XIX. Dürer Competition

E+ category

Final round (2026. February 6-8.) **Traditional round solutions**

E+1. A function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is called *magical* if for every n, the quantity $\sum_{d|n} f(d)$ is a power of two. Determine the smallest positive integer k for which there exists a magical function f such that each of the numbers $f(1), f(2), \dots, f(2026)$ is at most k.
$\mathbb{Z}^+$ denotes the set of positive integers. The powers of two are considered to be powers of 2 with nonnegative integer exponent.

Solution: The smallest such number is 512.

First solution: First we show that it cannot be smaller. Let $f^+(n) = \sum_{d|n} f(d)$.

Then $f^+(1) < f^+(2) < \ldots < f^+(1024)$ are different powers of two, so $f^+(1024) \geq 1024$, and $f^+(512) \leq \frac{1}{2} f^+(1024)$. From this $f(1024) = f^+(1024) - f^+(512) \geq \frac{1}{2} f^+(1024) \geq 512$.

Now we need a construction for this value. We write n as a product $\prod_{i=1}^{k} p_i^{\alpha_i}$ where p_i are pairwise different prime numbers, and $\alpha_i \in \mathbb{Z}^+$ for each $1 \leq i \leq k$. Using this, let $f(n) = 2^{\alpha_1+\alpha_2+\dots+\alpha_k-k}$.

If $n \leq 2026$ then $\alpha_1 + \alpha_2 + \ldots + \alpha_k \leq 10$ and for $n > 1$ we have $k \geq 1$, therefore $f(n) \leq 512$. Now we only need to show that $\sum_{d|n} f(d)$ will be a power of two for every n.

We will show this by induction with respect to the number of different prime factors of n.

If $n = p^\alpha$ then $\sum_{d|n} f(d) = \sum_{i=0}^{\alpha} f(p^i) = 1 + \sum_{i=1}^{\alpha} 2^{i-1} = 2^\alpha$, which is a power of two.

Now suppose that n has at least two different prime divisors. Then there exist a and b positive integers that are coprimes, for which $ab = n$ and both a and b have fewer prime factors than n does. (For example using the previous prime factorization $a = p_1^{\alpha_1}$, $b = \frac{n}{a}$ is a good choice.)

Note that if a and b are coprimes then $f(ab) = f(a)f(b)$, furthermore if $c \mid a$ and $d \mid b$ then c and d are also coprimes. Therefore $f^+(n) = \sum_{c|a} \sum_{d|b} f(cd) = \sum_{c|a} \sum_{d|b} f(c)f(d) = f^+(a)f^+(b)$, because the divisors of n are exactly the numbers that can be written as a product of a divisor of a and a divisor of b, and these products are pairwise different.

From the induction hypothesis $f^+(a)$ and $f^+(b)$ are powers of two, thus their product $f^+(n)$ is also a power of two. So we are done with proving the induction step.

Second solution: We prove the lower bound the same way as in Solution 1.

First let us define $f^+(n)$ the following way: if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ then $f^+(n) = 2^{\alpha_1+\dots+\alpha_k}$.

Here $f^+(n)$ is a power of two, so now we need to find a function f that has positive integer values everywhere, and leads to this f^+ function.

Let us use the Mobius inversion formula, which is

$$f(n) = \sum_{d|n} \mu(d)\, f^+\left(\frac{n}{d}\right), \qquad \mu(d) = \begin{cases} 0, & \text{if } \exists\, k: \ k^2 \mid d, \\ (-1)^k, & \text{if } d \text{ equals the product of } k \text{ different primes.} \end{cases}$$

From this

$$f(n) = \sum_{d|n} \mu(d) f^+\left(\frac{n}{d}\right) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} 2^{s-i} = 2^{s-k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \cdot 2^{k-i} = 2^{s-k}(2-1)^k = 2^{s-k},$$

where $s = \alpha_1 + \ldots + \alpha_k$, which is a nonnegative integer, so we are done.

