



E1. In a competition, there are a total of 20 participants across several different categories, and each participant competes in exactly one category. The numbers 2, 3, ..., 21 are assigned as bib numbers before the competition begins, with each participant receiving exactly one number. The organizer wants to distribute the bib numbers in such a way that no matter how the results turn out within each category, for any two categories, there will always be one person from each podium whose bib numbers are not relatively prime.

- Prove that the organizer can distribute the bib numbers in this way if there are 5 categories, each with 4 participants.
- Can the organizer distribute the bib numbers in this way if there are 4 categories, each with 5 participants?
- Does such a distribution of bib numbers exist if there are two categories, each with 10 participants?

On each podium, there are always three participants from the same category. The organizer knows which participant belongs to which category. Two different integers are relatively prime if their greatest common divisor is 1.

For example, in part b), the following distribution would not work: if the first category has bib numbers 4, 11, 14, 20, 21, and the second category has 3, 9, 10, 12, 15. In this case, it is not true that, regardless of the results, there will be one person from each podium whose bib numbers are not relatively prime. For instance, if the podium in the first category consists of participants with bib numbers 4, 11, and 14, and in the second category 3, 9, and 15, then any choice of one number from the first podium and one from the second will always be relatively prime.

Solution: a) We have 10 even and 10 odd bib numbers. Thus, if the organizer gives exactly 2 even bib numbers in each of the 5 categories, then by the pigeonhole principle there will be at least one even-numbered participant on every podium. Therefore, among the podium finishers of any two categories, the organizer can always find two participants with even bib numbers whose numbers are not relatively prime. Hence the organizer can distribute the bib numbers appropriately, for example as follows:

- Category A: 2, 3, 4, 5
- Category B: 6, 7, 8, 9
- Category C: 10, 11, 12, 13
- Category D: 14, 15, 16, 17
- Category E: 18, 19, 20, 21

b) Let there be 3 even-numbered participants in categories A and B; call these the even categories. Let there be 2 even-numbered participants in categories C and D; call these the odd categories. (Four even numbers in one category would be unnecessary, since by the pigeonhole principle already with 3 even numbers at least one even-numbered participant must end up on the podium.) Then if we look at categories A and B, the condition holds between them: there are 3 even bib numbers in each of these categories, so there will be at least one even-numbered participant on each podium whose bib numbers are not relatively prime. Thus we can only have a problem if, in some odd category, the 3 odd-numbered participants occupy the entire podium.

In the odd categories, distribute the three odd bib numbers in such a way that in each of the two odd categories, among these three numbers one is divisible by 3, one by 5 and one by 7. Then the bib numbers 11, 13, 17 and 19 must all be placed into the even categories. In addition, in each even category there should be one bib number divisible by 3 and one divisible by 5. These must be even numbers, since 11, 13, 17 and 19 all have to be in the even categories; they cannot be in the odd categories because of the divisibility requirements.

Therefore, if we take one odd and one even category, then either there will be one even-numbered participant on each podium, or in the odd category all three odd-numbered participants on the podium will have bib numbers divisible by 3, 5 and 7 respectively, and in the even category there will be at least one participant whose bib number is also divisible by either 3 or 5. Similarly, if we take the two odd categories, then either there will be an even bib number on each podium, or in one of the categories the three podium finishers are exactly the three odd bib numbers, with their bib numbers divisible by 3, 5 and 7 respectively. In that case there is also at least one bib number on the other podium that is divisible by one of these numbers. Thus in this case as well the condition is satisfied, and we have covered all possibilities. This theoretical distribution can indeed be realized; for example:

- Category A: 6, 10, 11, 14, 17
- Category B: 12, 13, 18, 19, 20
- Category C: 4, 5, 8, 9, 21
- Category D: 2, 3, 7, 15, 16



c) The numbers 11, 13, 17 and 19 are prime; each of them is relatively prime to any other bib number. Thus, if at least 3 of these bib numbers appear in the same category, then if exactly these 3 participants take the podium places, every bib number from the other category will be relatively prime to all of them, so in such a case the required distribution is impossible.

