XVIII. Dürer Competition

Final Round (07-09. 02. 2025.)

Traditional round - Solutions

E+ category

E+1. In triangle ABC, we have $\angle CAB = \angle CBA = 72°$. Point D lies on side AC such that $DA = AB$. We draw tangents from C to the circumcircle of triangle ABD, let the points of tangency be E and F. Prove that the midpoint of segment EF is also the circumcentre of triangle ABC.

First solution: Let G be the point on segment BC such that $BG = AB$. Furthermore, let K be the center of the circle passing through points A, B, D, G, E, F. Finally, let O be the intersection of the circles (AKD) and (BKG) different from K.

First, we prove that O is the circumcenter of triangle ABC. On one hand, by symmetry, O lies on the bisector of $\angle ACB$, so $\angle OCB = 18°$. On the other hand, we want to compute the angle $\angle OBC$:

$$\angle OBC = \angle OBG = \angle OKG = \frac{\angle DKG}{2} = \angle DBG = \angle ABC - \angle ABD = 72° - 54° = 18°,$$

since $\angle BAD = 72°$ and $AB = AD$, we obtain $\angle ABD = 54°$. Thus, $\angle OCB = \angle OBC$, which implies that $OC = OB$. By symmetry, $OB = OA$, so O is indeed the circumcenter of triangle ABC.

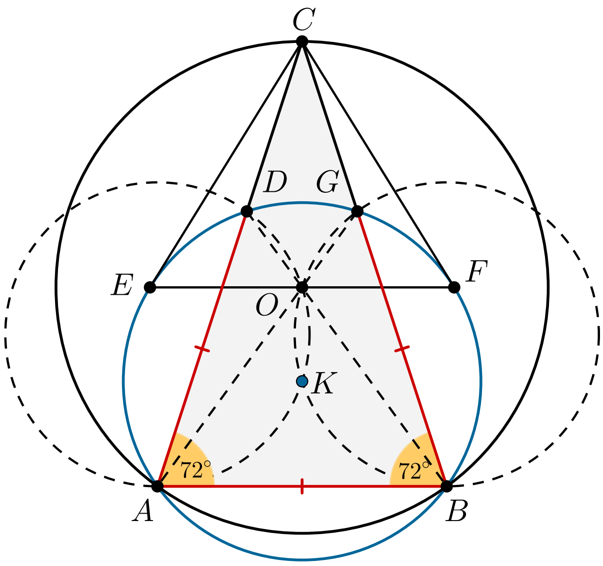
Next, we prove that O is the midpoint of segment EF. By symmetry, we know that the points C, O, K are collinear. Then, the power of point C with respect to the circle $ABDGEF$ is:

$$CF^2 = CG \cdot CB.$$

Further, the power of point C with respect to the circle $BGOK$ is:

$$CG \cdot CB = CO \cdot CK.$$

From this, we obtain $CF^2 = CO \cdot CK$, which implies that $\angle COF = \angle CFK = 90°$. By symmetry, we also know that $FO = EO$, and since $\angle EOF = 180°$, it follows that O is the midpoint of segment EF. This concludes the proof.



Second solution: Let us define O as the center of the circumcircle (ABC). Again, we observe that $O \in BD$, since by simple angle calculations we have

$$\angle OBA = \frac{180° - \angle AOB}{2} = \frac{180° - 2\angle BAC}{2} = 54° = \frac{180° - \angle BAD}{2} = \angle DBA.$$

We need to prove that C and O are inverses with respect to the circle (ABD). We invert the figure with respect to the circle (ABC). Under this inversion, the points A, B, C remain fixed, while O goes to infinity. After the inversion, the statement to be proven is that the images of C and O are inverses with respect to the image of the circle (ABD), that is, C is the center of the circle (ABD^*), where D^* denotes the inverse of D with respect to (ABC).

Since $CA = CB$, it suffices to show that $CD^* = CB$. By inversion, we have

$$\angle BD^*C = \angle OD^*C = \angle OCD = 18° = 72° - 54° = \angle DBC = \angle D^*BC.$$

Thus, the triangle BCD^* is indeed isosceles, which completes the proof.

