XVIII. Dürer Competition

Final Round (07-09. 02. 2025.)

Traditional round - Problems

E+ category

1. In triangle ABC, we have $\angle CAB = \angle CBA = 72°$. Point D lies on side AC such that $DA = AB$. We draw tangents from C to the circumcircle of triangle ABD, let the points of tangency be E and F. Prove that the midpoint of segment EF is also the circumcentre of triangle ABC.

2. Geronimo has thought of a polynomial P with integer coefficients. Thea wants to determine this polynomial. To do so, every minute she can say a rational number q, and Geronimo immediately tells her the value of $P(q)$.
a) Is there a polynomial P for which there exists a finite sequence of questions that Thea can ask, from which she can determine P?
b) Geronimo told Thea that the leading coefficient of P is 1. Prove that for every such P, there exists a finite sequence of questions that Thea can ask, from which she can determine P. Determine, as a function of P, the minimum number of such questions Thea needs to ask!
Thea determines the polynomial P if P is the only polynomial with integer coefficients that fits the information she has.

3. a) Is it true that for every positive integer N there exist N lines in the plane in general position such that every intersection point determined by these lines is at an integer distance from every line?
b) Do there exist infinitely many lines in the plane in general position with this property?
A set of lines is in general position if no two are parallel, and no three pass through the same point.

4. Let S be a finite nonempty subset of the positive integers, and let G be a connected tree graph on n vertices. For any vertices u and v, let $d(u, v)$ denote the graph-theoretical distance between the two vertices, that is, the number of edges of the unique path connecting u and v. A sequence $v_1, v_2, \ldots, v_n, v_{n+1}$ of vertices of G is called an *exploration* if $v_1 = v_{n+1}$, the vertices $v_1, v_2, \ldots, v_n$ are all distinct, and for every $1 \leq i \leq n$, the distance $d(v_i, v_{i+1})$ is in S. An exploration is *successful* if each number $s \in S$ appears the same number of times in the list $d(v_1, v_2), d(v_2, v_3), \ldots, d(v_n, v_{n+1})$. For which sets S does there exist a finite connected tree G with at least 2 vertices, which can be explored successfully?

5. A positive integer k is called *criminal* if there exist distinct positive integers m and n so that the number k has two digits in both its base m and its base n representation, and the two representations have the same two digits, but in reverse order. Prove that there exists a positive integer K so that every integer $k \geq K$ is criminal.
For positive integers b, k, the base b representation of k is the ordered tuple $(b_d, b_{d-1}, \ldots, b_1, b_0)$ of integers which satisfies $0 \leq b_i < b$ for all $i < d$, and $0 < b_d < b$, and furthermore $k = b_d \cdot b^d + b_{d-1} \cdot b^{d-1} + \ldots + b_1 \cdot b + b_0$. Then $(b_d, b_{d-1}, \ldots, b_1, b_0)$ are the digits, and the number of digits is $d + 1$. For example, the representation of 7 is $(2, 1)$ in base 3, and $(1, 2)$ in base 5, therefore 7 is criminal.

6. Game: Initially, an ordered pair of positive integers (n, k) is written on a sheet of paper. Two players are playing a game, taking turns alternately. In each turn, if the pair (a, b) is on the sheet and is not crossed out, then the player must cross out (a, b) and instead write $(a, b + 1)$ or $(a - b, b)$ on the sheet. The winner is the first player to write a pair in which at least one of the numbers is not positive.
Defeat the organisers twice in a row in this game! First, the organisers determine the value of n and k, then you get to choose whether you want to play as the first or the second player.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper. Each problem is worth 12 points. For a substantially different second solution or generalization, up to 2 extra points per problem might be awarded. The duration of the contest is 180 minutes. Good luck!

the organizers of the XVIII. Dürer Competition