numbers which are all at least k, and therefore $k | a_i - a_j$.

2nd solution:

There's such sequence, we'll show a construction, using the folloring lemma, calles Chinese Remaindeer Theorem:

If $m_1, m_2, \dots, m_k$ pairwise relative primes, $c_1, c_2, \dots, c_k$ arbitrary integres, than there exists a unique x integer, satisfying $0 \le x < m_1 \cdot m_2 \cdot \dots \cdot m_k$, and

$$\begin{aligned} x &\equiv c_1 \mod m_1 \\ x &\equiv c_2 \mod m_2 \\ &\dots \\ x &\equiv c_k \mod m_k. \end{aligned}$$

Let's observe, that it's sufficient to find a sequence, which is periodic mod every prime power, 'cause from here, we can use the Chinese remainder to show that for arbitrary positive integer it's periodic as well. Let $p_1, p_2, \dots$ be an assignment of the prime powers for which, if q is a prime $l < k$ positive integer numbers, then q^l is earlier than q^k. We'll construct the sequence $a_0, a_1, \dots$, so $a_0 = a_1 = 0$, and the $a_i = 0$ if and only if $i = 0$ or a 2 power (this isn't periodic), and the sequence has a period of 2^{k-1} modulo p_k.

Specially the seqence will be periodic with period 1 mod p_1, so every term is divisible by p_1.

We'll construct the sequence by induction, in the kth step we give the value of a_i forall $i \le 2^{k-1}$. We already did it to $k = 1$, now let's assume, that we did it for $k - 1 \ge 1$ and we want to construct them for k.

Let's construct the numbers $a_{2^{k-1}+1}, a_{2^{k-1}+2}, \dots, a_{2^k}$, according to our conditions. The required conditions are that forall $1 \le l \le k$ the sequence (a_i) is periodic modulo p_l and the cycle of the period is 2^{l-1}. So forall $2^{k-1} < m \le 2^k$ the expression $a_m - a_{m-2^{l-1}}$ must be divisible by p_l forall $1 \le l \le k$ integers. We always can construct such numbers, 'cause, if p_i and p_j are powers of the same prime numbers ($1 \le i < j \le k$), then $p_i \mid p_j$ and for the period as well, $2^{i-1} \mid 2^{j-1}$, so if $a_m \equiv a_{m-2^{j-1}} \pmod{p_j}$ is true, then it's trivial, that for congruency for i'll be true as well. Therefore, it is sufficient to observe the congruency with the largest power for each prime. So by the Chinese Remainde Theorem, we're able to construct such a_m. Furthermore, if $m \ne 2^k$, then let $a_m \ne 0$, while $a_{2^k} = 0$ is a good choice, cause all previous period divides 2^k, and $a_0 = 0$.

We finished the construction.

(Back to problems)

6. The 1st player has a winning strategy.

The number 1 cannot be selected in any ticket, the same holds for primes greater than 7 as well since there are not enough numbers not greater than 45 that are divisible by these. Therefore we cannot select : $1, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43$ that is 11 numbers.From the remaining 34 numbers we can fill out 5 tickets at most. We state that the first player can achive that with the right strategy, therefore filling out the last ticket and winning the game.

Let the first ticket be $5, 15, 30, 35, 45$ which leaves 6 numbers divisible by 6, 14 numbers divisible by 2 and not divisible by 3 , 6 numbers that are divisible by 2 but not by 3. Only 3 multiples of 5 and 5 multiples of 7 are left.

Now the second player can only select numbers that are divisible by 2 or 3 so the greatest common divisor can be $2, 3, 4$ or 6.

If the chosen numbers have a greatest common divisor that is not divisible by 3, then the 6 numbers that are divisible by 3 but not by 2 remain. If the starting player chooses these, there will still be $14+6-6=14$ even numbers left (there were 14 numbers divisible only by 2, and 6 numbers divisible by 6, but one ticket has already been filled in the meantime). Thus, they can still fill in at least two more completely even tickets.

If the second player fills his ticket with numbers whose greatest common divisor is divisible by 3, then there will still be $6+6-6=6$ numbers divisible by 3 left (there were 6 numbers divisible only by 3, and 6 numbers divisible by 6, but the second player filled a ticket).Therefore the starting player can fill in a ticket with these 6 numbers divisible by 3. This leaves 14 even numbers (those only divisible by 2), allowing them to fill in at least two more completely even tickets.

(Back to problems)

4.3.4 Category E$^+$

1. Let the common value of the fractions in the problem be denoted by s. We are going to use the following useful claim:
Claim: If $s=\frac{x}{y}=\frac{z}{w}$, then $s=\frac{x+z}{y+w}$ if $y+w\neq 0$, otherwise $x+z=y+w=0$.
Proof of claim: If $y+w\neq 0$, then $\frac{x}{y}=\frac{x+z}{y+w}\iff x(y+w)=y(x+z)\iff xw=yz$, which follows from the condition $\frac{x}{y}=\frac{z}{w}$. And if $y+w=0$, then $y=-w$, so $\frac{x}{y}=\frac{z}{w}\implies x=-z$, thus $x+z=0$. $\square$

Let's use the claim: if $a+b+c+d\neq 0$, then on one hand

$$s=\frac{a+2b+3c}{c+d}=\frac{c+2d+3a}{a+b}=\frac{(a+2b+3c)+(c+2d+3a)}{(c+d)+(a+b)}=\frac{4(a+c)+2(b+d)}{a+b+c+d},$$

on the other hand we have

$$s=\frac{b+2c+3d}{d+a}=\frac{d+2a+3b}{b+c}=\frac{(b+2c+3d)+(d+2a+3b)}{(d+a)+(b+c)}=\frac{2(a+c)+4(b+d)}{a+b+c+d}.$$

This can only hold if the two numerators are equal, that is $a+c=b+d$. Using this, we obtain

$$s=\frac{4(a+c)+2(b+d)}{a+b+c+d}=\frac{6(a+c)}{2(a+c)}=3.$$

Now, the fractions in the problem can be rewritten as equations: for example, $\frac{a+2b+3c}{c+d}=3\iff a+2b+3c=3c+3d\iff a+2b=3d$, and we can write this for the other fractions as well (changing the letters cyclically). We are going to show that all four numbers are equal. That is because if d is the largest among the four numbers, then $3d=a+2b\leq d+2d=3d$, but the equality only holds if $a=b=d$. Finally, because of this, we have $3c=d+2a=3d$, so $c=d$.

If $a+b+c+d$ would be 0, $a=b=c=d=0$, since $a=b=c=d$. Then, $a+b=0$, which is not allowed by the conditions.

In summary, $a=b=c=d=\lambda$ must hold, but $\lambda\neq 0$ (otherwise the denominators would also be 0). If, on the other hand, $a=b=c=d=\lambda\neq 0$, then indeed the value of every fraction is 3.

Second solution: We will present an alternative proof for $s = 3$. Let us rewrite the fractions as equations (like we did at the end of the previous solution), we just don't know their common value yet. For example, $s = \frac{a+2b+3c}{c+d} \iff a + 2b + 3c = sc + sd \iff a + 2b + (3-s)c - sd = 0$, thus we obtain a system of homogeneous linear equations consisting of four equations with four unknowns (for s).

$$\begin{array}{rcrcrcr}
a & + & 2b & + & (3-s)c & - & sd = 0 \\
-sa & + & b & + & 2c & + & (3-s)d = 0 \\
(3-s)a & - & sb & + & c & + & 2d = 0 \\
2a & + & (3-s)b & - & sc & + & d = 0
\end{array}$$

If this has a solution other than the trivial $(0,0,0,0)$, then the equations (as vectors) are linearly dependent, so the determinant of the associated matrix $\begin{pmatrix} 1 & 2 & 3-s & -s \\ 2 & 3-s & -s & 1 \\ 3-s & -s & 1 & 2 \\ -s & 1 & 2 & 3-s \end{pmatrix}$ has to be zero. Let us compute the determinant:

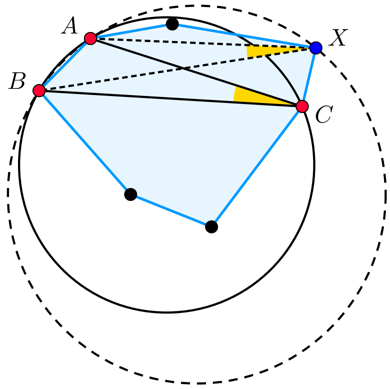
$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3-s & -s \\ 2 & 3-s & -s & 1 \\ 3-s & -s & 1 & 2 \\ -s & 1 & 2 & 3-s \end{vmatrix} &= 1 \cdot \begin{vmatrix} 3-s & -s & 1 \\ -s & 1 & 2 \\ 1 & 2 & 3-s \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 3-s & -s \\ -s & 1 & 2 \\ 1 & 2 & 3-s \end{vmatrix} \\
&+ (3-s) \cdot \begin{vmatrix} 2 & 3-s & -s \\ 3-s & -s & 1 \\ 1 & 2 & 3-s \end{vmatrix} + s \cdot \begin{vmatrix} 2 & 3-s & -s \\ 3-s & -s & 1 \\ -s & 1 & 2 \end{vmatrix} \\
&= \Big((3-s)^2 - 2s - 2s - 1 - s^2(3-s) - 4(3-s)\Big) \\
&- 2\Big(2(3-s) + 2(3-s) + 2s^2 + s + s(3-s)^2 - 8\Big) \\
&+ (3-s)\Big(-2s(3-s) + (3-s) - 2s(3-s) - s^2 - (3-s)^3 - 4\Big) \\
&+ s\Big(-4s - s(3-s) - s(3-s) + s^3 - 2(3-s)^2 - 2\Big) \\
&= 8s^3 - 24s^2 + 32s - 96 = 8(s-3)(s^2+4)
\end{aligned}$$

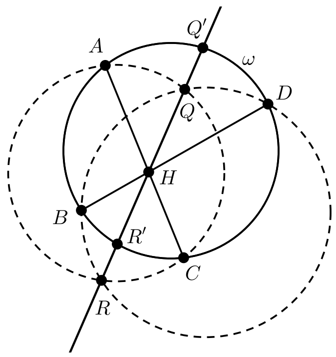
This polynomial can only take the value of 0 if $s = 3$ (because $s^2 + 4 > 0$).