E+2. Geronimo has thought of a polynomial P with integer coefficients. Thea wants to determine this polynomial. To do so, every minute she can say a rational number q, and Geronimo immediately tells her the value of $P(q)$.
a) Is there a polynomial P for which there exists a finite sequence of questions that Thea can ask, from which she can determine P?
b) Geronimo told Thea that the leading coefficient of P is 1. Prove that for every such P, there exists a finite sequence of questions that Thea can ask, from which she can determine P. Determine, as a function of P, the minimum number of such questions Thea needs to ask!
Thea determines the polynomial P if P is the only polynomial with integer coefficients that fits the information she has.

Solution: a) There is no such polynomial. Suppose that Thea can determine P based on the answers to some finite number of questions $q_1 = a_1/b_1, q_2 = a_2/b_2, \ldots, q_N = a_N/b_N$. We can see that polynomial $P^*(x) = P(x) + (b_1x - a_1) \cdot \ldots \cdot (b_Nx - a_N)$ is different from P but takes the same values at points $q_1, q_2, \ldots, q_N$, which contracticts the fact that Thea could determine P.

b) We will show that Thea needs at least n questions where n is the degree of P.

Firstly we prove that $n-1$ questions are not enough. Suppose that Thea can determine P based on the answers to questions $q_1 = a_1/b_1, q_2 = a_2/b_2, \ldots, q_{n-1} = a_{n-1}/b_{n-1}$. Now similarly to the first part, the polynomial $P^*(x) = P(x) + (b_1x - a_1) \cdot \ldots \cdot (b_{n-1}x - a_{n-1})$ is different to P but takes the same values at the asked points. We know P is of degree n and its leading coefficient is 1 and since the additional term is of degree at most $n-1$, therefore P^* is also of degree n and has 1 as its leading coefficient.

Now we will show that Thea can always determine P in n questions. Let the first question be $q_1 = 1/2$ and the answer to it $a_1/b_1 \cdot 2^k$, where $k \in \mathbb{Z}$ and both a_1, b_1 are odd positive integers. We claim that this means that the degree of P is $-k$. This is because if $R(x) = x^m + r_{m-1}x^{m-1} \cdots + r_1x + r_0$, where r_i are integers, then $R(1/2) \cdot 2^m$ is an integer, but $R(1/2) \cdot 2^{m-1}$ is not. This means that after the first question Thea knows n, the degree of P. Let the other questions be on values $q_i = 1/(i+1)$, where $2 \leq i \leq n$. Then the polynomial $P(x) - x^n$ is of degree at most $n-1$, and Thea knows its value in n points, therefore by using Lagrange-interpolation Thea can determine the polynomial $P(x) - x^n$, therefore also P.

E+3. **a)** Is it true that for every positive integer N there exist N lines in the plane in general position such that every intersection point determined by these lines is at an integer distance from every line?
b) Do there exist infinitely many lines in the plane in general position with this property?
A set of lines is in general position if no two are parallel, and no three pass through the same point.

Solution: **a)** We show that there exist N such lines for any arbitrarily large positive integer N. We define our lines in the form $a_ix + b_iy + c_i = 0$, where the values of a_i, b_i, c_i will be chosen later. Observe that the intersection point of the lines $a_ix + b_iy + c_i = 0$ and $a_jx + b_jy + c_j = 0$ is given by

$$\left(\frac{c_ib_j - b_ic_j}{b_ia_j - a_ib_j}, \frac{c_ia_j - a_ic_j}{b_ia_j - a_ib_j}\right),$$

where $b_ia_j - a_ib_j \neq 0$, since no two of our lines are parallel. Consequently, if we choose all a_i, b_i and c_i to be rational, the intersection points of any two lines will have rational coordinates.

