XVIII. Dürer Competition

Final Round (07-09. 02. 2025.)

Relay round - Problems

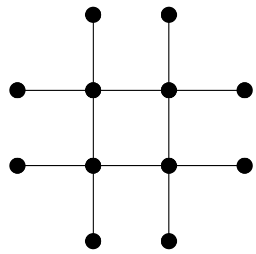
E⁺ category

E$^+$-1. How many ways are there to express 13860 as the product of two coprime positive integers? *Two expressions are not considered different if they consist of the same two numbers but in different order.*

(3 points)

E$^+$-2. Molli and Tamás are organizing a trip to the country of Hashtagonia which consists of 12 cities and 12 roads, shown as dots and segments in the figure. They both want to visit four cities, such that each city will be visited by at most one of them, and both Molli and Tamás will only visit each city at most once. How many ways are there to organize such a trip?
Two trips are considered to be different if at least one of Molli and Tamás visited different cities or the same cities but in a different order.

(3 points)



E$^+$-3. How many positive four-digit integers are there that consist of four different digits in descending order (from left to right), and the product of the outside two digits equals the product of the middle two?

(3 points)

E$^+$-4. In each January of the next 30 years, Scooby-Doo decides whether he should buy a new van for 9 Dürer dollars or get the current van serviced for x Dürer dollars where x denotes the number of years he has owned it. What is the minimal amount of Dürer dollars Scooby-Doo needs to spend on vans in the next 30 years if he has already decided to buy a new van next January?

(3 points)

E$^+$-5. How many ways are there to place 5 rooks on a 5×5 chessboard such that for all empty squares there exists a rook in its column or row?
Two cases are considered to be different if there exists a square which contains a rook in one case, but not in the other.

(4 points)

E$^+$-6. Let $ABCD$ be a trapezoid whose sides AB and CD are parallel. The sides AB, BC, CD and DA have lengths 128, 106, 5, and 65, respectively. Let X be the intersection of the internal angle bisectors at A and D, and let Y be the intersection of the internal angle bisectors at B and C. What is the length of segment XY?

(4 points)

E$^+$-7. The police have captured 16 suspects and placed them in a straight line. In each step, the police captain is allowed to swap two consecutive suspects. Find the least positive integer k such that no matter how the suspects are arranged initially, the captain can reach an order in at most k steps where no suspect stands directly between two taller or two shorter suspects.
All suspects have different heights.

(4 points)

E$^+$-8. There are two integers written on the board. In each step, Dani deletes one of the numbers and replaces it with the sum of the two numbers. He continues this until the number 42 appears on the board for the first time. How many different ways could Dani obtain 42, if initially the board contains the number 1 twice?
Two ways of obtaining 42 *are considered the same if in each step Dani deletes the same value. For example, deleting* 1 *in the first step counts as one way, no matter which* 1 *he deletes.* (4 points)

E$^+$-9. Anett and Andris live in a country where the registration number of each car is a 3-digit number (between 000 and 999), and each car has a different registration number. While travelling on the highway, they play the following game: Andris thinks of a secret number between 000 and 999, and every time a car passes them, for each digit in that car's registration number, he tells Anett which one of the following three statements is true:

- The digit of the registration number is equal to the secret number's digit in the same position.
- The digit of the registration number appears in the secret number, but not in the same position.
- The digit of the registration number does not appear in the secret number.

Andris has also told Anett that all three digits of the secret number are distinct. At least how many different cars must pass them so that Anett can surely determine the secret number?

(5 points)

E$^+$-10. In the top row of a two-row table, Benedek wrote several consecutive natural numbers greater than 1 next to each other, in increasing order. Then, he wrote the smallest prime factor of each number in the cell below. Afterwards, he erased the original numbers. Then for each number remaining in the table, it was true that to either its left or its right there were at least as many numbers in the table as the value of the number itself. At least how many numbers did Benedek write in the top row to begin with?