(Back to problems)

2. We will show that these sets P are exactly the ones where the points lie on a circle or on a line. It is clear that these sets satisfy the condition of the problem. Now suppose that there is such a set P that also satisfies the problem but not all of the points are on a circle or on a line. Then the convex hull of P will have at least 3 sides.

Now let us regard the circles determined by the vertices of the convex hull (these all belong to $S(P)$), and let (one of the) largest one be ω. Notice that every point of P is contained inside or on the boundary of ω. Since suppose that the point $X \in P$ falls outside of ω. Let $A, B, C \in P$ be three points on ω. Then since A, B, C, X are all points on the convex hull, we can assume that the quadrilateral $ABCX$ is convex. Since the triangle ABC has an acute angle at either A or C, let us assume that $ACB\sphericalangle$ is acute. Since C and X fall onto the same side of line AB, this means that the directed angle $AXB\sphericalangle$ is smaller than the directed angle $ACB\sphericalangle$, since the radius of circle (AXB) is smaller than that of (ACB) (since the sine of the angle belonging to the same chord is smaller), which is a contradiction.





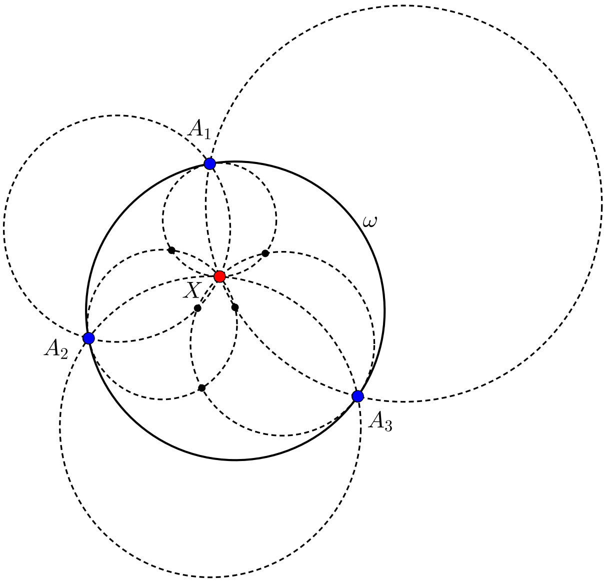
Now we will show that ω cannot have more than three points of P on its boundary. Suppose indirectly that $A, B, C, D \in \omega$ in this order (meaning that $ABCD$ is a convex quadrilateral). Firstly we will show that there is more than one point inside ω. Indeed, if M was the only such point, then circles (ABM) and (CDM) cannot have another point of intersection, therefore they are tangent. But then there has to be a point $E \in P$ on the circle $ABCD$, and circles (ABM) and (CEM) cannot be tangent, which is a contradiction.

Similarly circles $(BCM), (DAM)$ are also tangent. This means that the quadrilateral $ABCD$ is a parallelogram (and since convex, a rectangle). Since if there are 2023 points on the circle, then we can choose A, B, C, D in a way that they do not form a parallelogram (rectangle). Therefore that there are at least two points inside ω. This means that there is a point Q inside ω that is not the intersection of lines AC and BD. Then circles $(ACQ), (BDQ)$ exist and they have a second intersection $R \neq Q$ since they cannot be tangent as they have a common interior point (for example $AC \cap BD$). By the definition of ω, R is also in the interior of ω. Let the line QR intersect ω in points Q' and R'. The pairwise radical axes of circles $\omega, (ACQ), (BDQ)$ are lines AC, BD, QR, therefore they are concurrent in point H, the radical centre. But then

$$HQ' \cdot HR' = HA \cdot HC = HQ \cdot HR < HQ' \cdot HR',$$

which is impossible, we have reached a contradiction.

Therefore there are exactly three points of P on ω, let these be A_1, A_2 and A_3. Let X be a fixed interior point and Y any interior point different from X. Then the circle (XYA_i) cannot intersect ω in a fourth point (as there are exactly three points on ω). This leaves two possibilities: either it is tangent to ω or passes through another point A_j. Now look at circles (A_iA_jX) $(1 \leq i, j \leq 3)$ where (A_iA_iX) is the circle passing through A_i, X and tangent to ω. These six circles have at most $\binom{6}{2}$ intersection points inside ω that are different from X, but every point Y is a point of this kind. Since there are 2020 such points Y inside ω, we have reached a contradiction again and therefore concluded the proof.



(Back to problems)

3. For the solution, see Category E Problem 4.

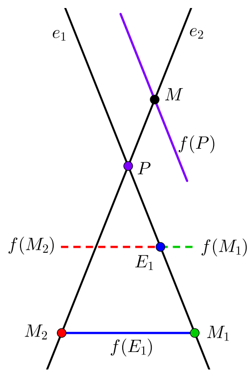
(Back to problems)

4. a) Suppose indirectly that f is a surjective polarising function. Let us take two parallel lines, let these be l_1 and l_2. Then there are two (disctint) points L_1, L_2 for which $f(L_1) = l_1$ and $f(L_2) = l_2$. But since the line L_1L_2 has a preimage, let K be such a point (meaning that $f(K) = L_1L_2$). Now by the condition since $L_1, L_2 \in f(K)$, $K \in f(L_1), f(L_2)$. But $f(L_1) \cap f(L_2) = l_1 \cap l_2 = \emptyset$, which is a contradiction.

b) Let f be the function that maps point (a, b) to the line defined by the equation $x = by - a$. It is clear that this function is injective, let us now show that it is polarising. Suppose that point (c, d) is on line $f((a, b))$, meaning that $c = bd - a$. By rearranging this we get that $a = db - c$, therefore (a, b) is also on $f((c, d))$.

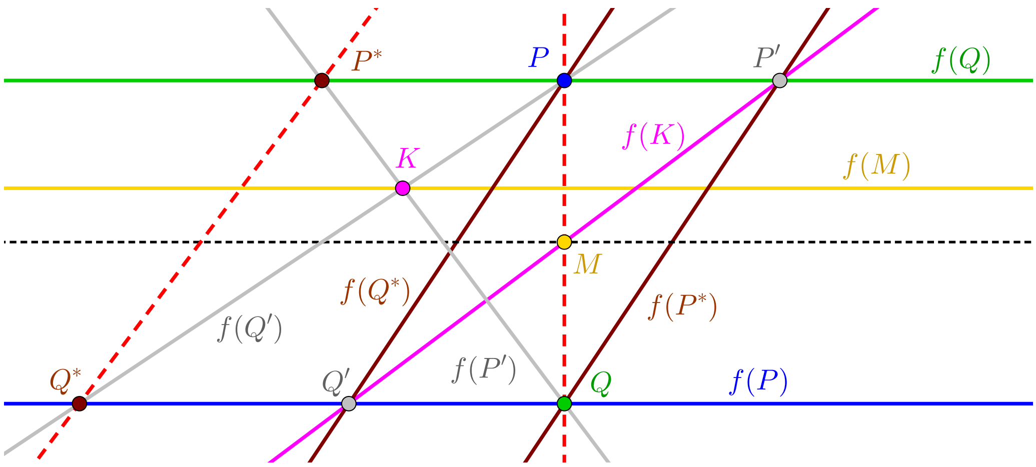
c) Now suppose indirectly that there is no such point. The key will be that f cannot be far from being surjective. Let us call the line e *lonely* if it has no preimage, meaning that it is not the image of any point. Then if $E_1, E_2 \in e$, then lines $f(E_1), f(E_2)$ cannot intersect, since if they did and $M = f(E_1) \cap f(E_2)$, then by the conditions of polarising $E_1, E_2 \in f(M)$, meaning that $f(M) = e$ which is impossible. Therefore the images of the points of a lonely line are different (by injectivity) and pairwise parallel lines.

Now let us regard a point P and its pencil of lines $\mathcal{S}$ (the set of lines passing through P). We will show that there is at most one line from $\mathcal{S}$ that does not have a preimage. Suppose that lines e_1 and e_2 are both like this. Then the image of every point on e_1 is a line in the direction $\mathbf{u}_1$, while the image of every point on e_2 is a line in the direction $\mathbf{u}_2$. But since $P \in e_1, e_2$, it implies that $\mathbf{u}_1 \parallel \mathbf{u}_2 \parallel f(P)$. Suppose that $e_1 \parallel f(P)$, then the point $M = e_2 \cap f(P)$ exists. Since $M \in f(P)$, therefore $P \in f(M)$. But since $M \in e_2$, it is also true that $f(M) \parallel f(P)$, meaning that $f(M) = e_2$, but this is a contradiction as we supposed that e_2 has no preimage. Therefore neither of e_1 or e_2 are parallel to $f(P)$. Let $E_1 \in e_1$ be a point for which $P \notin f(E_1)$ (such point exists by injectivity and by parallelity). Then points $M_1 = f(E_1) \cap e_1$, $M_2 = f(E_1) \cap e_2$ exist and are distict. But these two intersection points are on $f(E_1)$, therefore by the conditions of polarising-ness $E_1 \in f(M_1), f(M_2)$. But since both intersection points lie on $f(E_1)$, it means that $E_1 \in f(M_1), f(M_2)$. But since these two lines are both parallel to $f(P)$ and pass through E_1, it follows that $f(M_1) = f(M_2)$ but this contradicts the injectivity.



With this f is indeed very close to being surjective: in every pencil of lines through a point P there is at most one lonely line. Now we will show that in every pencil of lines there is exactly one lonely line. If P is not such a point, then the line through P parallel to $f(P)$ would have a preimage Q which would lie on $f(P)$, but then line PQ is not lonely as it passes through P, but its preimage would have to lie both on $f(P)$ and $f(Q)$ but it is impossible as these lines are parallel.

Now we know that in every pencil of lines there is exactly one lonely line. Observe furthermore that the lonely lines must be parallel since if two of them had an intersection, then their intersection would have two lonely lines in its pencil. Let us now take an arbitrary point P and consider the line through P that is parallel to $f(P)$ (which is different from $f(P)$ as we supposed indirectly). This line cannot be lonely as otherwise for any point $Q \in f(P)$ the lonely line through would be parallel to $f(P)$, therefore would be the same as $f(P)$, which is not lonely. Therefore there is a point Q for which $f(Q)$ is the line passing through P and parallel to $f(P)$. And as we have seen, line PQ has to be lonely.