Note: in both solutions, the construction is the same as setting each $f(n)$ to the smallest value where $f^+(n)$ is a power of two, one by one for each n in increasing order.

E+2. Sauron erased some, possibly infinitely many lattice points from the infinite unit square lattice such that the Euclidean distance between any two erased points is at least d, where d is a fixed positive number. Gandalf wants to visit all the remaining points along the lattice lines. In each step, he can only move to an adjacent remaining point, and he visits each of them exactly once. They noticed that no matter where Gandalf starts, he cannot visit all the remaining lattice points in this way. Determine all the possible values of d for which this can happen.
Two remaining lattice points are adjacent if their distance is 1.

Solution: It is possible for all values of d.

For each even d, we will construct a set of lattice points such that the minimal distance between its points is d, but if Sauron erases this set, Gandalf can't traverse all the remaining points.

Let Sauron erase point (a, b) if and only if $d|a$ and $d|b$.

Color the lattice black and white like a chessboard. Since d is even, all the erased points have the same colour.

Assume towards contradiction that Gandalf can traverse the remaining points.

Let a *segment* of Gandalf's path be a set of lattice points which were visited consecutively during the walk. On each segment, the squares are alternately white and black, so the difference of their number is at most 1.

Consider a $dn \times dn$ grid for arbitrary $n \in \mathbb{Z}^+$, which doesn't contain Gandalf's starting point. The intersection of this square with Gandalf's path is a disjoint union of segments, whose endpoints lie on the boundary of the square, so there are at most $2(dn - 1)$ such segments. This means that the total difference of the number of black and white non-erased points in this square is at most $2(dn - 1)$.

On the whole square this difference is at most 1, and on the erased points it is exactly n^2, so $n^2 - 1 \leq 2(dn - 1)$.

If n is large enough, $n^2 - 1 > 2dn - 2$, which is a contradiction, so Gandalf cannot visit all the remaining points.

E+3. Let H be the orthocentre of triangle ABC, and let M be the midpoint of BC. Let D be a point on the line BC such that $DH \perp AM$, and let E be the reflection of M with respect to B. Assume that the circle with diameter BE and the circumcircle of triangle AHD intersect at two points, let them be X and Y. Prove that X, Y and M are collinear.