Now consider the other case, when the two categories each contain 2 of these 4 prime bib numbers. By symmetry, we may assume that category A contains bib numbers 11 and 13, while category B contains 17 and 19. In one of the categories, some participant must have bib number 7; we may assume this is in category A. Then in category B, all of the remaining eight bib numbers would have to be divisible by 7, because if the podium in category A consists of bib numbers 11, 13 and 7, and in category B the podium includes the participants with bib numbers 17 and 19, then the third podium finisher in B must always have a bib number divisible by 7. Therefore, the remaining eight bib numbers in category B would all have to be divisible by 7.

However, there are not that many bib numbers divisible by 7; only 14 and 21 are. Thus there must be some bib number in category B which is not divisible by 7, and in this case the organizer cannot distribute the bib numbers to the participants in a way that satisfies the conditions.



XIX. Dürer Competition

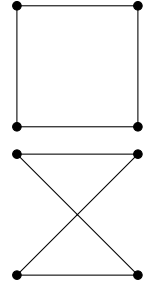
E
category

First round (2025. November 14.)

Solutions

E2. In an elven city, a sociologist wants to map the residents' acquaintance network. For privacy reasons, the network does not include names. It is known that for any three residents A , B , and C , the following holds: if A and B do not know each other, and B and C do not know each other, then A and C do not know each other either. Furthermore, if two residents know each other, then they have the same number of acquaintances. For which numbers of residents is the number of possible networks odd?

Acquaintance is mutual. Two networks are considered identical if, based solely on the networks, it is possible that exactly the same pairs of residents know each other. For example, the two networks shown in the figure are considered identical.



Solution: Think of the acquaintance network as a graph whose vertices represent the residents, and two vertices are connected by an edge exactly when the corresponding residents know each other. Take a graph that satisfies the conditions of the problem, and consider a vertex v of this graph. Let the degree of v be k , and the set of its neighbours be the vertex set A , and let the remaining vertices form the set B .

If we take a vertex $a \in A$ and a vertex $b \in B$, then these two are connected. Indeed, if they were not, then a and b would not be connected, and b and v would also not be connected, so a and v could not be connected either, which is impossible.

Thus, every vertex in A is connected to every vertex in B . Since all vertices in A are connected to v , they each have degree k , and the vertices in B are connected to all vertices in A , so their degrees are also k . But the vertices in B already have k neighbours (the set A), so they have no additional neighbours. Therefore, v together with the vertices of B spans an empty graph.

This argument applies to any v , so every vertex, together with its non-neighbours, forms an empty induced subgraph, and it is connected to all other vertices. That is, the complement of the graph is a disjoint union of complete graphs. Since every vertex has degree k , the sizes of the complete graphs are identical, and thus they are divisors of n .

Moreover, it can be verified that for any divisor m of n , the complement of the union of $\frac{n}{m}$ complete graphs of size m satisfies the conditions. Therefore, the number of n -vertex tarantula graphs equals the number of divisors of n . This number is odd exactly when in the prime factorisation of n every prime appears with an even exponent, i.e., when n is a perfect square.



E3. By a *partition* of a regular n -gon we mean drawing $n - 3$ diagonals that do not intersect each other. A partition is called *tricky* if among the resulting $n - 2$ triangles, exactly two are isosceles. For which integers $n \geq 4$ does a regular n -gon admit a tricky partition?

Solution: First, we show that for even n a tricky partition exists. Let the vertices of the regular n -gon be $a_1, \dots, a_n$ in this order around the polygon. Choose the vertex a_1 of the n -gon and draw all $n - 3$ diagonals incident to it. This gives a partition of the n -gon. We will show that this is a tricky partition.