Therefore all lonely lines are parallel to PQ. Now we will show that if M is the midpoint of segment PQ, then $f(M)$ passes through M. Clearly $f(M) \parallel f(P), f(Q)$ as the line PMQ is lonely. Now let us take a point K on $f(M)$ that is not on line PQ. Then $f(K)$ passes through M and intersects lines $f(P)$ and $f(Q)$ in points Q' and P' respectively. Since M is a midpoint, the quadrilateral $PP'QQ'$ is a parallelogram. Let $Q^* = PK \cap f(P)$ and $P^* = QK \cap f(Q)$. Notice that $f(P')$ will be the line QK and therefore $f(P^*)$ is the line $P'Q$. Similarly we can show that $f(Q^*)$ is the line PQ'. But since $P'Q \parallel PQ'$, therefore the line P^*Q^* has to be lonely since otherwise its preimage would lie on both lines (and this cannot happen as they are parallel). Therefore $P^*Q^* \parallel PQ$, which implies that PQQ^*P^* is a rectangle, therefore K, the intersection of diagonals lies on the midsegment, meaning that $M = f(M)$ and this is a contradiction.

(Back to problems)

5. Part **a)** can be proven on its own, but here we are presenting a solution that works for both cases. Throughout the solutions the equations will be written modulo p. We will show that there are p^{k-1} ordered k-tuples. This can be conjectured by looking at small values of k.

The case $p = 2$ is easy as the first $k-1$ terms can be chosen arbitrarily and there is only one choice for a_k, meaning that there are 2^{k-1} solutions. From now on suppose that p is an odd prime.

The idea is the following: we will regard all series $(a_1, a_2, ..., a_{k-1})$ and observe how many suitable a_k exist.

We can observe the following:

- It is known that is q is a nonzero quadratic residue class module p, then there are exactly two numbers c and $-c$ that satisfy $x^2 = q$. Therefore if there is a suitable a_k for a series $(a_1, ..., a_{k-1})$, then there are exactly two of them which are multiples by -1 modulo p. This happens exactly when $p - \sum_{i=1}^{k-1} a_i^2$ is a nonzero quadratic residue. If $p - \sum_{i=1}^{k-1} a_i^2$ is a quadratic non-residue, then there is no suitable a_k and if $p - \sum_{i=1}^{k-1} a_i^2 = 0 \pmod p$ then there is only $a_k = 0$.
- It is also known that if l is a quadratic non-residue modulo p, then for any $p \nmid k$ exactly one of k and $l \cdot k$ will be a quadratic residue.

We will find a bijection ϕ on

$$S = \{(a_1, a_2, ..., a_{k-1}) \mid \forall i \ : \ a_i \in \{0, 1, ..., p-1\}\}$$

for which the following holds: for all $\mathbf{x} \in S$ either in both of $\mathbf{x}$ és $\phi(\mathbf{x})$ the sum of the squares of the elements is 0, meaning that there is only one suitable a_k, or only one of them has suitable a_k, but that one has 2. Then for all pairs of $(\mathbf{x}, \phi(\mathbf{x}))$ there are exactly two solutions, meaning that in total there are p^{k-1} solutions.

Clearly there is a quadratic non-residue l for which $l-1$ is a quadratic residue. Let n be a number for which $n^2 = l - 1$. Since $l \neq 1$, $n \neq 0$.

Now we are defining the bijection:

$$\phi : S \to S$$

$$\phi((a_1, a_2, ..., a_{k-1})) = (na_1 + a_2, a_1 - na_2, na_3 + a_4, a_3 - na_4, ..., na_{k-2} + a_{k-1}, a_{k-2} - na_{k-1})$$

meaning that if $\mathbf{a} = (a_1, a_2, ..., a_{k-1})$ and $\phi(\mathbf{a}) = \mathbf{b} = (b_1, b_2, ..., b_{k-1})$ then $b_{2i+1} = na_{2i+1} + a_{2i+2}$ és $b_{2i+2} = a_{2i+1} - na_{2i+2}$.

Firstly let us show that this indeed a $S \to S$ bijection. For this what is needed is that for any element $\mathbf{b} = (b_1, b_2, \dots b_{k-1}) \in S$ there is exactly one $\mathbf{a} \in S$ for which $\phi(\mathbf{a}) = \mathbf{b}$. To show this we need that there is only one choice of a_{2i+1}, a_{2i+2} for which $b_{2i+1} = na_{2i+1} + a_{2i+2}$ and $b_{2i+2} = a_{2i+1} - na_{2i+2}$. It is enough to show this for $i = 0$ only, as it follows for all other i. By multiplying the first equation by n and summing with the second one we get that $nb_1 + b_2 = (n^2+1)a_1 = la_1$, therefore the only choice is $a_1 = \frac{nb_1+b_2}{l}$. From this $a_2 = b_1 - na_1$. The solution is indeed unique, therefore ϕ is a bijection.

Then

$$(na_{2i+1} + a_{2i+2})^2 + (a_{2i+1} - na_{2i+2})^2 =$$
$$= n^2 a_{2i+1}^2 + 2na_{2i+1}a_{2i+2} + a_{2i+2}^2 + a_{2i+1}^2 - 2na_{2i+1}a_{2i+2} + n^2 a_{2i+2}^2 =$$
$$= (n^2+1)(a_{2i+1}^2 + a_{2i+2}^2) = l(a_{2i+1}^2 + a_{2i+2}^2),$$

meaning that $\sum_{i=1}^{k-1} b_i^2 = l \sum_{i=1}^{k-1} a_i^2$. From this we get that $p - \sum_{i=1}^{k-1} b_i^2 = l\left(p - \sum_{i=1}^{k-1} a_i^2\right)$, therefore either both of them are zero or (by observation 2) only one of them is a quadratic residue, meaning that ϕ is as we desired. Therefore the number of solutions is p^{k-1}.

(Back to problems)

6. Answer: The second player wins if $n = 7$, or $n = 8m + 1$ or $n = 8m + 4$.

Restatement: Consider the following, different game: A board contains the number 4 initially. Two players aternate writing new numbers on the board, either a 4, a 3, or two instances of the number 2, except during the first turn, when First can choose from a set S, where S is fixed, and $S \in \{\{\emptyset\}, \{\emptyset, \{2\}\}, \{\{3\}, \{2\}\}, \{\{4\}, \{3\}, \{2, 2\}\}\}$. After d such moves, the players have a new moveset:
-Delete a 4, and write a 3, or
-Delete a 4, and write a 2, or
-Delete a 3, and write a 2, or
-Delete a 3, or
-Delete a 2.
The player who can't make a move loses.

Claim: The new game is equivalent to the one in the problem, according to the following parametrisation:
-If $n = 4m + 1$, then $S = \{\emptyset\}$,
-If $n = 4m + 2$, then $S = \{\emptyset, \{2\}\}$,
-If $n = 4m + 3$, then $S = \{\{3\}, \{2\}\}$,
-If $n = 4m + 4$, then $S = \{\{4\}, \{3\}, \{2, 2\}\}$. Also let

$$d = \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Notice that the largest k for which there is a size k move decreases by 4 every time, and the sub-table a player chooses is always contained in the sub-table chosen on the previous turn.

The numbers on the board then correspond to the sizes of the sub-tables generated with size at most 4.

2. phase: In the new game, call phase 2 the stage when only the new moveset is available. Let (x, y, z) be the state, where x denotes the number of 4s, y the number of 3s and z the number of 2s. Then $x \geq 0$, $y \geq 0$, $z \geq 0$ always holds. Then the legal moves are:

$$\begin{gathered}(x, y, z) \to (x-1, y+1, z)\\ (x, y, z) \to (x-1, y, z+1)\\ (x, y, z) \to (x, y-1, z+1)\\ (x, y, z) \to (x, y-1, z)\\ (x, y, z) \to (x, y, z-1)\end{gathered}$$

Let the starting state in phase 2 be (x_0, y_0, z_0).

Claim: The player who comes second in phase 2 wins, if y_0 ans z_0 are even, first wins otherwise.

This is easy to see: let (x, y, z) be a winning state, is y and z are even. From a non-winning state, one can always make a legal move, and in fact, make a move that leads to a winning state.

1. phase: Let's consider the cases where d is odd or even.

Even d: We'll show that if $n = 4m+2$ or $n = 4m+3$, then First wins, otherwise Second does. Notice that phase two is also started by First, so their aim is to make either y_0 or z_0 odd.
-$n = 4m + 1$: First is forced to write nothing. Second should then write a 4, and copy First after that.
-$n = 4m + 2$ or $n = 4m + 3$: First writes a 2. The parity of the number of 2s remains odd, so First wins.
-$n = 4m + 4$: Second should copy what First does.

Odd d: We'll show First always wins.
-$n = 4m + 1$: First writes nothing, then copies second.
-$n = 4m + 2$: First writes nothing. The number of 2s remains even, with their last move, First insures that the number of 3s is even as well.
-$n = 4m + 3$ or $n = 4m + 4$: First writes a 3. The number of 2s remains even, with their last move, First ensures, that the number of 3s is even as well. In case $d = 1$, First can't do so, and Second wins in this case.