Now, we use the fact that the distance of a point (x_0, y_0) from the line $ax+by+c = 0$ is given by $\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$. We choose the triples $(a_i, b_i, \sqrt{a_i^2 + b_i^2})$ to be primitive Pythagorean triples. This ensures that any rational-coordinate point will be at a rational distance from all N lines (provided that c is also rational). Furthermore, the slopes a_i/b_i of the lines will be distinct due to the primitiveness, meaning that no two of our lines are parallel. Now, we need to ensure that no three lines pass through a single point. Fortunately, this can be easily arranged since we have only finitely many lines, and we can freely choose the constants c_i as any rational numbers. Finally, since we have only finitely many distances, we can scale the configuration by the least common multiple of the denominators of these distances, making all distances integer-valued.

b) We prove that infinitely many such lines cannot exist. We will prove by contradiction, assume that there are infinitely many lines, let one of them be line e and let A and B be intersection points on e. Now let f be a line in the construction different from e. Let f' be the line parallel to f passing through A. Now the circle with centre B and tangent to f' has an integer radius (it is the difference of the distance of B and A from f, which are both integers) and this radius is less than $|AB|$. This means that there are only finitely many such possible circles, but each such circle corresponds to two directions of lines, meaning that in total there can be only finitely many directions of lines, which is contradicting the fact that the lines are in general position.

E+4. Let S be a finite nonempty subset of the positive integers, and let G be a connected tree graph on n vertices. For any vertices u and v, let $d(u, v)$ denote the graph-theoretical distance between the two vertices, that is, the number of edges of the unique path connecting u and v. A sequence $v_1, v_2, \dots, v_n, v_{n+1}$ of vertices of G is called an *exploration* if $v_1 = v_{n+1}$, the vertices $v_1, v_2, \dots, v_n$ are all distinct, and for every $1 \le i \le n$, the distance $d(v_i, v_{i+1})$ is in S. An exploration is *successful* if each number $s \in S$ appears the same number of times in the list $d(v_1, v_2), d(v_2, v_3), \dots, d(v_n, v_{n+1})$. For which sets S does there exist a finite connected tree G with at least 2 vertices, which can be explored successfully?

Solution: We will show that there exists a suitable graph G if and only if there is an odd element is S.

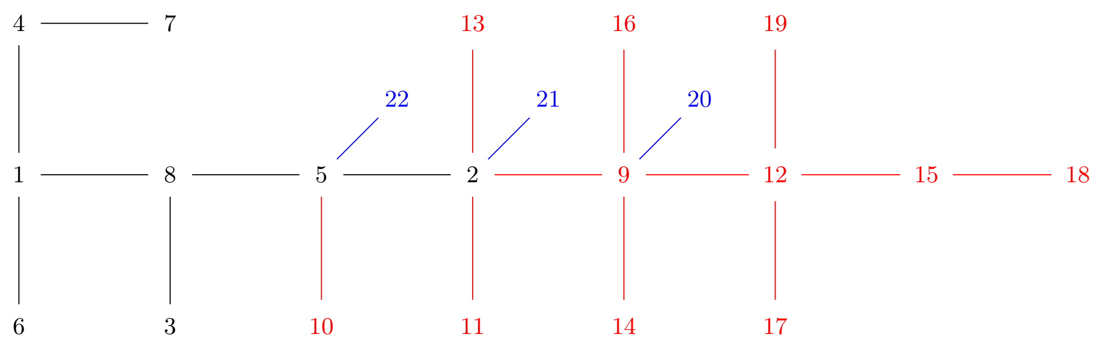
The condition is necessary: If S contains only even numbers then let's colour the tree graph with two colours, black and white, such that neighbouring vertices are of different colours. If the first vertex in the exploration is white, then we can never reach the black ones with only even jumps, therefore the whole graph cannot be explored.

Constructing G: Now we can assume that S contains an odd element.

First if $a > 1$, $b \ge 1$ are two different elements of S and $a + b - 2 \notin S \setminus \{a, b\}$, furthermore there is a successfully explorable graph G' for set $S' = S \setminus \{a, b\} \cup \{a + b - 2\}$, then there is one for S as well. Let $v_1, v_2, \dots, v_{|G'|}, v_1$ be a successful exploration in G'. We know that $|S'| \cdot k = |G'|$, where is k is the number of times we used one of the elements of S. Now for every step of size $a+b-2$ let us make the following modification: if we are stepping from $u \in G'$ to $v \in G'$, let the shortest path between them (of length $a + b - 2$) be the vertices $u, w_1, w_2, \dots, w_{a+b-3}, w_{a+b-2} = v$. Then we will add a leaf $\bar{w}$ to vertex w_{a-1}, and modify the exploration by stepping from u to $\bar{w}$, and from $\bar{w}$ to v. Since the lengths of the shortest paths between these vertices are a and b respectively, therefore we created one step of length a and one of length b instead of one of length $a + b - 2$.