It is immediately clear that the diagonal from a_1 to a_3 determines the isosceles triangle $a_1a_2a_3$, and similarly the diagonal to a_{n-1} determines the isosceles triangle $a_{n-1}a_na_1$, so the partition already has two isosceles triangles. Notice that every other triangle has exactly one side which is also a side of the polygon, and its other two sides are diagonals of the polygon. Since every diagonal is longer than a side of the polygon, such a triangle can only be isosceles if the two diagonals forming its sides have equal length. Let the triangle's vertices be a_1, a_j, a_{j+1} . The diagonals a_1a_j and a_1a_{j+1} can only have equal length if a_j is the k -th vertex from a_1 in one direction, and a_{j+1} is the k -th vertex from a_1 in the other direction for some positive integer k . However, in this case the number of sides of the polygon is $2k + 1$, since apart from a_ja_{j+1} we counted every side exactly once. But n is even and $2k + 1$ is odd, so this is impossible. Therefore, there are exactly two isosceles triangles in the partition, so we obtain a tricky partition for any even number n .

Now we show that for odd n , no tricky partition exists. First, we prove the following lemma: In any partition of a convex n -gon with $n \geq 4$, there are at least two triangles that have two of their sides also sides of the original polygon. The proof is by induction on n . It is clear for $n = 4$. Assume it holds for all $k < n$, i.e., any partition of any convex k -gon contains two such triangles. Now we prove it for n . Take any diagonal of the n -gon. This divides the polygon into two smaller convex polygons with k_1 and k_2 vertices, where $k_1, k_2 < n$. If $k_1 = 3$, then that region is a triangle having two of its sides as sides of the original n -gon. If $k_1 \geq 4$, we may apply the induction hypothesis to the k_1 -gon, so in any partition of it, there are two triangles whose two sides are sides of the k_1 -gon. Since the sides of the k_1 -gon are sides of the original n -gon except for the diagonal we drew, one of these triangles must contain two original n -gon sides (since the diagonal can only be part of one triangle). The same holds for the k_2 -gon on the other side of the diagonal, so we have found one such triangle on each side. Thus, we have proved the lemma for any partition containing this diagonal, but since we chose the diagonal arbitrarily, the argument applies to any diagonal, hence the lemma holds. We are now ready to prove the main statement. (We could also prove by pigeon hole principle, as there are $n - 2$ triangles, n sides of the polygon, but there cannot be a triangle with three sides of the polygon.)

Take a regular n -gon with odd $n \geq 5$ and any of its partitions. We know that there are two triangles $a_{i-1}a_i a_{i+1}$ and $a_{j-1}a_j a_{j+1}$ which contain two sides of the n -gon, hence they are isosceles and even congruent, so the diagonals $a_{i-1}a_{i+1}$ and $a_{j-1}a_{j+1}$ have equal length.

Case 1. Suppose the two triangles share a vertex, say $a_{i-1} = a_{j+1}$, i.e. $i - 1 = j + 1$. Since the polygon has an odd number of sides, the number of sides on the path along the polygon's edges from a_{i+1} to a_{j-1} not containing a_i is odd. Then the partition becomes a partition of the polygon obtained by cutting off a_i and a_j along the diagonals $a_{i-1}a_{i+1}$ and $a_{j-1}a_{j+1}$. Note that in this $(n - 2)$ -gon, every side is a side of the original n -gon except for $a_{i-1}a_{i+1}$ and $a_{j-1}a_{j+1}$. By the lemma, there must be a triangle containing two sides, so to avoid having another isosceles triangle, one such triangle must contain $a_{i-1}a_{i+1}$ and the other must contain $a_{j-1}a_{j+1}$. Again we obtain two triangles sharing a vertex, and the same argument applies now to an $(n - 4)$ -gon, and so on, until we reach a triangle. This triangle has two sides which are diagonals of the original polygon, and these are equal in length, so we get a new isosceles triangle. Thus the partition contains at least three isosceles triangles, so it cannot be tricky.

Case 2. If the initial triangles $a_{i-1}a_i a_{i+1}$ and $a_{j-1}a_j a_{j+1}$ do not share a vertex, then the polygon sides not belonging to either triangle determine two disjoint paths. Each path connects one vertex of the first triangle to one vertex of the second: one connects a_{i-1} to a_{j+1} , the other a_{i+1} to a_{j-1} . Since the polygon has an odd number of sides, the lengths of these two paths have different parity. As in the previous case, the partition of the $(n - 2)$ -gon obtained by cutting along $a_{i-1}a_{i+1}$ and $a_{j-1}a_{j+1}$ must contain two triangles with two sides, and they must contain the two diagonals along which we cut, otherwise the partition would not be tricky. All sides of this $(n - 2)$ -gon are sides of the original polygon except for the two diagonals, which again have the same length. In this case there may be more than one way the two triangles can appear, but the property still holds. Moreover, the two paths between the triangles continue to have different parity, and as we move to smaller and smaller polygons, at least one path always becomes shorter. Repeating this argument, the parity difference