(5 points)

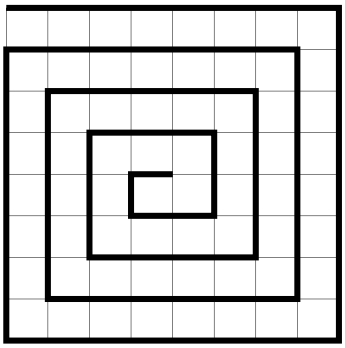
E$^+$-11. In the coordinate system of the plane, consider the set points for which both coordinates are positive integers less than 46. Call a square beautiful if all of its vertices belong to this set, and its sides are parallel to the axes. How many grid points are there for which the number of beautiful squares that do not have this point as a vertex is divisible by 13?

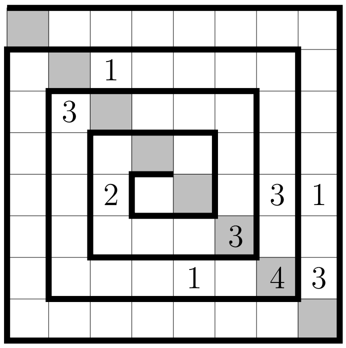
(5 points)

E$^+$-12. Let A, B, C and D be four distinct points in three-dimensional space. It is known that no three of the points lie on the same line, and there exist infinitely many spheres tangent to all four segments AB, BC, CD and DA at an internal point. Let the lengths of these four segments be a, b, c, and d, respectively. It is known that they are distinct one-digit positive integers. How many possibilities are there for the ordered quadruple (a, b, c, d)?

(5 points)

E$^+$-13. Csabi, the snail, has a house which consists of 8×8 squares, just like the figure on the left. Csabi is learning the positive integers at the moment; he only knows them up to 4. His brother is helping with his studies by placing the integers $1, 2, 3, 4$ into some of the squares in Csabi's house. He would like to this so that when reading the numbers in the non-empty squares in the spiral starting at the entrance of the house, the numbers appear in the order $1, 2, 3, 4, 1, 2, 3, 4, 1, 2, \ldots, 4$. Furthermore, to keep Csabi's house balanced, he wants to place each one of $1, 2, 3, 4$ exactly once in each column and each row. So far, he has already placed the integers shown on the figure on the right, help him finish the job! Once he has finished, what will be the product of the numbers in the non-empty gray squares?





(6 points)

E$^+$-14. Let x be a real number satisfying $\sin^{10}(x) + \cos^{10}(x) = \frac{11}{36}$. What is the value of $\sin^{12}(x) + \cos^{12}(x)$? **Please submit the sum of the numerator and the denominator in the simplified form of the fraction.**

(6 points)

E$^+$-15. Csongi is drawing line segments one by one into his squared notebook, with one endpoint at the origin and the other endpoint at a grid point in the first quadrant (where both coordinates are non-negative). The first segment goes to $(0, 1)$, and the second one goes to $(1, 1)$. Let us denote the angle of segment s with the x axis by $\varphi(s)$. If the first $n \geq 2$ segments are $s_1, \ldots, s_n$, Csongi then picks s_{n+1} as the shortest segment t for which $\varphi(t)$ is strictly between $\varphi(s_n)$ and $\varphi(s_{n-1})$. What is the remainder of the square of the length of the 666^{th} line segment when divided by 66?
In each step there is a unique segment with minimal length.

(6 points)

E$^+$-16. The Baker Street tram station has a strict schedule of trams arriving exactly every 10 minutes. Unfortunately, the trams are often late; the delay is uniformly distributed between 0 and t minutes, where $t < 10$ is some unknown fixed constant. The delays of different trams are independent of each other, and there is a precise screen at the station that always displays the delay of the next arriving tram (hence this is unchanged until the next tram arrives, and at that point, it switches to the delay of the following tram). Áron - independently of any delays of trams - arrives at the station at random with uniform distribution between 12:00 and 13:00. The expected value of the delay displayed when he arrives is exactly 4 minutes. What is the value of t? **It is given that t can be uniquely written as $a + \sqrt{b}$, where a and b are integers and b is positive. You need to give $a + b$ as your answer!**

(6 points)