This modification will result in a new tree graph as we only added leaves. Now by induction on $|S|$, if $|S| \ge 3$, and S contains an odd element, then let a be the largest odd element and let b be an arbitrary even element is S. The set $S' = S \setminus \{a, b\} \cup \{a + b - 2\}$ has fewer elements than S and contains an odd element, therefore we are done. If S contains only odd elements, then let a and b two of them, then performing the same operation the resulting set S' still contains at least one odd element.
If $|S| = 2$ and there is an even element in S, then after forming S' we are done, therefore we now only need to consider the cases where S consists of one or two odd numbers.

If $|S| = 1$, firstly we show a construction for $S = \{3\}$:



It is clear that for by continuing the red and blue pattern we can achieve for any l that for every vertex there is another vertex at a distance at least $l + 1$. From here on let $l(G)$ denote the smallest such number for G.

Using induction suppose that for some odd k we have found a suitable graph G for $S = \{k\}$ and $l(G) \ge k + 1$, we will show that for $S = \{k + 2\}$ there exists a suitable G'. In G add a leaf to the starting vertex of G, let this be the starting vertex in G'. Now we go through the vertices of G as follows: if we want to reach a new leaf attached to a vertex $x \in V(G)$ and we are currently in a new leaf attached to $y \in V(G)$, then consider a series of vertices $y, v_1, v_2, \dots, x$ of G which is in the original successful exploration, put a new leaf at each of these vertices and go through these leaves in order. Since for every vertex x in G there is a vertex at distance of $k + 1$ (since $l(G) \ge k + 1$), we first need to reach a new leaf at this vertex and then step to x. Afterwards we perform this for every vertex we need to reach a new leaf which if of distance $k+2$ of the starting vertex. We have only added leaves to G therefore G' is still a tree graph and $l(G') \ge l(G)$.

Therefore in order to obtain a graph for $S = \{k\}$, we need to start from a graph G that is suitable for $S = \{3\}$ and for which $l(G) \geq k+1$. After performing the steps from the previous paragraph we obtain a suitable graph for $S = \{k\}$.

Since for $S = \{1\}$ a tree with two vertices is sufficient, we have shown that the all of the one-element sets S work.

Finally we need to consider the case where $S = \{p, q\}$ and $p < q$ are both odd. If $p > 1$, consider a graph G for $S = \{q\}$. Since q is odd and a tree graph is always bipartite, G has to have an even number of vertices (since an exploration alternates between the partitions), meaning that there are an even number of q-steps in the successful exploration. Now instead of step $v_i \to v_j$ with shortest path $v_i, a_1, a_2, \ldots, a_{q-1}, v_j$ we can add a leaf x to vertex a_{p-1} and add a leaf y to a_1, and perform the steps $v_i \to x \to y \to v_j$. With this we have increased the number of p-steps by 2, repeating this a few times we can achieve that there are the same number of p and q-steps.

If $S = \{1, 2k+1\}$ then let the graph be:

With this we have provided a construction for all cases.

E+5. A positive integer k is called *criminal* if there exist distinct positive integers m and n so that the number k has two digits in both its base m and its base n representation, and the two representations have the same two digits, but in reverse order. Prove that there exists a positive integer K so that every integer $k \geq K$ is criminal.
For positive integers b, k, the base b representation of k is the ordered tuple $(b_d, b_{d-1}, \ldots, b_1, b_0)$ of integers which satisfies $0 \leq b_i < b$ for all $i < d$, and $0 < b_d < b$, and furthermore $k = b_d \cdot b^d + b_{d-1} \cdot b^{d-1} + \ldots + b_1 \cdot b + b_0$. Then $(b_d, b_{d-1}, \ldots, b_1, b_0)$ are the digits, and the number of digits is $d+1$. For example, the representation of 7 is $(2,1)$ in base 3, and $(1,2)$ in base 5, therefore 7 is criminal.