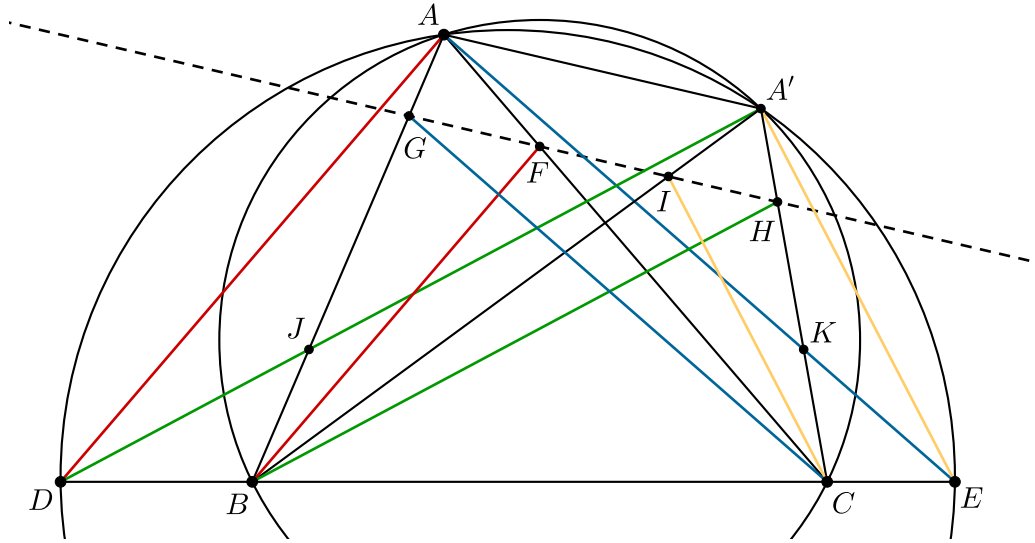
eventually forces us into the configuration of Case 1, which we already know cannot yield a tricky partition. Thus for odd $n \geq 5$, a regular n -gon has no tricky partition.

Second solution:

We outline another alternative solution as well. We may assume that there are not three triangles in the partition that each contain two sides of the regular n -gon. Then, by simple counting, in any partition every triangle must have at least one side which is also a side of the regular n -gon. Then, moving from one isosceles triangle adjacent to one pair of neighboring sides toward another along the diagonals used in the partition, the lengths of the diagonals increase step by step monotonically for a while, then decrease step by step monotonically (except at the turning point). Since initially the two small diagonals were equal in length, the diagonals can be equal again at the turning point only if n is even.

E4. Let ABC be a triangle and let D and E be two points on line BC such that the order of the four points is D, B, C, E . Let X denote the intersection point of the circumcircles of triangles ABC and ADE which is different from A . The line parallel to AD through B intersects line AC in point F , and the line parallel to AE through C intersects line AB in point G . Similarly the line parallel to XD through B intersects line XC in point H , and the line parallel to XE through C intersects line XB in point I . Show that points F, G, H, I lie on a line.

Solution: Let us make a colourful figure for better visibility. We will denote the point X by A' instead. Let J be the intersection of AB and DA' , and let K be the intersection of AE and $A'C$.

