XIX. Dürer Competition

E+

category

First round (2025. November 14.) **Solutions**

E+1. Tyrion and Littlefinger play a game that begins with 2026 empty boxes numbered from 0 to 2025. In each round, the next player in turn chooses a box with index i that does not yet contain a number, and then the other player chooses an integer a_i and places it in the box. The players alternate choosing boxes, with Tyrion going first. The game ends when every box contains a number. If $x = 5202$ is a solution for the equation $a_{2025}x^{2025} + a_{2024}x^{2024} + \ldots + a_1x + a_0 = 0$, then Tyrion wins, otherwise Littlefinger wins. Who has a winning strategy?
Throughout the game, both players know the contents of all boxes.

Solution: Tyrion has a winning strategy.

We prove by induction that Tyrion can guarantee the following property of the polynomial at each step:

Claim If k is the largest index of any empty box, pretend that we place 0 in all the still-empty boxes and substitute 5202 into the polynomial. Then Tyron can make sure this value is divisible by 5202^k.

Note that Tyron will place the integer in the last remaining empty box. Let k denote the index of this box. If, by placing 0 in this box, the polynomial on the left-hand side evaluated at 5202 is divisible by 5202^k, then Tyrion can choose a_k so that $-a_k \cdot 5202^k$ is exactly this value. Thus 5202 is a solution for the equation.

Proof of Claim The statement holds initially, as the 0 polynomial evaluated at 5202 is divisible by 5202.

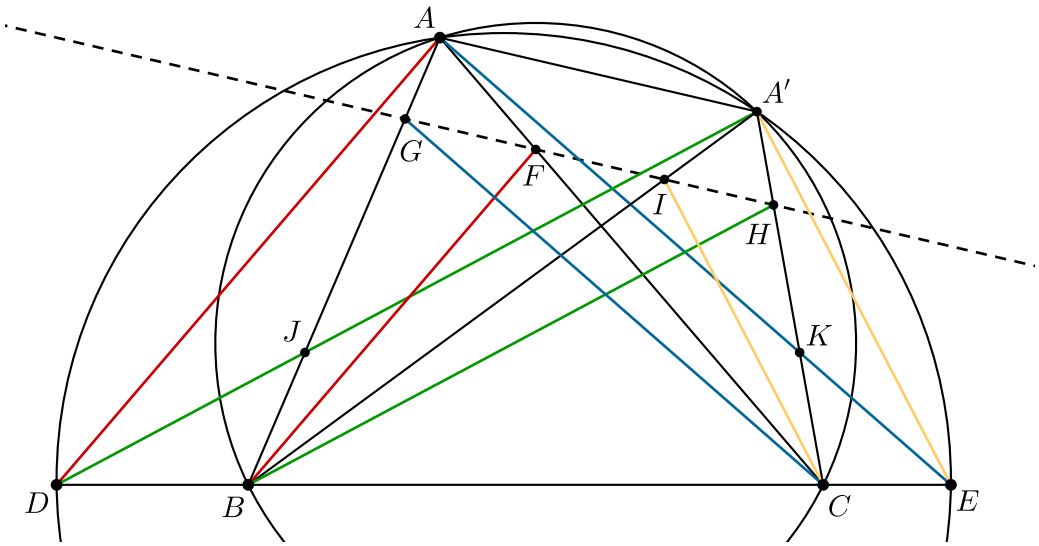
We need to check two cases based on whose turn it is.

If Littlefinger chooses the box and Tyrion the integer, then Tyrion should place 0 in the box. Then the index of the highest-index empty box cannot increase, and the polynomial does not change after placing any zeros.

If Tyron chooses the box and Littlefinger the integer, Tyron should choose the empty box with the largest index. Let k denote this index. Then, regardless of which integer Littlefinger chooses, the index of the highest-index empty box decreases. Before this step, it was k, so ignoring this box, the sum in question is divisible by 5202^k. During this step, the sum changes by $a_k \cdot 5202^k$, so it will still be divisible by 5202^k, and thus by all lower powers as well. Therefore, Tyrion can ensure the above property in this case too, and hence he indeed has a winning strategy.

E+2. Let ABC be a triangle and let D and E be two points on line BC such that the order of the four points is D, B, C, E. Let X denote the intersection point of the circumcircles of triangles ABC and ADE which is different from A. The line parallel to AD through B intersects line AC in point F, and the line parallel to AE through C intersects line AB in point G. Similarly the line parallel to XD through B intersects line XC in point H, and the line parallel to XE through C intersects line XB in point I. Show that points F, G, H, I lie on a line.

Solution: Let us make a colourful figure for better visibility. We will denote the point X by A' instead. Let J be the intersection of AB and DA', and let K be the intersection of AE and $A'C$.



First, we prove that GI and FH are parallel to AA'. Because GI and FH were produced in the same way, it suffices to prove it for one of them: let us prove it for GI.