(Back to problems)

4.4 Final round – day 2

4.4.1 Tables

#	ANS	Problem	P
C-1	4	Erik selected some edges of a cube in such a way	3p
C-2	26	Zorka wanted to know how old Captain Panna is,	3p
C-3	372	What is the only 3-digit number that is divisible by 4	3p
C-4	24	Albert, the captain, sailed his ship 15 km in a straight line	3p
C-5	84	Kartal fills the fields of a 5×5 table	4p
C-6	5	A digital display can show an integer	4p
C-7	105	A criminal octopus has eight tentacles.	4p
C-8	40	In the Kingdom of Letters, an election is held	4p
C-9	273	We glued together 25 standard dice in the manner	5p
C-10	648	What is the smallest positive integer such that	5p
C-11	14	On the circles of the diagram there are 9 ants	5p
C-12	3000	On the beach of the island of Óxisz, a sandcastle	5p
C-13	120	A number is called a *duck number*	6p
C-14	575	We glued together a $5 \times 7 \times 9$ cuboid from small cubes	6p
C-15	2563	Fill in the grid with the digits $1, 2, 3, 4, 5, 6$	6p
C-16	194	Ibolya, Hanga, Kamilla, and Rózsa are distributing flowers	6p

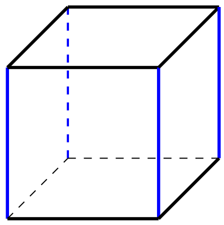
#	ANS	Problem	P
D-1	26	Zorka wanted to know how old Captain Panna is,	3p
D-2	50	Leila, the one-eyed pirate, was in her ship	3p
D-3	84	Kartal fills the fields of a 5×5 table	3p
D-4	5	In the board game called Quadropoly, Csaba ended up in jail.	3p
D-5	273	We glued together 25 standard dice in the manner	4p
D-6	888	A swordfish has a special relationship	4p
D-7	66	Given rectangles $ABCD$ and $AEFG$ such that E	4p
D-8	648	What is the smallest positive integer such that	4p
D-9	120	A number is called a *duck number*	5p
D-10	14	On the circles of the diagram there are 9 ants	5p
D-11	575	We glued together a $5 \times 7 \times 9$ cuboid from small cubes	5p
D-12	45	Captain Morgan threw 9 darts aiming at the board	5p
D-13	180	At an individual maths competition there were 3 students	6p
D-14	31	Dorka wrote some different positive integers on a piece of paper	6p
D-15	2563	Fill in the grid with the digits $1, 2, 3, 4, 5, 6$	6p
D-16	62	The 3×4 grid shown on the diagram	6p

#	ANS	Problem	P
E-1	5	In the board game called Quadropoly, Csaba ended up in jail.	3p
E-2	84	Kartal fills the fields of a 5×5 table	3p
E-3	5	A digital display can show an integer	3p
E-4	273	We glued together 25 standard dice in the manner	3p
E-5	66	Given rectangles $ABCD$ and $AEFG$ such that E	4p
E-6	8128	A number is called a *duck number*	4p
E-7	888	A swordfish has a special relationship	4p
E-8	14	On the circles of the diagram there are 9 ants	4p
E-9	1232	Let N be the smallest positive integer such that	5p
E-10	180	At an individual maths competition there were 3 students	5p
E-11	45	Captain Morgan threw 9 darts aiming at the board	5p
E-12	2563	Fill in the grid with the digits $1, 2, 3, 4, 5, 6$	5p
E-13	399	We glued together a $3 \times 3 \times 5$ cuboid	6p
E-14	64	The circle k with centre A has a radius	6p
E-15	35	Six villages, Arka, Bőcs, Cák, Dég, Ete and Füzér	6p
E-16	2600	The altitudes of an acute triangle are	6p

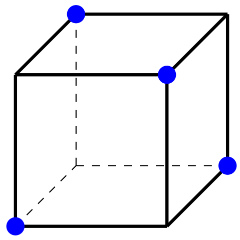
#	ANS	Problem	P
E^+-1	8128	A number is called a *duck number*	3p
E^+-2	888	A swordfish has a special relationship	3p
E^+-3	14	On the circles of the diagram there are 9 ants	3p
E^+-4	1232	Let N be the smallest positive integer such that	3p
E^+-5	31	Dorka wrote some different positive integers on a piece of paper	4p
E^+-6	45	Captain Morgan threw 9 darts aiming at the board	4p
E^+-7	8064	At an individual maths competition there were 4 students	4p
E^+-8	1041	We glued together a $3 \times 5 \times 5$ cuboid from small cubes	4p
E^+-9	115	Gabi asked Beni when his birthday was.	5p
E^+-10	64	The circle k with centre A has a radius	5p
E^+-11	2563	Fill in the grid with the digits $1, 2, 3, 4, 5, 6$	5p
E^+-12	2600	The altitudes of an acute triangle are	5p
E^+-13	35	Six villages, Arka, Bőcs, Cák, Dég, Ete and Füzér	6p
E^+-14	1139	What is the value of $\sum\limits_{k=1}^{17} \frac{1}{k(k+1)(k+2)(k+3)}$?	6p
E^+-15	117	Benjamin thought of a real number x	6p
E^+-16	50	A positive integer n is called *infernal*	6p

4.4.2 Category C

1. Erik can choose 4 edges, for example the following way:



We will show that he couldn't have selected more edges. For this consider the following 4 corners of the cube:



All edges of the cube connect to one of these corners and for every corner he can choose at most one edge that connects to it. This means that the maximum number of edges is indeed 4.

(Back to problems)

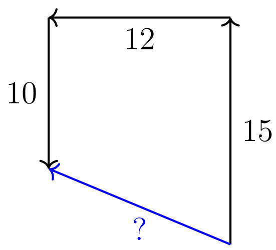
2. Let x denote the age of Captain Panna and y the age of Tamás (in years). Panna is 25 years older than Tamás, so $x = y + 25$. Tamás will be will be half as old as Panna is now, so $y + 12 = \frac{x}{2}$. Solving the system of equations, we get $x = 26$, so Panna is 26 years old.

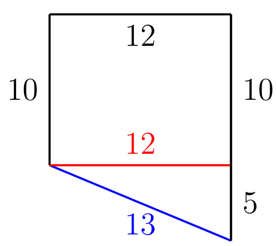
(Back to problems)

3. Let the 3-digit number be $\overline{xyz}$, where $x > 0$ and $x, y, z \leq 9$. We know that $x = z + 1$ and $x \cdot z + 1 = y$. If $x \geq 4$, then $z \geq 3$, so $y \geq 4 \cdot 3 + 1 > 9$, which is impossible. Therefore only these cases are possible: $x = 1, y = 1, z = 0$, or $x = 2, y = 3, z = 1$, $x = 3, y = 7, z = 2$ (The value of x determines y and z). Out of these, only 372 is divisible by 4, thus the answer is 372.

(Back to problems)

4. Albert took the following route (in black) instead of the shortest blue route:





After drawing the red segment, we get a right angle triangle with legs 5 and 12, meaning that the length of the blue segment is $\sqrt{12^2 + 5^2} = 13$ km. Albert's route was $15+12+10 = 37$ km long, this means that he could have saved $37 - 13 = 24$ km.

(Back to problems)

5. Let's observe that no matter how we select a row and a column, they will always have one cell in common. This value gets added once and subtracted once, meaning that it doesn't count towards the difference. Now we need to take the 4 remaining cells of the biggest row of Benedek (we need these to be as big as possible) and subtract the 4 remaining cells the smallest column of Dani (we need these to be as small as possible). The biggest difference that can be obtained is $25 + 24 + 23 + 22 - (1 + 2 + 3 + 4) = 25 + 24 + 23 + 22 - 10 = 84$.

(Back to problems)

6. Due to the faulty segment the numbers 5, 9 and 6, 8 appear the same, this means that 5-6-7-8-9 appears the same as 9-8-7-6-5. If the number at the units place increases from 5 to 9 or if it decreases from 9 to 5, it cannot be determined which is the direction of the numbers, since the digits on the higher places also don't change during this time. Therefore it is possible that the display showed five numbers while Béla was watching.

In any case where the number at the units place is different from 5-6-7-8-9, we can tell in one step which direction the counting goes, as the other numbers cannot be confused with any other. This means that Béla couldn't have watched more than five numbers.

(Back to problems)

7. The first handcuffed pair of tentacles can be chosen $\binom{8}{2}$ ways, the second pair out of the remaining 6 tentacles can be chosen $\binom{6}{2}$ ways, the third pair $\binom{4}{2}$ ways and the remaining two tentacles will be the last pair. Since the order in which we choose the pairs doesn't matter, we need to divide by the possible orders of the pairs which is 4!. Altogether the tentacles of the octopus can be handcuffed $\frac{\binom{8}{2}\cdot\binom{6}{2}\cdot\binom{4}{2}}{4!} = 105$ ways.

(Back to problems)

8. We will show that $x = 10$ and $y = 4$. Then $x \cdot y = 40$.

If we want to reach the least possible number of main votes, then let there be two groups where everyone supports A and 6 groups where only one person supports A. Then A gets 4 main votes.

Less is not possible as if A gets at most 3 main votes, then have to be at least 5 groups which A doesn't get a main vote from, therefore at most one person can support A in these groups. If there is a sixth such group, then in the remaining two groups there would have to be 5 supporters of A each, which leads to 4 main votes. If A gets the main votes from 3 groups, then at least 11 people support them from these groups, therefore there has to be a group where at least 4 people support A, which means 2 main votes. Therefore in either case A receives at least 4 main votes.

If we want to react the largest possible number of main votes for A, then let's regard the case when there are 5 groups where 3 people support A and one person each supports B and C. It doesn't matter what the other three groups look like is only one supporter of A in them, which is not worth a main vote. Thus in this case A gets two main votes from the first five groups, which is 10 in total.

Now let's assume that it is possible that A gets at least 11 main votes, then they get two main votes from at least three groups. In these groups at least 3 people support A. If they get two main votes from exactly three groups, then it the remaining five groups there are at most 7 supporters, which is at most 3 main votes, which is 9 altogether. If A gets two main votes from exactly 4 groups, then in the remaining groups at most 4 people support them, which gives at most two main votes, which is 10 altogether. Finally, if A gets two main votes from five groups, then in the remaining groups they only have one supporter which is not worth a main vote.

Therefore the answer is 40.

(Back to problems)

9. Since the faces glued together have the same number of dots and the "stacks" of cubes consist of an odd number of cubes (1, 3 or 5), therefore the sum of the number of the dots on the two ends of any stack of cubes is 7. The stacks are in three different directions and in each direction there are 13 of them. Each visible face of the cubes is the end of exactly one stack of cubes. This means that the number of visible dots is $7 \cdot 3 \cdot 13 = 273$.

(Back to problems)

10. It is known that a number is a perfect square if any only if in its prime factorisation every power is divisible by 2, and that a number is a perfect cube if any only if in its prime factorisation every power is divisible by 3.

Let n be the smallest positive integer satisfying the conditions of the problem. Since $n/2$ is a perfect cube, this means that in the prime factorisation of n the power of 2 is odd and the power of 3 is even. Similarly since $n/3$ is a perfect cube, in the prime factorisation of n the power of 2 is divisible by 3 and the power of 3 has a remainder of 1 when divided by 3. This means that n is of the form $n = k \cdot 2^3 \cdot 3^4 \cdot 2^{6a} \cdot 3^{6b}$ where k, a and b are integers.