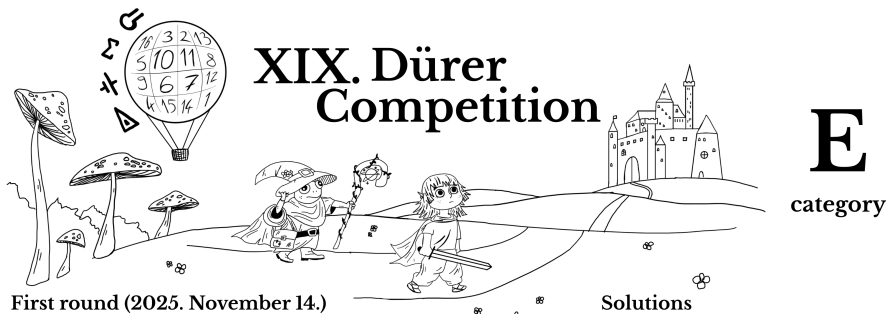


First, we prove that GI and FH are parallel to AA' . Because GI and FH were produced in the same way, it suffices to prove it for one of them: let us prove it for GI .

We show that we can scale $AA'E$ into GIC from the centre B . We prove it by the scaling, which has centre B and sends E to C . If we denote the image of A as X , then because of the scaling, we have $\angle XCB = \angle AEB$ and that X is on AB , so we have $X = G$. Similarly, the image of A' is I . This shows that from centre B , we can scale $AA'E$ into GIC , and thus AA' is parallel to GI , as desired. (An alternative way to prove that AA' is parallel to GI , is simply by using Desargues' theorem for the triangles $AA'E$ and GIC .)

So we got that GI and FH are both parallel to AA' , so it is enough to prove that GH is also parallel to AA' . As $AA'CB$ is cyclic, it is equivalent to $GHCB$ being cyclic. Since DA' and EA are parallel to the diagonals of $GHCB$, and the two quadrilaterals also share the common sides AB and $A'C$, the cyclicity of $GHCB$ is equivalent to the cyclicity of $JKAA'$, so now it suffices to prove the latter. Because of inscribed angles, it is enough to have $\angle JAK = \angle JA'K$. As $BAA'C$ is cyclic, we have $\angle BAC = \angle BA'C$, so it is enough to have $\angle DA'B = \angle CAE$. But actually it is quite a known lemma: if $DAA'E$ and $BAA'C$ are cyclic such that $DBCE$ are collinear, then $\angle DA'B = \angle CAE$. The proof is simple from angle-chasing because of inscribed angles:

$$\angle CAE = \angle CAA' - \angle EAA' = \angle CBA' - \angle EDA' = (180^\circ - \angle DBA') - \angle EDA' = \angle DA'B$$



E5. The role-playing game D.Ü.R.E.R. is played by n players sitting around a circular table. The role of the storyteller is determined as follows. Initially, the oldest player holds a fair die. In one step, the player who currently has the die rolls it, and if the result is d , the die is passed d times one seat to the left. This step is repeated k times. The player who holds the die at the end becomes the storyteller. Determine all values of n for which there exists a k such that the above method of selection is fair.

A selection method is fair if every player has the same probability of becoming the storyteller.

Solution:

If there exists a number k for n players such that after k steps everyone has the same chance of being the storyteller, i.e., the remainder of the sum of the k numbers rolled is equally likely to be $0, 1, \dots, n-1$, then we call k a *fair number* for n . If a number is fair for a given n , then every positive integer greater than it will also be fair, since the last roll has a $\frac{1}{6}$ chance of giving the remainder of the sum $1, 2, \dots, 6$. If, without the last throw, we obtained all remainders with the same probability, i.e., the same number of throws belonged to each remainder, then the last roll only multiplies the number of corresponding throws for each remainder by six. Thus, if a number is fair for a given n , then every positive integer greater than it will also be fair.

If there is a fair number k_m for a number m , then it will also be suitable for every divisor a of m . In the case of $m = ab$, every remainder modulo m can be written in the form $ca + d$ where $0 \leq c < b$, and these are obtained with equal probability when rolling k_m times. Thus, we obtain every remainder of a with the same probability, namely b times the probability obtained for m , since the remainder of every remainder of the form $ca + d$ is d .

After k rolls, a total of 6^k different sequences of rolls could have been created. In order to obtain every remainder of n with equal probability, $n|6^k$ is required. Thus, the prime factors of n can only be 2 and 3. One roll is a fair selection for 1, 2, 3, and 6 people, so there is a corresponding k for these (every positive integer is a fair selection for these).