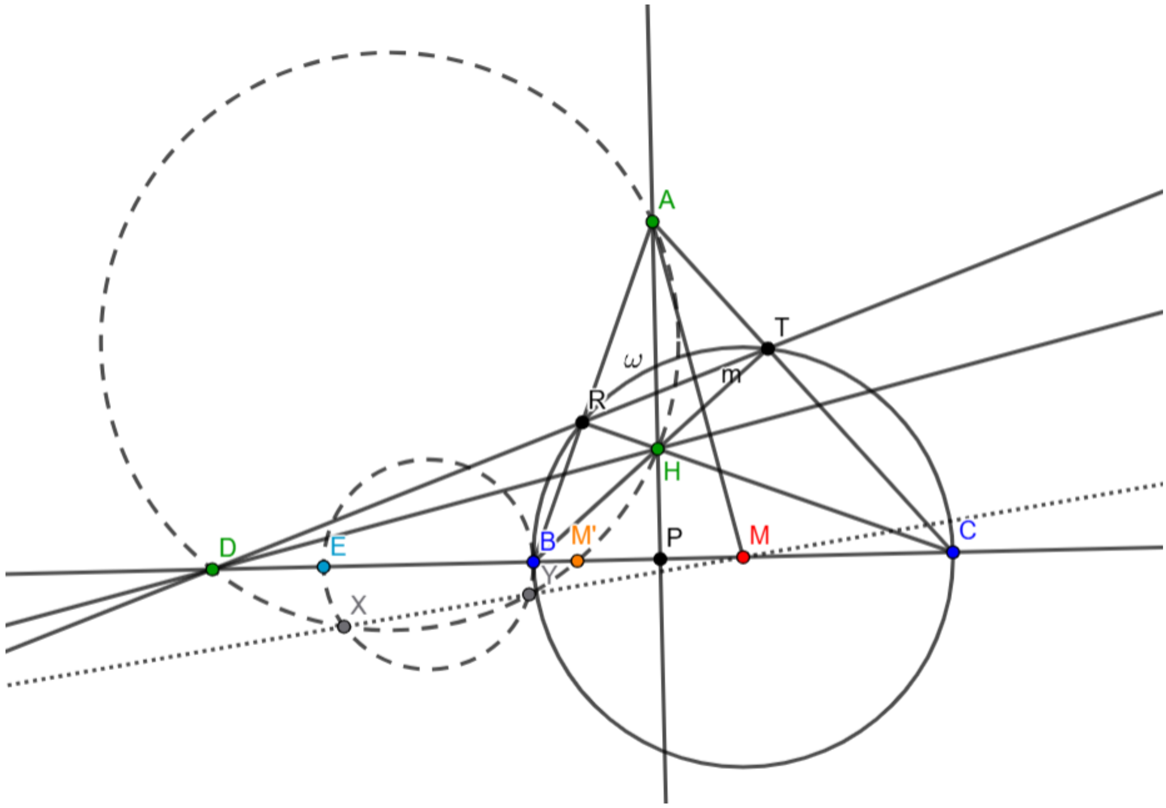
Solution: Let ω be BC's Thales circle, thus M will be the its midpoint. Futhermore the points T and R lie on ω, which are the feet of the altitudes from B and C. Let D' be the intersection of the lines TR and BC, we will show now that D' is in fact D. Drawing the lines of sides and diagonals in the cyclic quadrilateral $BCTR$, the intersections will be A, H and D'. Thus A's polar respect to ω is $D'H$, this yields that it is perpendicular to AM. This means that $D' = D$. Similarily the polar of D with respect to ω is AH, therefore M is the orthocenter of triangle ADH. Now let M' be the reflection of M onto AH. Then M' lies on the circle ADH since the reflection of a triangle's orthocenter onto its side will be on the circumcircle of the triangle. Let P be the foot of the altitude AH. Now $(BCDP) = -1$, thus the powers from M to the circles yields

$$MP \cdot MD = MB \cdot MC$$

Multiplying both sides by 2, we have

$$MM' \cdot MD = ME \cdot MC$$

Thus the power of M to the circles $AHDM'$ and BC's Thales circle are the same, So M' is on the radical axis of the circles, therefore it is on the line XY.



E+4. We label the vertices of a graph in the following manner: to each vertex, we assign a positive integer not larger than its degree. We say that a simple, connected graph is *beautiful* if for every such labeling, there exists a walk in the graph whose endpoints may be arbitrary, and which visits each vertex exactly as many times as its label. What is the minimum number of edges a beautiful graph with $n > 1$ vertices can have?

During a walk in a graph, we may visit vertices and edges multiple times.

Solution: Such a graph has at least

$$e(n) = \begin{cases} k(k-1)+1 & \text{if } n = 2k \\ k^2+1 & \text{if } n = 2k+1 \end{cases}$$

edges.

We will first show that this can be achieved. Note that $e(n)$ is the number of edges in a graph with n vertices where the edges are split as evenly as possible between two classes, which each form a clique, and the two cliques are connected by a single edge. We will show that in general, if we connect a clique with $n+1$ vertices and one with $m+1$ vertices with a single edge, the resulting graph will be nice.

Let $A = \{a_0, a_1, \ldots, a_n\}$ and $B = \{b_0, b_1, \ldots, b_m\}$, where the subgraphs spanned by A, and B are complete, and there is an edge between a_0 and b_0. Let the degree of vertex x be $d(x)$, let the value written on this vertex be $1 \le f(x) \le d(x)$.

First create a list containing the values a_i for $1 \le i \le n$, each exactly $f(a_i)$ times, such that two adjacent vertices are not the same. To do this, choose the i for which $f(a_i)$ is maximal and write a_i exactly $f(a_i)$ times. Then insert the values a_j for $i \ne j$ in any order, each at most $f(a_j)$ times, resulting in a list with alternating values of a_i and other vertices.