Solution:

Let $k \geq 2$ be a non-criminal integer. We claim that if d is a positive integer with $d^5 < k$, then $d \mid k$.

Proof: Fix the value of k, and the proof proceeds by induction on d, $d = 1$ is trivial. Let $d^5 < k$ and suppose for all $d' < d$ we have $d' \mid k$, and $d \nmid k$. Let r be the remainder of k when divided by d, so $1 \leq r \leq d-1$. Let $\ell = \mathrm{LCM}(d, r) < d^2$. Since $d \mid k-r$ and $r \mid k-r$, we have $\ell \mid k-r$ and $r \mid k-\ell$. If we take $m = \frac{k-r}{\ell}$ and $n = \frac{k-\ell}{r}$, then $k = m \cdot \ell + r = n \cdot r + \ell$. These are the base m and n representations of k, provided that the following 4 inequalities hold:

$$\ell < m,\ r < m,\ \ell < n \text{ and } r < n.$$

We already know that $\ell \geq d > r$, and $m < n$, since this is equivalent to the statement $(k-r)r < (k-\ell)\ell$, which follows form the fact that $x \mapsto (k-x)x$ is a strictly increasing function up to $x < k/2$, and $r < \ell < d^2 < k^{2/5} \leq k/2$, if $k \geq 2$. So if $\ell < m$, all 4 inequalities hold, and k is criminal. We know that

$$k > d^5 \geq d^4 + d \geq \ell^2 + r$$

so, $m = \frac{k-r}{\ell} > \ell$, contradiction.

Hence, we are only left to prove that there exists some K positive integer such that for all $k \geq K$ there exists some $d < k^{1/5}$ with $d \nmid k$. Let K be the product of the first 10 primes, suppose that there exists some $k \geq K$ not satisfying the above. Then all of the first 10 primes divide k, so let the exponent of p in k be $e_p \geq 1$. Then $k \geq \prod_{i=1}^{10} p_i^{e_{p_i}}$, so there exists some $p \in \{p_1, \ldots, p_{10}\}$, such that $p^{e_p} < k^{1/10}$. But then $d = p^{2e_p} < k^{1/5}$, with $d \nmid k$, contradiction. Hence the statement of the problem holds with $K = \prod_{i=1}^{10} p_i = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 = 6469693230$.

Note: A list of all non-criminal integers:

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 18, 20, 24, 32, 48, 60, 72, 168, 720.$$

E+6. Game: Initially, an ordered pair of positive integers (n, k) is written on a sheet of paper. Two players are playing a game, taking turns alternately. In each turn, if the pair (a, b) is on the sheet and is not crossed out, then the player must cross out (a, b) and instead write $(a, b + 1)$ or $(a - b, b)$ on the sheet. The winner is the first player to write a pair in which at least one of the numbers is not positive.

Defeat the organisers twice in a row in this game! First, the organisers determine the value of n and k, then you get to choose whether you want to play as the first or the second player.

Solution: Let us call a position a winning position, if starting from there the second player has a winning strategy, otherwise call it a losing one. We will determine for all positions (a, b) if they are winning or losing.

Clearly it is a losing position if $a \leq b$. If $a \leq 2b$, then if someone subtracts from a, they lose, if both of them increase b, then the person reaching $a = b$ will lose. Therefore in this case the winning positions are when $a - b$ is odd.

Now we will show that if $a > 2b$, then a even, b odd is a winning position, and it is a losing position if they have the same parity. We can reach a winning position starting from (even, even) by always increasing b. From (odd, odd) we can always move to (even, odd) by using $a - b$. Note that these lead to winning positions even when $a \leq b$. Finally we need to prove that the positions (odd, even) are winning, meaning we can only move to losing positions from them, which is clear, as one of the steps leads to (odd, odd), the other one to (even, even).

At last we have the case where $a > 2b$ and a is even, b is odd. Then by increasing b we get to a losing position. If both players decrease a, the winning player is whoever brings a below $2b$ first, therefore here the winning positions are when $\lfloor \frac{a}{b} \rfloor$ is odd.