Finally, we see that for $n = 4$ and $n = 9$, there are no fair numbers, i.e., none of their multiples. Let us indirectly assume that there are fair numbers for them. In neither case does 1 roll distribute well, so we can assume that $j \geq 1$ rolls do not distribute well, but $j+1$ does. For $n = 4$, let j_i be the number of sequences consisting of j throws for which the sum is i ($i = 0, 1, 2, 3$, and the operations are to be understood modulo 4). Then we know that $(j+1)_i = j_i + j_{i+1} + 2j_{i+2} + 2j_{i+3}$ and this must be equal for all i . That is, $0 = (j+1)_{i+1} - (j+1)_i = j_i - j_{i+2}$, so $j_0 = j_2$ and $j_1 = j_3$ must be true. But j throws well for 2, i.e., $j_0 + j_2 = j_1 + j_3$, so $j_0 = j_2 = j_1 = j_3$, so even $j \geq 1$ had to roll well for $n = 4$, which is a contradiction.

Similarly, for $n = 9$, we can write down the value of $(j+1)_i$, where $i = 0, 1, \dots, 8$ and we calculate modulo 9. $(j+1)_i = j_{i-1} + j_{i-2} + j_{i-3} + j_{i-4} + j_{i-5} + j_{i-6}$ and this must take on the same value for all i . That is, $0 = (j+1)_{i+1} - (j+1)_i = j_i - j_{i-6}$ for all i . From this, $j_0 = j_3 = j_6$, $j_1 = j_4 = j_7$ and $j_2 = j_5 = j_8$. But $j \geq 1$ throws well for 3, i.e., $j_0 + j_3 + j_6 = j_1 + j_4 + j_7 = j_2 + j_5 + j_8$, meaning that all j_i must also be equal. Therefore, even j throws should have drawn well for $n = 9$, which is a contradiction.

n cannot be divided by either 4 or 9, so there are no other suitable n values apart from 1, 2, 3, and 6. Only for $n = 1, 2, 3, 6$ does a positive integer k exist for which the selection is fair.

Second solution:

By raising the polynomial $\frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6)$ to the k th power, we obtain a polynomial in which the coefficient of x^i is exactly equal to the probability that the sum of the throws is i . This means that a draw is good if the exponents are sorted into groups according to the remainder n and the sum of the coefficients is the same in each group. Then, substituting an n -th primitive unit root ε_n into this polynomial, using $\varepsilon_n^n = 1$, we get

$$\left(\frac{1}{6}(\varepsilon_n + \varepsilon_n^2 + \varepsilon_n^3 + \varepsilon_n^4 + \varepsilon_n^5 + \varepsilon_n^6) \right)^k = \frac{1}{n}(\varepsilon_n + \varepsilon_n^2 + \dots + \varepsilon_n^n).$$

Writing the right side as a product, if $\varepsilon_n \neq 1$

$$\left(\frac{1}{6}(\varepsilon_n + \varepsilon_n^2 + \varepsilon_n^3 + \varepsilon_n^4 + \varepsilon_n^5 + \varepsilon_n^6) \right)^k = \frac{1}{n} \varepsilon_n \frac{\varepsilon_n^n - 1}{\varepsilon_n - 1} = 0.$$

This means that

$$0 = \frac{1}{6}(\varepsilon_n + \varepsilon_n^2 + \varepsilon_n^3 + \varepsilon_n^4 + \varepsilon_n^5 + \varepsilon_n^6) = \frac{1}{6} \varepsilon_n \frac{\varepsilon_n^6 - 1}{\varepsilon_n - 1},$$



Which gives $\varepsilon_n^6 - 1 = 0$, so the n th primitive unit root is also a sixth unit root, i.e. if $\varepsilon_n \neq 1$, the only possible solutions are $n = 2, 3, 6$, and in this case $k = 1$ indeed gives a fair selection. We still need to look at $\varepsilon_n = 1$ separately, since we assumed $\varepsilon_n \neq 1$ during the calculation. In this case, $n = 1$ and every k gives a fair draw. Only for $n = 1, 2, 3, 6$ there exists a positive integer k for which the selection is fair.