We show that we can scale $AA'E$ into GIC from the centre B. We prove it by the scaling, which has centre B and sends E to C. If we denote the image of A as X, then because of the scaling, we have $XCB\angle = AEB\angle$ and that X is on AB, so we have $X = G$. Similarly, the image of A' is I. This shows that from centre B, we can scale $AA'E$ into GIC, and thus AA' is parallel to GI, as desired. (An alternative way to prove that AA' is parallel to GI, is simply by using Desargues' theorem for the triangles $AA'E$ and GIC.)

So we got that GI and FH are both parallel to AA', so it is enough to prove that GH is also parallel to AA'. As $AA'CB$ is cyclic, it is equivalent to $GHCB$ being cyclic. Since DA' and EA are parallel to the diagonals of $GHCB$, and the two quadrilaterals also share the common sides AB and $A'C$, the cyclicity of $GHCB$ is equivalent to the cyclicity of $JKAA'$, so now it suffices to prove the latter. Because of inscribed angles, it is enough to have $JAK\angle = JA'K\angle$. As $BAA'C$ is cyclic, we have $BAC\angle = BA'C\angle$, so it is enough to have $DA'B\angle = CAE\angle$. But actually it is quite a known lemma: if $DAA'E$ and $BAA'C$ are cyclic such that $DBCE$ are collinear, then $DA'B\angle = CAE\angle$. The proof is simple from angle-chasing because of inscribed angles:

$$CAE\angle = CAA'\angle - EAA'\angle = CBA'\angle - EDA'\angle = (180^\circ - DBA'\angle) - EDA'\angle = DA'B\angle$$

XIX. Dürer Competition

E+

category

First round (2025. November 14.) **Solutions**

E+3. A goblin has eaten all positive integers n for which n divides the sum of squares of every positive integer less than n that is also coprime to n. Prove that there exist 2025 consecutive positive integers, all of which have been eaten by the goblin!

Solution: Let $f(n)$ be the sum of the squares of the positive integers less than n that are coprime to n. First, we consider the case when $n = p^k$ for some p prime and $k \in \mathbb{N}$. We can write

$$\begin{aligned} f(p^k) &= \sum_{i=1}^{p^k-1} i^2 - \sum_{j=1}^{p^{k-1}-1} (pj)^2 \\ &= \frac{(p^k-1)p^k(2p^k-1)}{6} - p^2\frac{(p^{k-1}-1)p^{k-1}(2p^{k-1}-1)}{6} \\ &= \frac{p^k}{6}\left((p^k-1)(2p^k-1) - (p^{k-1}-1)p(2p^{k-1}-1)\right) \end{aligned}$$

The expression in the parentheses is congruent to 1 modulo p, so $v_p(f(p^k)) = k - v_p(6)$, where $v_p(a)$ denotes the exponent of p in the prime factorization of a. This means that the goblin ate p^k if and only if $p \neq 2, 3$, and for $p = 2, 3$, we have $p^{k-1} | f(p^k)$.

Let $n = p^k \cdot M$, where p does not divide M. By the Chinese Remainder Theorem, for every i smaller than and coprime to p^k and j smaller than and coprime to M, there is exactly one ℓ smaller than and coprime to $p^k \cdot M$, for which $\ell \equiv i \pmod{p^k}$ and $\ell \equiv j \pmod{M}$. So for each such i there are exactly $\phi(M)$ such ℓ where $\phi(M)$ denotes the number of positive integers not larger than and coprime to M. From this, we get

$$f(p^k \cdot M) \equiv \phi(M) \cdot f(p^k) \pmod{p^k}.$$

If we write n in the form $\prod_{i=1}^{m} p_i^{k_i}$, then the goblin ate n if and only if for all i, $\phi(n/p_i^{k_i}) \cdot f(p_i^{k_i})$ is divisible by $p_i^{k_i}$. If $p_i \neq 2, 3$, then the second factor is divisible by $p_i^{k_i}$, so the product is too. If $p_i \leq 3$, then we need $p_i | \phi(n/p_i^{k_i})$. Since $\phi(ab) = \phi(a)\phi(b)$ for a and b coprime, this is equivalent to some $j \neq i$ satisfying $p_i \mid \phi(p_j^{k_j}) = p_j^{k_j-1}(p_j - 1)$, which means $p_i \mid p_j - 1$. For $p_i = 2$, this is equivalent to n having another prime divisor; for $p_i = 3$, n having a prime divisor of the form $3k + 1$.

So the goblin ate every n except those that are powers of two, or divisible by 3 but not by any prime of the form $3k + 1$. In particular, the goblin ate all the numbers that are divisible by a prime of the form $3k + 1$.

Using this, we can find 2025 consecutive numbers such that the goblin ate all of them. Since there are infinitely many primes of the form $3k + 1$, we can pick 2025 of them, let these be $p_1, p_2, \ldots, p_{2025}$. According to the Chinese Remainder Theorem, there exists an n such that $n + i \equiv 0 \pmod{p_i}$ for all i, since p_i are pairwise coprime. For this n, we know that $n + 1, n + 2, \ldots, n + 2025$ all have a prime divisor of the form $3k + 1$, so the goblin ate all of them.