Clearly we get the smallest possible n when $k = 1$ and $a = b = 0$, then $n = 2^3 \cdot 3^4 = 648$, which satisfies the conditions of the problem.

(Back to problems)

11. Every ant on a given circle goes either clockwise or counterclockwise. When an ant changes circle then its direction also switches and if it doesn't change circle, then its direction on the circle doesn't change. Therefore the possible directions of a given ant on the circles is uniquely determined and has a checkerboard pattern.

The two ants in the second row would go clockwise on the top left circle, while the rest would go counterclockwise. Since we know that ants going in the same direction cannot meet, only ants in opposite directions can meet. If every ant is on the same circle, all the possible encounters happen.

Therefore the answer is 14, the two ants in the first group can meet all the 7 other ants.

(Back to problems)

12. The volume of a rectangular section of the moat is 1 m $\cdot$ 2 m $\cdot$ 35 cm= 1 m $\cdot$ 2 m $\cdot$ 0,35 m = 0,7 m^3.

At the vertices of the sandcastle we have the following angles: $60° + 90° + 120° + 90° = 360°$, this means that the angles of the rhombus shaped moat are $60°, 120°, 60°, 120°$ and since its sides are 1 m long, the rhombus is the same as if two equilateral triangles of sides of 1 m were glued together. The altitude of one such triangle is $\frac{\sqrt{3}}{2}$ m. The depth of the moat at this section is $\sqrt{12}$ dm $= \frac{\sqrt{12}}{10}$ m $= \frac{2\sqrt{3}}{10}$ m. This means that the volume of a rhomboid section of the moat is 1 m $\cdot \frac{\sqrt{3}}{2}$ m $\cdot \frac{2\sqrt{3}}{10}$ m $= \frac{2\cdot 3}{2\cdot 10}\mathrm{m}^3 = \frac{3}{10}\mathrm{m}^3 = 0,3\mathrm{m}^3$.

There are three rectangular and three rhomboid sections, therefore their total volume is $3 \cdot (0,3 + 0,7)\mathrm{m}^3 = 3\mathrm{m}^3$.

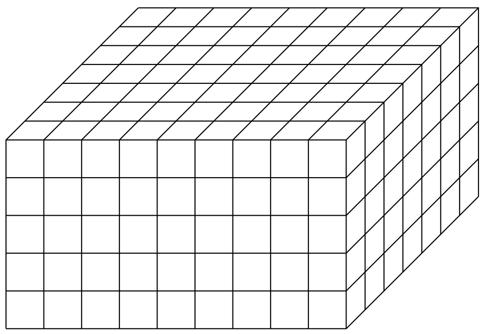
We know that one m^3 equals 1000 liters, therefore the total volume of water in the moat is 3000 liters.

(Back to problems)

13. Altogether there are 2^8 8-digit numbers consisting of only digits 1 and 2. If we pair these with their reverse, exactly one of them (the smaller one) will be a duck number, except when the number and its reverse are the same. These can be counted the following way: the first 4 digits can be chosen arbitrarily and these determine the last 4 digits, therefore there are 2^4 of them. In conclusion the total number of duck numbers is $\frac{2^8-2^4}{2} = 120$.

(Back to problems)

14. Let us the regard the cuboid the following way:



Anita obtained the resulting cuboid by removing some layers of yellow cubes from the bottom, top, front, back, left and right sides of the original cuboid. Since the red cube is not on the surface, this means that from the bottom and the top she could have removed at most one layer each, from the front and back at most two layers each and from the left and right at most three layers each. Since not removing any layers from a given direction is also a possibility, this means that altogether she could have chosen the number of layers to remove in $2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 = 576$ ways. Since this counts the case where no cubes got removed, in total she could have obtained 575 different cuboids.

(Back to problems)

15.

	3		3	1		
5				6		
2	2	4	6	5	3	1
5				4		
				3		
				2		
3				1	2	

I. Next to the 1, only a 2 or a 3 can be placed due to the difference condition, but the 2 is excluded by the 2 already in the column. In the 4th column, the numbers must follow each other in order due to the difference condition, and if they were reversed, a 2 and a 6 would end up next to each other in the last row, which is not allowed due to the 3 difference in the row. In the 2nd row, the first three places must be filled with 2, 4, and 6, in this order, because the 2 and the 6 cannot be neighbors, and the 2 cannot be next to the 5.

	3		3	1		
5				6	1	
2	2	4	6	5	3	1
5	1	6		4		
	4			3		
				2		
3				1	2	

II. Since a difference of 5 can only come from $6-1$ among these numbers, the 1 and 6 must be next to each other in the 1st row. However, the 1 cannot be placed in the 3rd column next to the 6 because of the 3 above the column. Due to the 5 before the 3rd row, the 1 and 6 must be next to each other there, and because of the 1, they must be to the left of the 4. Considering the column constraints, the 6 must be in the 2nd column, and subsequently, the 1 cannot be in the 3rd column. The 4th element in the 1st column cannot be 2 or 3, so it must be 4 due to the column constraint.

	3		3	1		
5				6	1	
2	2	4	6	5	3	1
5	1	6		4		
	4	1	2	3		
			1	2		
3	6			1	2	

III. In the 3rd column, the 1 can only be placed in two positions due to the other 1s. If it is placed in the 4th position, then, considering the 3rd column and the 6th row constraints, only 3 and 4 can occupy the 5th and 6th positions. Thus, the 3rd position would have to be 2, which cannot be next to a 6. Therefore, the 1 must be placed in the 5th position of the 3rd column, and accordingly, in the 4th position of the 2nd column. For the 3rd column, the 4th position cannot be a 3 or 4, so it must be a 2. In the last row, considering the previously placed 6s and the row constraint, a 6 can only be placed in the 1st position.

	3		3	1		
5		2		6	1	
2	2	4	6	5	3	1
5	1	6	3	4		2
	4	1	2	3		
			1	2	4	
3	6			1	2	

IV. In the 3rd row, based on the previously placed 2s, the 2 can only go into the last position. Due to the earlier placement of 2s, in the 1st row, the 2 can only be placed in the 2nd position. Similarly, due to the previously placed 4s, in the 5th column, the 4 can only be placed in the 5th position. Additionally, because of the previously placed 3s, in the 3rd row, the 3 can only be placed in the 3rd position.

	3		3	1		
5	3	2	5	6	1	4
2	2	4	6	5	3	1
5	1	6	3	4	5	2
	4	1	2	3	6	5
	5	3	1	2	4	6
3	6	5	4	1	2	3

V. In the 3rd column, under the 1, due to the column constraint, only a 4 can be placed, after which the remaining missing 4 can also be entered. From here, the remaining empty fields can be filled in a few steps.

Therefore the desired four-digit number is 2563.

(Back to problems)

16. First we need to determine in which round and at which house they finish giving out flowers. Since Ibolya and Kamilla give flowers to every second house and there are an even number of houses, they will only give flowers to the houses with an odd number, and these houses will receive a flower already in the first round.

Therefore we need to determine which of the even numbered houses will receive a flower last. On the inner side they can only receive from Hanga, on the outer side only from Rózsa.

In the first round Rózsa gives flowers to the houses whose number has a remainder 1 when divided by 5, and since 26 is the last such house, she starts the next round at house number 5 and gives flowers to the houses whose number is divisible by 5. In the following rounds she gives flowers to the houses with the remainder 4, 3 and 2 respectively, therefore on the outer side the last house to receive a flower is the one with the highest number that has a remainder 2 when divided by 5, which is 22.

Similarly on the inner side Hanga gives out flowers to the houses with remainders 1, 2 and 0 when divided by 3, meaning that by the end of the third round, all houses on the inner side will have received a flower.

Thus the last house to receive a flower is house 22 on the outer side, in step $4 \cdot 26 + 22 = 126$.

In this time Ibolya and Kamilla both give out $\frac{126}{2} = 63$ flowers, Hanga gives out $\frac{126}{3} = 42$, and Rózsa $\frac{125}{5} + 1 = 26$, since she gives a flower in step 126.

The four girls altogether give out $63 + 63 + 42 + 26 = 194$ flowers.

The names of the girls are flowers: Hanga - heather, Rózsa - rose, Kamilla - chamomile, Ibolya - viola.

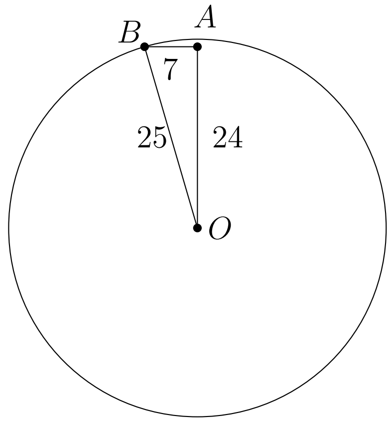
(Back to problems)

4.4.3 Category D

1. For the solution, see Category C Problem 1.

(Back to problems)

2. Let the center of the sea (where Leila is setting off from) be O. From here she goes straight for 24 km and gets to point A. After turning left by 90 degrees, she goes straight another 7 km and reaches point B, which is on the shore of the circular sea. Triangle OAB has a right angle at A and by the Pythagorean theorem we get that $OB = 25$ km. The diameter of the sea is exactly double its radius, therefore the answer is 50.



(Back to problems)

3. For the solution, see Category C Problem 5.

(Back to problems)

4. When throwing just one die, the probability of any number is $\frac{1}{6}$. When using two dice, we can distinguish 36 cases that all have the same probability, for example the probability that the first die is 3 and the second one is 4 is $\frac{1}{36}$. Since the numbers on each die are between 1 and 6, their sum is between 2 and 12. Csaba gets out of jail if the sum of the two numbers is 4, 8 or 12.
$4 = 1 + 3 = 2 + 2 = 3 + 1$ – the probability of the sum being 4 is $\frac{3}{36}$
$8 = 2 + 6 = 3 + 5 = 4 + 4 = 5 + 3 = 6 + 2$ –the probability of the sum being 8 is $\frac{5}{36}$
$12 = 6 + 6$ – the probability of the sum being 12 is $\frac{1}{36}$
Altogether: $\frac{3}{36} + \frac{5}{36} + \frac{1}{36} = \frac{9}{36} = \frac{1}{4}$, where the sum of the numerator and the denominator is $1 + 4 = 5$.