This is possible because $f(a_i) \le d(a_i) = n$, $f(a_j) \ge 1$, and because there are exactly $n-1$ indices j such that $i \ne j$, so for example we could use every such index one time. If there are any a_j values left which we did not use $f(a_j)$ times, then we insert these in an arbitrary order between two values that are not a_j. This can be done because the length of the list is at least $2(f(a_j)-1)$, so vertices can be inserted in at least $2(f(a_i))$ positions, and at most $2(f(a_j)-1) \le 2(f(a_i)-1)$ positions can't be inserted into.

Now insert a_0 exactly $f(a_0)$ times either at the end of the list or between numbers in the list without inserting between two neighbouring numbers twice. This is possible because the list is at least n elements long and $f(a_0) \le d(a_0) = n+1$. Create the list for the B graph in a similar manner such that it's first element is b_0. Finally concatenate the two lists, resulting in a walk in G that satisfies the conditions.

Now we move on to the proof of the lower bound. Note that if we write $d(a_i)$ on a vertex, and 1 on all of its neighbours, then this vertex has to be one of the endpoints of a good walk, because otherwise the occurences of a_i in the walk together have at least $d(a_i)+1$ neighbours, which are all the neighbours of a_i. But there is only $d(a_i)$ of them, and each can occur only 1 time in the walk, which is a contradiction.

Now if we have three independent vertices, and we write their degree on them, and 1 on all the other vertices, then any good walk should have all three as endpoints, which is impossible, so the graph cannot be beutiful.

From Turán's (or Mantel's) theorem, the complement of the graph has minimum number of edges if and only if it is the complete bipartite graph on $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ points. Since our graph has to be connected, our construction is optimal.

E+5. Let $P(x)$ be a polynomial with nonnegative real coefficients, and $P(0) = 0$. S uppose that if $0 \leq x \leq 1$, then $P(-2025x) \geq -2025P(x)$. Let $x_1, \dots, x_{2026}$ be real numbers whose sum is nonnegative, and assume that $-2025 \leq x_i \leq 1$ holds for all x_i. Prove that

$$P\left(\frac{x_1 + \ldots + x_{2026}}{2026}\right) \leq \frac{P(x_1) + \ldots + P(x_{2026})}{2026}.$$

Solution: Suppose that numbers $x_1, x_2, \dots, x_k$ are negative, and $x_{k+1}, x_{k+2}, \dots, x_{2026}$ are non-negative. If all the numbers are non-negative, let $k = 0$.

For all $1 \leq i \leq k$, $-2025 \leq x_i < 0$, so $0 < -\frac{x_i}{2025} \leq 1$. Let $y_i = -\frac{x_i}{2025}$. Using the given inequality for $y_1, \dots, y_k$, we get $P(x_i) \geq -2025P(y_i)$. Summing this for all y_i gives

$$P(x_1) + P(x_2) + \ldots + P(x_k) \geq -2025\left(P(y_1) + P(y_2) + \ldots + P(y_k)\right)$$

Note that

$$P(y_1) + P(y_2) + \ldots + P(y_k) \leq P(y_1 + y_2 + \ldots + y_k),$$

because the terms on the right side are those on the left side, plus a number of other terms with non-negative coefficients, and all y_i are non-negative. From this, we get

$$P(x_1) + \ldots + P(x_{2026}) \geq -2025((P(y_1) + \ldots + P(y_k)) + P(x_{k+1}) + \ldots + P(x_{2026}) \geq$$

$$\geq -2025P(y_1 + y_2 + \ldots + y_k) + (k-1)P(0) + P(x_{k+1}) + \ldots + P(x_{2026})$$

using that $P(0) = 0$.

