E+
category

Final round (2026. February 6-8.) Traditional round problems

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1. A function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is called *magical* if for every n, the quantity $\sum_{d|n} f(d)$ is a power of two. Determine the smallest positive integer k for which there exists a magical function f such that each of the numbers $f(1), f(2), \ldots, f(2026)$ is at most k.
$\mathbb{Z}^+$ denotes the set of positive integers. The powers of two are considered to be powers of 2 with nonnegative integer exponent.

2. Sauron erased some, possibly infinitely many lattice points from the infinite unit square lattice such that the Euclidean distance between any two erased points is at least d, where d is a fixed positive number. Gandalf wants to visit all the remaining points along the lattice lines. In each step, he can only move to an adjacent remaining point, and he visits each of them exactly once. They noticed that no matter where Gandalf starts, he cannot visit all the remaining lattice points in this way. Determine all the possible values of d for which this can happen.
Two remaining lattice points are adjacent if their distance is 1.

3. Let H be the orthocentre of triangle ABC, and let M be the midpoint of BC. Let D be a point on the line BC such that $DH \perp AM$, and let E be the reflection of M with respect to B. Assume that the circle with diameter BE and the circumcircle of triangle AHD intersect at two points, let them be X and Y. Prove that X, Y and M are collinear.

4. We label the vertices of a graph in the following manner: to each vertex, we assign a positive integer not larger than its degree. We say that a simple, connected graph is *beautiful* if for every such labeling, there exists a walk in the graph whose endpoints may be arbitrary, and which visits each vertex exactly as many times as its label. What is the minimum number of edges a beautiful graph with $n > 1$ vertices can have?
During a walk in a graph, we may visit vertices and edges multiple times.

5. Let $P(x)$ be a polynomial with nonnegative real coefficients, and $P(0) = 0$. S uppose that if $0 \le x \le 1$, then $P(-2025x) \ge -2025P(x)$. Let $x_1, \ldots, x_{2026}$ be real numbers whose sum is nonnegative, and assume that $-2025 \le x_i \le 1$ holds for all x_i. Prove that

$$P\left(\frac{x_1 + \ldots + x_{2026}}{2026}\right) \le \frac{P(x_1) + \ldots + P(x_{2026})}{2026}.$$

6. Game: At the start of the game, there are eight positive integers on the first level, and a positive integer k is given, which is at most the sum of the eight numbers. The players take turns alternately, and in each turn, the current player erases two numbers from the same level, and writes their sum to the next level. The winner is the player who writes a number greater than or equal to k first.
Defeat the organisers twice in a row in this game! First, the organisers determine the eight numbers and k, then you get to choose whether you want to play as the first or the second player.

Please write all the solutions on separate pages. Make sure to write the name of your team and the category on every paper.
Each problem is worth 12 points. For a substantially different second solution or generalization, up to 2 extra points per problem might be awarded. The duration of the contest is 180 minutes. Good luck!

the organisers of the XIX. Dürer Competition