(Back to problems)

5. For the solution, see Category C Problem 9.

(Back to problems)

6. Since none of the digits are divisible by 8, therefore 0 and 8 cannot appear in the number. Furthermore we know that the product of the three digits is divisible by 8, which means that at least two of the digits are even, otherwise we wouldn't have enough factors of 2 in their product. Since the sum of the digits is also divisible by 8, it must be even and this means that the third digit also has to be even. Therefore the numbers consist of the digits 2, 4 and 6. The sum of the digits is divisible by 8, therefore it has to be either 8 or 16. From this we get that the number is either made up of two 6s and a 4 or two 2s and a 4. These numbers are 224, 242, 422, 466, 646, among which only 224 and 664 are divisible by 8, therefore these are the favourite numbers of the swordfish. The sum of the favourite numbers of the swordfish is $224 + 664 = 888$.

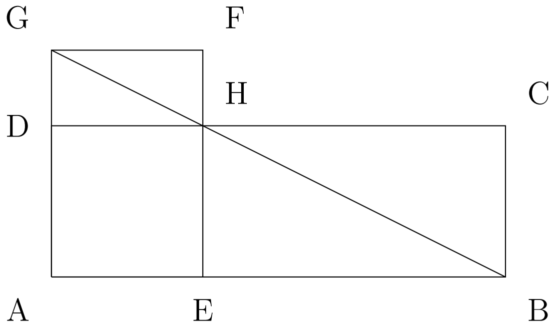
(Back to problems)

7. Let A_1 denote the area of rectangle $ABCD$ and let A_2 be the area of rectangle $AEFG$. Let x be the length of segment EB and y be the length of segment DG.

The area of the square $AEHD$ is 36, this means that $AE = AD = 6$ and that $A_1 = 6 \cdot (6+x)$ and $A_2 = 6 \cdot (6 + y)$, furthermore since $A_1 = 10 \cdot A_2$ we get that $6 \cdot (6 + x) = 10 \cdot 6 \cdot (6 + y)$, meaning that $y = \frac{x-54}{10}$.

Since points B, H, G are collinear, the triangles DHG are EBH congruent, meaning that $\frac{EH}{EB} = \frac{DG}{DH}$, which means that $\frac{6}{x} = \frac{y}{6}$ or in other words $xy = 36$. Now using that $y = \frac{x-54}{10}$ we obtain $x \cdot \frac{x-54}{10} = 36 \Rightarrow x^2 - 54x - 360 = 0$.

This is the same as $(x-60)(x+6) = 0$ and since $x > 0$, the only possible solution is $x = 60$. Therefore the answer is $AB = 6 + x = 66$.



(Back to problems)

8. For the solution, see Category C Problem 10.

(Back to problems)

9. For the solution, see Category C Problem 13.

(Back to problems)

10. For the solution, see Category C Problem 11.

(Back to problems)

11. For the solution, see Category C Problem 14.

(Back to problems)

12. Firstly let's suppose indirectly that the captain did not hit the X. If all 9 darts hit the 20 sector, then the sum is 180. For any other score he would need to hit the 10 at least once, which would give at most 170 in total.

Therefore Captain Morgan needs to hit the sector X at least once.

First let's count the number of values of X which are divisible by 5. According to the previous reasoning 5, 10 and 20 are not possible. We claim that every other possibility which is not greater than 175 is possible.

If X is of the form $20k + 15$ $(0 \le k \le 8)$, then 175 can be achieved with a throw of X and at most 8 20s.

If X is of the form $20k$ and $k \neq 1$, then with a throw of X, one 5, one 10 and at most six 20s we get 175.

If X is of the form $20k + 5$ $(1 \le k \le 8)$, then with a throw of X, one 10 and at most seven 20s we get 175.

Finally, if X is of the form $20k + 10$ $(1 \le k \le 8)$, then with a throw of X, one 5 and at most seven 20s we get 175.

Thus X can be any number divisible by 5 up to 175 except for 5, 10 and 20, which are 32 possibilities.

If X is not divisible by 5, then since all the other throws and the total score is divisible by 5, it is only possible if he threw X five times. Then the possible values of X can be determined by looking at the sum of the other 4 throws, subtracting that from 175 and dividing by 5.

It is not hard to see that every integer which is not greater than 80 and is divisible by 5 is a possible option for the sum of the other 4 throws, except for 75.

Thus the numbers 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 80 are possible from at most 4 throws, for which the corresponding values of X are: 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 19. These are 16 numbers, but 35, 30, 25 have been counted before, therefore we only get 13 new solutions.

Therefore there are $32 + 13 = 45$ possible values for X.

(Back to problems)

13. The teacher can only say for sure that class B has a higher total score than class A, if when ordered within class, every student in class B is ranked higher than the student with the same rank in class A. For example, the best student in class B is ranked higher than the best student in class A and so on. Without distinguishing the students within the classes, then there are five possible orderings according to this rule (a denotes a student from class A, b is one from class B): *bbbaaa*, *bbabaa*, *bbaaba*, *babbaa*, *bababa*. Since we need to distinguish the students within the classes, the 3 students in a class can be ordered in $3! = 6$ ways, meaning that we need to multiply 5 by $6 \cdot 6 = 36$. Therefore there are 180 rankings where the teacher can say for sure that the sum of the scores is higher in 12.b than in 12.a.

(Back to problems)

14. If Lili thought of a prime number, then it can only be obtained as a product if the prime number is among the ones written down by Dorka. Furthermore the same is true if Lili thought of a square of a prime number, since only distinct numbers can be multiplied.

The prime numbers between 2 and 100 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, in total 25 numbers. Squares of primes between 2 and 100 are: $4, 9, 25, 49$, a total of 4 numbers.

If for a prime p its fourth power is smaller than 100, then it can only be obtained as $p \cdot p^3$ or p^4, therefore at least one of p^3 and p^4 has to be written down on the paper. This is true for $p = 2, 3$, therefore at least 31 numbers have to be written down by Dorka. If she writes the before mentioned 29 numbers (the primes and their squares) and 16 and 81, then any number can be obtained.

(Back to problems)

15. For the solution, see Category C Problem 15.

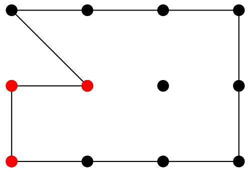
(Back to problems)

16. Let us distinguish the two inner points and the ten points on the perimeter. If we look at the polygon and number the grid points clockwise and we take the two inner points out, then we get the points on the perimeter in their original clockwise order as the polygon is not self-intersecting.

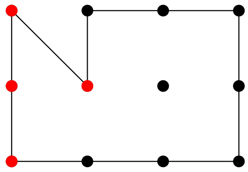
Therefore the only thing differentiating the possible polygons in the position of the two inner points compared to the other 10 points.

Let's consider cases based on which two points on the perimeter the left inner point is between. The diagram is symmetric with respect to the horizontal line through the middle, therefore we only need to consider cases where the left inner point goes between two points that are not in the bottom row.

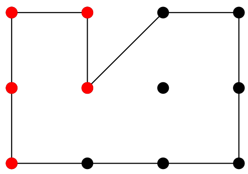
1. If it is connected to the two points on the left side:
   Then the right inner point can be inserted between any two other points except two reds, since in those cases it would be self-intersecting. This gives 9 possible polygons.



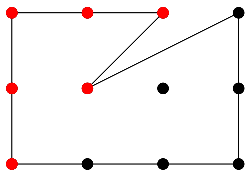
2. If it is connected to the top left corner and the point to the right of it:
   Then the right inner point can be inserted between any two other points except the four reds. This gives 8 polygons.



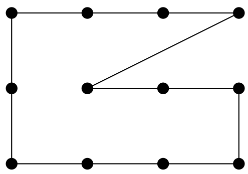
3. If it is connected to the two top middle points:
   Then the right inner point can be inserted between any point except between two reds. There are 7 such polygons.



4. If it is connected to the two rightmost top points:
   Similarly here the right inner point cannot be inserted between red points, this is 6 possible polygons.



5. If it is connected to the top right corner and the point below it:
   Then there is only one possible polygon.



Altogether these are $9 + 8 + 7 + 6 + 1 = 31$ possibilites, but the same reasoning can be applied to the right inner point, therefore there are twice as many total possible 12-gons, 62.

(Back to problems)

4.4.4 Category E

1. For the solution, see Category D Problem 4.

(Back to problems)

2. For the solution, see Category C Problem 5.

(Back to problems)

3. For the solution, see Category C Problem 6.

(Back to problems)

4. For the solution, see Category C Problem 9.

(Back to problems)

5. For the solution, see Category D Problem 7.

(Back to problems)

6. Altogether there are 2^{14} 14-digit numbers consisting of only digits 1 and 2. If we pair these with their reverse, exactly one of them (the smaller one) will be a duck number, except when the number and its reverse are the same. These can be counted the following way: the first 7 digits can be chosen arbitrarily and these determine the last 7 digits, therefore there are 2^7 of them. In conclusion the total number of duck numbers is $\frac{2^{14}-2^7}{2} = 8128$.

(Back to problems)

7. For the solution, see Category D Problem 6.

(Back to problems)

8. For the solution, see Category C Problem 11.

(Back to problems)

9. In the problem we are looking for the smallest integer n for which $n = 2 \cdot k^2 = 3 \cdot l^3 = 5 \cdot m^5$ for some integers k, l and m. It is known that if a sth power is divisible by prime p, then it is also divisible by p^s.

Since the half, third and fifth of n are all integers, n must be divisible by 2, 3 and 5. Because of the uniqueness of the prime factorisation, the power of 2 has to equal in l^3 and m^5, and since these are 3rd and 5th powers, the power of 2 in n must be divisible by 15. Similarly it can be shown that n is divisible by 3^{10} and 5^6.

From this follows that n is divisible by $2^{15} \cdot 3^{10} \cdot 5^6$ and it can be seen that $n = 2^{15} \cdot 3^{10} \cdot 5^6$ satisfies the conditions of the problem. The number of its divisors is $(15+1) \cdot (10+1) \cdot (6+1) = 1232$.