Note that $P(x)$ is convex on $\mathbb{R}^+$, because its second derivative is a polynomial with non-negative coefficients, so it is non-negative on R^+. Therefore by using Jensen's inequality

$$(k-1)P(0) + P(x_{k+1}) + P(x_{k+2}) + \ldots + P(x_{2026}) \geq 2025P\left(\frac{x_{k+1} + x_{k+2} + \ldots + x_{2026}}{2025}\right).$$

Now let $X = \frac{x_{k+1}+x_{k+2}+\ldots+x_{2026}}{2025}$ and $Y = y_1 + y_2 + \ldots + y_k$. Then we only need to show that

$$\frac{2025P(X) - 2025P(Y)}{2026} \geq P\left(\frac{2025X - 2025Y}{2026}\right),$$

since $2025X - 2025Y = x_1 + x_2 + \ldots + x_{2026}$. For this we first need that

$$\frac{2025P(X) - 2025P(Y)}{2026} \geq \frac{2025}{2026}P(X - Y),$$

since $P(X - Y) + P(Y) \leq P(X)$, and we have shown before that $P(a) + P(b) \leq P(a + b)$ holds for any positive numbers a, b. Furthermore

$$\frac{2025}{2026}P(X - Y) \geq P\left(\frac{2025}{2026}(X - Y)\right),$$

since by writing both sides as a polynomial of $X - Y \geq 0$, on the left side each coefficient is multiplied by $\frac{2025}{2026}$, while on the right side the coeffcient of x^i is multiplied by $\left(\frac{2025}{2026}\right)^i$, which is less than equal to $\frac{2025}{2026}$ for all $i \geq 1$. But since we know that $P(0) = 0$, the constant term of P is zero, therefore each term is larger on the left hand side than on the right, and this is enough. By combining the last two inequalities, we arrive to the problem's statement.

E+6. Game: At the start of the game, there are eight positive integers on the first level, and a positive integer k is given, which is at most the sum of the eight numbers. The players take turns alternately, and in each turn, the current player erases two numbers from the same level, and writes their sum to the next level. The winner is the player who writes a number greater than or equal to k first.

Defeat the organisers twice in a row in this game! First, the organisers determine the eight numbers and k, then you get to choose whether you want to play as the first or the second player.

Solution: Let the numbers on the first level be $a_1 \geq a_2 \geq \cdots \geq a_8$. On the second, third and fourth level, let the numbers in the order they are written be b_1, b_2, b_3, b_4, c_1, c_2 and d.

We state that the second player has a winning strategy if and only if $a_1 + a_2 + a_7 + a_8 < k$ and $a_3 + a_4 + a_5 + a_6 \geq k$.

Assume that one of these conditions is not satisfied. We will show that the first player has a winning strategy.

The first player should write $b_1 = a_1 + a_2$ as his first move. If he does not win by this, then the second player writes the number $b_2 \geq a_7 + a_8$. He cannot win by doing this, as $b_2 \leq b_1$. In the next move, the first player should write $c_1 = b_1 + b_2 \geq a_1 + a_2 + a_7 + a_8$. If it is at least k, then the first player wins. If it is less than k, then by the assumption, we know that $a_3 + a_4 + a_5 + a_6 < k$. During the next three moves, the largest number obtainable is $c_2 = b_3 + b_4 \leq a_3 + a_4 + a_5 + a_6 < k$, so the first player indeed wins with $d = c_1 + c_2 \geq k$.

Now we will show that the second player wins if both the above conditions hold. As $a_1 + a_2 < k$, we know that one can only win with the numbers c_1, c_2 or d.

Let the first player write b_1, then the second player should write the smallest b_2 possible. Now the first player can either write $c_1 = b_1 + b_2 \leq a_1 + a_2 + a_7 + a_8 < k$ or a b_3. If he writes b_3, then the second player should write $c_1 = b_1 + b_2 < k$ and vica versa. So the next number which can win is c_2. We can see that c_2 will be written down in the sixth move, so the second player writes it. As it does not contain neither a_7 or a_8 (as they are already in b_1 or b_2), we know that $c_2 = b_3 + b_4 \geq a_3 + a_4 + a_5 + a_6 \geq k$, so the second player indeed wins.