XIX. Dürer Competition

E+ category

First round (2025. November 14.) **Solutions**

E+4. Let p be a fixed prime number. We call a nonempty finite subset H of the plane *p-fantastic* if it is possible to write positive integers on the points of H such that all of the following three conditions hold:

- Not all points of H lie on a single line.
- There exists a point in H on which a number not divisible by p is written.
- If a line contains at least two points of H, then the sum of the numbers written on the points on the line is divisible by p.

Determine the minimum size of a p-fantastic set of points as a function of p.
Exactly one number is written on each of the points of H, and there are no numbers on other points.

Solution: The smallest size of such a set is $n = p + 1$.

Solution. First, we construct a p-fantastic set of size $p + 1$ that is not contained in a line, then show that there is no such set of size at most p.

Construction for $n = p + 1$. Let $S = \{(0, i) : 0 \le i \le p - 1\} \cup (1, 0)$. Then S has size $p + 1$ and is not contained in a line. Assign the label $+1$ to the points $\{(0, i) : 0 \le i \le p-1\}$ and -1 to $(1, 0)$. It is straightforward to verify that this labelling satisfies all conditions.

Proof that $n \le p$ is impossible. The cade of $p = 2$ is trivial, so we can assume that $p \ge 3$. Now suppose that S is p-good of size at most p, not contained in a line, and there is a good labelling $f : S \to \mathbb{Z}$. First, we show that p divides the sum of the labels, $s = \sum_{P \in S} f(P)$. Let L be the set of lines that contain at least two points from S, and for $\ell \in L$ let $c(\ell)$ be the number of points on the line ℓ. We can write:

$$0 \equiv \sum_{\ell \in L} \left[(c(\ell) - 1) \sum_{P \in \ell} f(P) \right] = \sum_{P \in S} \left[f(P) \sum_{\ell : P \in \ell} (c(\ell) - 1) \right] = \sum_{P \in S} f(P)(n - 1) = s(n - 1) \pmod{p}$$

As $3 \le n \le p$, we can divide by $n - 1$ modulo p, hence we get that $p \mid s$.

Now take an arbitrary point $P \in S$, and look at the lines going through P. The sum of the labels along each line is divisible by p, and the sum of these sums is exactly $s + (k - 1)f(P)$, where k lines go through P. But $2 \le k \le p-1$, so $p \nmid k-1$. As $p \mid s$, we have $p \mid f(P)$, so every label is divisible by p, but that is a contradiction. This finishes the proof.

XIX. Dürer Competition

E+
category

First round (2025. November 14.) **Solutions**

E+5. Is there a bounded infinite sequence of real numbers $a_1, a_2, \ldots$ for which $|a_i - a_j| > \frac{1}{|i-j|}$ holds for all $i \neq j$?
An infinite sequence of real numbers is called bounded if there exists a real number M for which $|a_i| \leq M$ holds for all i.

Solution: We will show that the sequence $a_n = 3\sqrt{2} \cdot \{n \cdot \sqrt{2}\}$ has this property. We need to verify that for any $i > j$, we have $|3\sqrt{2} \cdot \{i \cdot \sqrt{2}\} - 3\sqrt{2} \cdot \{j\sqrt{2}\}| > \frac{1}{i-j}$,, which, after dividing through, reduces to showing that

$$|\{i \cdot \sqrt{2}\} - \{j \cdot \sqrt{2}\}| > \frac{1}{3\sqrt{2}(i-j)}$$

The absolute value of the difference of the fractional parts is either $\{i\sqrt{2}\} - \{j\sqrt{2}\} = \{(i-j)\sqrt{2}\}$, if $\{i\sqrt{2}\} > \{j\sqrt{2}\}$, or $\{j\sqrt{2}\} - \{i\sqrt{2}\} = 1 - \{(i-j)\sqrt{2}\}$.

From now on let $||x||$ denote the distance between the real number x and the nearest integer, that is, $||x|| = \min(\{x\}, 1 - \{x\})$. From this it is clear that $|\{i \cdot \sqrt{2}\} - \{j \cdot \sqrt{2}\}| \geq ||(i-j)\sqrt{2}||$, so it suffices to show that $||(i-j)\sqrt{2}|| > \frac{1}{3\sqrt{2}(i-j)}$. For simplicity, denote $i - j$ by k, and let ℓ be the closest integer to $(i-j)\sqrt{2}$. Suppose that $||k\sqrt{2}|| = |k\sqrt{2} - \ell| \leq \frac{1}{3\sqrt{2}k}$. Multiplying both sides by $k\sqrt{2} + \ell$ gives $|2k^2 - \ell^2| \leq \frac{k\sqrt{2}+\ell}{3k\sqrt{2}}$. The left-hand side is a nonzero integer, so at least 1. On the other hand, $k\sqrt{2} + \ell \leq k\sqrt{2} + k\sqrt{2} + 1 < 3k\sqrt{2}$, since $k \geq 1$. Therefore, the right-hand side is less than 1, a contradiction.