(Back to problems)

10. For the solution, see Category D Problem 13.

(Back to problems)

11. For the solution, see Category D Problem 12.

(Back to problems)

12. For the solution, see Category C Problem 15.

(Back to problems)

13. First let's count how many cuboids are possible without further restrictions: In each direction the two ends of the cuboid can be chosen arbitrarily, meaning that in the vertical direction there are $\binom{6}{2} = 15$ ways and in the other two directions $\binom{4}{2} = 6$ ways, therefore altogether there are $15 \cdot 6 \cdot 6 = 540$ possible sub-cuboids.

If we subtract from this the number of cuboids containing red on the surface, we will get the answer of the problem.

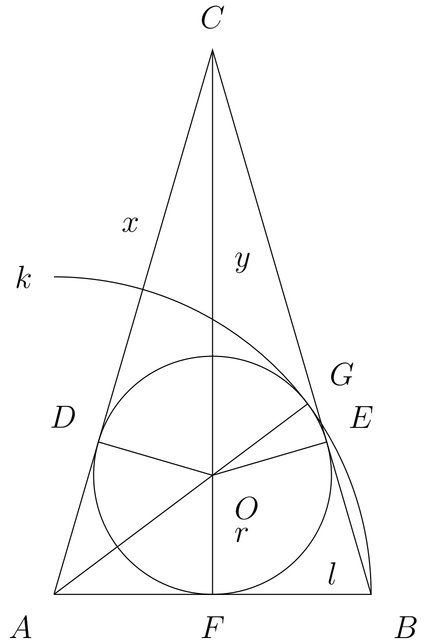
In order to have red on the surface, the cuboid must contain the red cube, the number of such cuboids is $\left(\frac{5+1}{2}\right)^2 \cdot \left(\frac{3+1}{2}\right)^2 \cdot \left(\frac{3+1}{2}\right)^2 = 144$, since the bottom of the cuboid can be one of the bottom 3 layers and the top of the cuboid can be one of the top 3 layers. Similarly in the other two directions the red cube has to be on the right side of the cut, meaning that the coordinates of the sides can be chosen 2 ways in these directions. All the choices for the sides are independent, therefore they need to be multiplied.

Not all of these are bad for us, since we want to keep the ones that contain the red cube on the inside. These are the cuboids that contain the middle $3 \times 3 \times 3$ cube, in the two remaining directions we have two choices (either keep that layer or not), giving us $2^2 = 4$ such cuboids.

Therefore the number of wrong cuboids is $144 - 4 = 140$, and the number of right ones is $540 - 140 = 400$, but this includes the case when no cubes are removed, therefore the answer is $400 - 1 = 399$.

(Back to problems)

14. Consider the following diagram:



Since F is the midpoint of segment AB, we know that $AF = 7$. Triangle AFO is right-angled (since segment AB is tangent to circle l in F), therefore according to the Pythagorean theorem $7^2 + r^2 = AO^2$. Circles l and k are tangent in G, therefore points A, O, G are collinear, therefore $AO + OG = AG$, thus $\sqrt{7^2 + r^2} + r = 14 \Rightarrow 7^2 + r^2 = (14 - r)^2 = r^2 - 28r + 196 \Rightarrow 28r = 147 \Rightarrow r = \frac{21}{4}$. Since the tangent segments are equal we know that $AD = AF$, $CD = CE$, $BF = BE$. Let $y = OC$, $x = CD$, $K = 14 + (7 + x) + (7 + x)$ (the perimeter of triangle ABC). Since triangle ABC is equilateral, therefore $m_{AB} = r + y$, thus $2 \cdot T_{ABC} = r \cdot K = r \cdot (28 + 2x) = 14 \cdot (r + y) \Rightarrow y = \frac{1}{7}rx + r$. Triangle CDO is right-angled, therefore $r^2 + x^2 = y^2 = \left(\frac{1}{7}rx + r\right)^2 = \frac{1}{49}r^2x^2 + \frac{2}{7}r^2x + r^2 \Rightarrow (1 - \frac{1}{49}r^2)x^2 = \frac{2}{7}r^2x$, since $x \neq 0$, thus $x = \frac{\frac{2}{7}r^2}{1-\frac{1}{49}r^2} = \frac{\frac{2}{7}\cdot\frac{21}{4}^2}{1-\frac{1}{49}\cdot\frac{21}{4}^2} = \frac{\frac{63}{8}}{(1-\frac{9}{16})} = \frac{\frac{63}{8}}{\frac{7}{16}} = 18$. Therefore $K = 14 + (7 + 18) + (7 + 18) = 64$.

(Back to problems)

15. Aron traveled 39 minutes longer than Alex. Based on the conditions, Aron could have waited a maximum of 4 minutes at the stop with buses every 5 minutes (since if he just missed the previous bus by 1 minute, he would have to wait 4 minutes for the next one).Similarly, it can be reasoned that at the stop with buses every 7 minutes, he could have waited a maximum of 6 minutes; at the stop with buses every 9 minutes, a maximum of 8 minutes; at the stop with buses every 11 minutes, a maximum of 10 minutes; and at the stop with buses every 12 minutes, a maximum of 11 minutes. Therefore Aron could have waited a maximum of $4+6+8+10+11 = 39$ minutes. However, this is only possible if Alex caught every bus exactly on time (meaning he had to wait 0 minutes due to transfers).

We want to determine the frequency of the bus service between each pair of towns. Let's first examine the bus service between Et and Fuse.Suppose the frequency is n minutes. That means Aron had to wait $n - 1$ minutes at Et so he arrived there after waiting $39 - (n - 1)$ minutes. This number must be divisible by n since Alex waited 0 minutes and the buses run every n minutes from that point.Among the given bus frequencies (5, 7, 9, 11, 12 minutes), only the 5-minute frequency satisfies this property. Therefore, the bus between Ete and Füzér runs every 5 minutes, and Aron waited $39 - 5 + 1 = 35$ minutes before reaching Et.

Let's examine the frequency of the buses between Deck and Et. Following the previous reasoning, if the buses run every D minutes then $35 - D + 1$ must be divisible by D. From the remaining frequencies this condition is only satisfied by 9 and 12. (and in these cases Aron waited $35 - 9 + 1 = 27$ or $35 - 12 + 1 = 24$ minutes before reaching Deck) . Let's consider the 12-minute case: Then, for the bus from Chap to Deck, which runs every C minutes, it most hold that C divides $24 - C + 1$. From the frequencies only 5 satisfies this, but we know that the 5-minute bus runs between Et and Fuse.Therefore, the buses between Deck and Et run every 9 minutes, and Aron waited 27 minutes before arriving at Deck.

For the bus running every C minutes between Chap and Deck the condition $C|(27 - C + 1)$. Among the remaining three bus frequencies, this condition is only satisfied by the 7-minute frequency, meaning Aron waited 21 minutes for the transfer to reach Chap. For the bus running every B minutes betweeen Bock and Chap the condition $B|(21 - B + 1)$ must be satisfied. Therefore B can only be 11 minutes That leaves the frequency between Arc and Bock to be 12 minutes.

Ben started from Arc 60 minutes after Alex, so he didn't have to wait for the 12-minute bus there. This means he arrived in Bock 60 minutes after Alex, where the buses depart every 11 minutes. Therefore, Ben arrived in Chap 66 minutes after Alex. The buses from Chap run every 7 minutes, so Ben reached Deck 70 minutes after Alex. From Deck, buses run every 9 minutes, which means Ben arrived in Et 72 minutes after Alex. Finally, with the buses from Et running every 5 minutes, Ben reached Fuse 75 minutes after Alex, which is 35 minutes after Aron.

(Back to problems)

16. Let's denote the sides of the triangle and their altitudes by a, b, c, m_a, m_b, m_c. The least common multiple of the altitudes is $\text{lcm} = 2^3 \cdot 3^2 \cdot 5^2 \cdot 13$.

Then let x such that $2 \cdot A = a \cdot m_a = b \cdot m_b = c \cdot m_c = \text{lcm} \cdot x$. From this we obtain for the sides $a = 2^3 \cdot 5 \cdot x = 40x$, $b = 3 \cdot 13 \cdot x = 39x$, $c = 5^2 \cdot x = 25x$, meaning that the half of the perimeter is $s = \frac{a+b+c}{2} = 52x$.

Then from Heron's formula we get that $2^2 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot x = \frac{1}{2}\text{lcm} \cdot x = \frac{1}{2}a \cdot m_a = A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{52x \cdot 12x \cdot 13x \cdot 27x} = \sqrt{2^4 \cdot 3^4 \cdot 13^2 x^2} = 2^2 \cdot 3^2 \cdot 13 \cdot x^2$, meaning that $x = 5^2 = 25$. From this we get that the perimeter is $P = 2s = 2 \cdot 52 \cdot 25 = 2600$.

(Back to problems)

4.4.5 Category E$^+$

1. For the solution, see Category E Problem 6.

(Back to problems)

2. For the solution, see Category D Problem 6.

(Back to problems)

3. For the solution, see Category E Problem 7.

(Back to problems)

4. For the solution, see Category E Problem 8.

(Back to problems)

5. For the solution, see Category D Problem 14.

(Back to problems)

6. For the solution, see Category D Problem 10.

(Back to problems)

7. The teacher can only say for sure that class B has a higher total score than class A, if when ordered within class, every student in class B is ranked higher than the student with the same rank in class A. For example, the best student in class B is ranked higher than the best student in class A and so on.

Without distinguishing the students within the classes, then the number of possible orderings according to this rule is exactly the 4th Catalan number, 14. They can also be easily listed: *bbbbaaaa*, *bbbabaaa*, *bbbaabaa*, *bbbaaaba*, *bbabbaaa*, *bbababaa*, *bbabaaba*, *bbaabbaa*, *bbaababa*, *babbbaaa*, *babbabaa*, *babbaaba*, *bababbaa*, *babababa* (here a denotes a student from class A, b is one from class B). Since we need to distinguish the students within the classes, we need to multiply by $4! \cdot 4!$. Therefore there are $14 \cdot 4! \cdot 4! = 8064$ rankings where the teacher can say for sure that the sum of the scores is higher in 12.b than in 12.a.

(Back to problems)

8. First let's count how many cuboids are possible without further restrictions: In each direction the two ends of the cuboid can be chosen arbitrarily, meaning that in the vertical and long direction there are $\binom{6}{2} = 15$ ways and in the third direction $\binom{4}{2} = 6$ ways, therefore altogether there are $15 \cdot 15 \cdot 6 = 1350$ possible sub-cuboids.

If we subtract from this the number of cuboids containing red on the surface, we will get the answer of the problem.

In order to have red on the surface, the cuboid must contain the red cube, the number of such cuboids is $\left(\frac{5+1}{2}\right)^2 \cdot \left(\frac{5+1}{2}\right)^2 \cdot \left(\frac{3+1}{2}\right)^2 = 324$, since the bottom of the cuboid can be one of the bottom 3 layers and the top of the cuboid can be one of the top 3 layers. Similarly in the other two directions the red cube has to be on the right side of the cut, meaning that the coordinates of the sides can be chosen 3 and 2 ways depending on the direction. All the choices for the sides are independent, therefore they need to be multiplied.

Not all of these are bad for us, since we want to keep the ones that contain the red cube on the inside. These are the cuboids that contain the middle $3 \times 3 \times 3$ cube, in the four remaining directions we have two choices (either keep that layer or not), giving us $2^4 = 16$ such cuboids.

Therefore the number of wrong cuboids is $324 - 16 = 308$, and the number of right ones is $1350 - 308 = 1042$, but this includes the case when no cubes are removed, therefore the answer is $1042 - 1 = 1041$.

(Back to problems)

9. A year consists of 12 months. Beni's birthday couldn't have been on the $5., 6., 7., 10., 11.$ and 12. months of the year, as they don't have a common power with any of the other positive integers up to 12 (and their own powers are all different).

If Beni was born in January, then he said 1 to Gabi, from which she couldn't have told which day he was born. These are 31 possibilities.

Other than this, the following dates will be unclear:

- If Beni was born on an even day in February, then he would have said the same as if he was born on half the number of days in April.
- If Beni was born on an even day in March, then he would have said the same as if he was born on half the number of days in September.
- If Beni was born on a day divisible by 3 in February, then he would have said the same as if he was born on third the number of days in August.
- If Beni was born on a day divisible by 3 in April, then he would have said the same as if he was born on two thirds the number of days in August.

This altogether 31 days in January, 19 days in February, 15 day in March, 20 days in April, 15 in August and 15 in September, in total 115 days.

(Back to problems)

10. For the solution, see Category E Problem 12.

(Back to problems)

11. For the solution, see Category C Problem 15.

(Back to problems)

12. For the solution, see Category E Problem 16.

(Back to problems)

13. For the solution, see Category E Problem 15.

(Back to problems)

14. Firstly note that

$$\frac{2}{k(k+1)(k+2)(k+3)} = \frac{(k+1)(k+2)-k(k+3)}{k(k+1)(k+2)(k+3)} = \frac{1}{k(k+3)} - \frac{1}{(k+1)(k+2)},$$

$$\frac{1}{k(k+3)} = \frac{1}{3}\cdot\frac{(k+3)-k}{k(k+3)} = \frac{1}{3}\cdot\left(\frac{1}{k}-\frac{1}{k+3}\right),$$

$$\frac{1}{(k+1)(k+2)} = \frac{(k+2)-(k+1)}{(k+1)(k+2)} = \frac{1}{k+1}-\frac{1}{k+2}.$$

Using this the formula from the problem can be transformed:

$$\sum_{k=1}^{17}\frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{2}\left(\frac{1}{3}\cdot\sum_{i=1}^{17}\frac{1}{k} - \frac{1}{3}\cdot\sum_{i=1}^{17}\frac{1}{k+3} - \sum_{i=1}^{17}\frac{1}{k+1} + \sum_{i=1}^{17}\frac{1}{k+2}\right) =$$

$$\frac{1}{2}\left(\frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}-\frac{1}{18}-\frac{1}{19}-\frac{1}{20}\right)-\frac{1}{2}+\frac{1}{19}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{3}\cdot\frac{11\cdot 2\cdot 3\cdot 5\cdot 19 - 2\cdot 5\cdot 19 - 2^2\cdot 3^2\cdot 5 - 3^2\cdot 19}{2^2\cdot 3^2\cdot 5\cdot 19} - \frac{1}{2}+\frac{1}{19}\right) =$$

$$= \frac{6270-190-180-171}{20520} - \frac{5130}{20520} + \frac{540}{20520} = \frac{1139}{20520}.$$

Therefore the answer is 1139.

(Back to problems)

15. It is easy to see that $B = 100$, since for any $a \in \{1, 2, \ldots, 100\}$ for both $y = \sqrt[4]{a}$ and $y = -\sqrt[4]{a}$ we get that $\lfloor y^4 \rfloor = a$, but the sign of y^3 is different in the two cases, and so is the sign of $\lfloor y^3 \rfloor$. Thus for any $a \in \{1, 2, \ldots, 100\}$ the value of y cannot be uniquely determined and no other a is possible.

Now let a be a number for which $a \in \{1, 2, \ldots, 100\}$ and $a = \lfloor y^3 \rfloor$ does not uniquely determine $\lfloor y^2 \rfloor$.

After rearranging the equations $a = \lfloor y^3 \rfloor \iff \sqrt[3]{a} \le y < \sqrt[3]{a+1}$ and $b = \lfloor y^2 \rfloor \iff \sqrt{b} \le y < \sqrt{b+1}$. Therefore we are looking for the number of numbers $a \in \{1, 2, \ldots, 100\}$ for which the interval $[\sqrt[3]{a}, \sqrt[3]{a+1})$ is intersected by more than one intervals of the form $[\sqrt{b}, \sqrt{b+1})$, in other words where there is an integer b for which $\sqrt{b} \in (\sqrt[3]{a}, \sqrt[3]{a+1})$, meaning that $b \in (a^{2/3}, (a+1)^{2/3})$.

An integer b can be in at most one such interval, as the intervals are pairwise disjoint. Furthermore the length of the intervals is less than 1:

$$(a+1)^{2/3} < a^{2/3} + 1^{2/3}$$

After cubing both sides

$$(a+1)^2 < a^2 + 3a^{4/3} + 3a^{2/3} + 1$$
$$2a < 3a^{4/3} + 3a^{2/3}$$
$$2/3 < a^{1/3} + a^{-1/3}$$

Since a is positive, we get the inequality from the AM-GM inequality.

Therefore for every such interval there is exactly one integer b which lies within the interval and for every $b \in 1, 2, \ldots, \lfloor 101^{2/3} \rfloor$ b is either in one of the intervals or is and enpoint to one. Such a b is an endpoint to an interval is and only if $b = a^{2/3}$. Therefore the number of such bs if the number of possible as which are perfect cubes or $|\{1, 8, 27, 64\}| = 4$. Therefore $A = \lfloor 101^{2/3} \rfloor - 4 = 17$.

Thus $A + B = 117$.

(Back to problems)

16. An infernal number cannot be larger than 100, since for every number larger than 101 its factorial is divisible by 101, but the numerator is not, therefore the quotient wouldn't be an integer.

Let's split every factorial in the numerator as $1 \le i \le 100$-re i!-t $1 \cdot 2 \cdot \ldots \cdot i$. Altogether there will be 100 1-s, 99 2s, ... and 1 100 term. Then:

$$1! \cdot 2! \cdot \ldots \cdot 100! = 1^{100} \cdot 2^{99} \cdot 3^{98} \cdot \ldots \cdot 100^1 = 1^{100} \cdot (2 \cdot 2^{98}) \cdot 3^{98} \cdot (4 \cdot 4^{96}) \cdot \ldots \cdot (98 \cdot 98^2) \cdot 99^2 \cdot (100 \cdot 100^0) =$$
$$= (1^{100} \cdot 2^{98} \cdot 3^{98} \cdot 4^{96} \cdot \ldots 99^2 \cdot 100^0) \cdot (2 \cdot 4 \cdot \ldots \cdot 98 \cdot 100) =$$
$$= (1^{100} \cdot 2^{98} \cdot \ldots \cdot 100^0) \cdot (2 \cdot 1) \cdot (2 \cdot 2) \cdot \ldots \cdot (2 \cdot 49) \cdot (2 \cdot 50) = (1^{100} \cdot 2^{98} \cdot \ldots \cdot 100^0) \cdot 2^{50} \cdot 1 \cdot 2 \cdot \ldots \cdot 50 =$$
$$= (1^{100} \cdot 2^{98} \cdot \ldots \cdot 100^0) \cdot 2^{50} \cdot 50!$$

.

Since an integer on an even power is always a perfect square, the numerator of the fraction can be written as the product of a perfect square and 50!. We can already see that 50 is an infernal number.

An infernal number is certainly less than 53, since in the numerator there are an even number 53s and of the integers not greater than 100 exactly the ones less than 53 will have an even number of 53s in their factorial (as $2 \cdot 51 < 100$). Furthermore every infernal number has to be between 47 and 93 since in the numerator the prime 47 appears in 47^{54} and 94^7, altogether and odd (51) number of times and of the integers not greater than 100 exactly the ones not less than 47 and less than 94 will have an odd number of 47s in its factorial.

Therefore only the numbers $47, 48, 49, 50, 51$, or 52 can be infernal. We already know that 50 is infernal.

We get the following:

$$\frac{1! \cdot 2! \cdot \ldots \cdot 100!}{47!} = \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{50!} \cdot 50 \cdot 49 \cdot 48; \quad \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{48!} = \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{50!} \cdot 50 \cdot 49;$$

$$\frac{1! \cdot 2! \cdot \ldots \cdot 100!}{49!} = \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{50!} \cdot 50; \quad \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{51!} = \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{50!} \cdot \frac{1}{51};$$

$$\frac{1! \cdot 2! \cdot \ldots \cdot 100!}{52!} = \frac{1! \cdot 2! \cdot \ldots \cdot 100!}{50!} \cdot \frac{1}{51 \cdot 52}$$

It is known that the product of a positive square q and integer z is a perfect square if and only if z is also a perfect square. Furthermore if $\frac{q}{z}$ is and integer, then it is a square if and only if z is also a square. It can be easily checked that of the numbers $50 \cdot 49 \cdot 48$; $50 \cdot 49$; 50; 51 és $51 \cdot 52$ none of them are square, therefore none of the numbers 47; 48; 49; 51 and 52 are infernal.

In summary: The only infernal number is 50, therefore the sum of them is also the same.

(Back to problems)